FREDHOLM AND LOCAL SPECTRAL THEORY, WITH APPLICATIONS TO MULTIPLIERS

Fredholm and Local Spectral Theory, with Applications to Multipliers

by

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A mia madre e a mia sorella Mariella

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Preface

A significant sector of the development of spectral theory outside the classical area of Hilbert space may be found amongst at multipliers defined on a complex commutative Banach algebra A. Although the general theory of multipliers for abstract Banach algebras has been widely investigated by several authors, it is surprising how rarely various aspects of the spectral theory, for instance Fredholm theory and Riesz theory, of these important classes of operators have been studied. This scarce consideration is even more surprising when one observes that the various aspects of spectral theory mentioned above are quite similar to those of a normal operator defined on a complex Hilbert space.

In the last ten years the knowledge of the spectral properties of multipliers of Banach algebras has increased considerably, thanks to the researches undertaken by many people working in local spectral theory and Fredholm theory. This research activity recently culminated with the publication of the book of Laursen and Neumann [**214**], which collects almost every thing that is known about the spectral theory of multipliers.

The beautiful book of Laursen and Neumann has as a main motivation that of providing a modern introduction to local spectral theory and a particular emphasis is placed on the applications of the general local spectral theory to convolution operators on group and measure algebras. This book contains also several results on Fredholm theory. However, most of the aspects of Fredholm theory which are developed in this book are those that may be particularly approached through the methods of local spectral theory.

In this book I will try, in a certain sense, to reverse this process. Indeed, our first major concern is with the Fredholm theory, in particular the interested reader will find here a distinct flavor of this theory, which is emphasized by the chapters devoted to the Kato decomposition, to the abstract Fredholm theory in semi-prime Banach algebras, and by the chapter devoted to inessential operators between Banach spaces.

A second concern of this monograph is that of showing how the interplay between Fredholm theory and local spectral theory is significant and beautiful. It should be clear that relative to the part on local spectral theory, the content is rather limited, and the notions developed in this monograph are those which are of interest for the applications to multipliers of commutative semi-simple Banach algebras.

The deep interaction between the Fredholm theory and local spectral theory becomes evident when one consider the so called single-valued extension property. This property, which dates back to the early days of local spectral theory, was first introduced in 1952 by Dunford and has received a systematic treatment in the classical Dunford and Schwartz book [**97**]. It also plays an important role in the book of Colojoară and C. Foiaş [**83**], in Vasilescu [**309**], and in the recent book of Laursen and Neumann [**214**]. This single-valued extension property is one of the major unifying themes for a wide variety of linear bounded operators in the spectral decomposition property. This property is, for instance, satisfied by normal operators on Hilbert spaces and more generally by decomposable operators. Examples of non-decomposable operators having this property may be found amongst the class of all multipliers of semi-simple Banach algebras. In fact, every multiplier of a semi-prime Banach algebra has the single-valued extension property and there exist examples of convolution operators which are not decomposable.

A third aim of this book is to present some important progress, made in recent years, in the study of perturbation theory for classes of operators which occur in Fredholm theory. This subject is covered by several excellent books, such as the monographies by Kato [**183**], by Heuser [**160**], by Caradus, Pfaffenberger, and Yood [**76**], by Pietsch [**263**], but naturally they do not present the more recent results. Some of these new results solve very old open problems in operator theory and this book may be intended as a complement of the books mentioned above.

For those unacquainted with the subject matter, examples and motivations for certain definitions are mentioned in order to give some feeling for what is going on.

Now we describe in more detail the architecture of this monograph. This book consists of seven chapters.

The first chapter is devoted to the Kato decomposition for bounded operator on Banach spaces. This decomposition property arises from the classical treatment of perturbation theory of Kato [**182**], and its flourishing has greatly benefited from the work of many authors in the last ten years, in particular from the work of Mbekhta [**226**], [**227**], Müller [**240**], Rakočević [**274**] and Schmoeger [**290**]. The operators which satisfy this property form a class which includes the class of semi-Fredholm operators. This is an important result of Kato and here we present the proof given by West [**324**]. We shall develop, quite systematically, the properties of some important subspaces which play an important role in this decomposition theory, such as the analytical core, the quasi-nilpotent part, and the Kato resolvent of an operator. We shall also introduce the concepts of semi-regular, essentially semi-regular, and Kato type of operators. All these concepts generate distinguished spectra, and particular emphasis is placed on the spectral mapping theorems for these spectra.

The second chapter deals with a localized version of the single-valued extension property at a point and relates it, in the third chapter, to the finiteness of some classical quantities associated with an operator. These properties, such as the ascent and the descent of an operator, are the basic bricks of Fredholm theory, so the third chapter may be viewed as the part of the book in which the interaction between local spectral theory and Fredholm theory comes into focus. In fact, for semi-Fredholm operators the single-valued extension property at a point may be characterized in several ways, and some of these characterizations lead to the localization of some distinguished part of the spectra originating from Fredholm theory. Moreover, some classical spectral mapping theorems related to semi-Browder spectra and to the Browder spectrum, as well as the classification of the open connected components of the Fredholm, or the semi-Fredholm, region being able to be obtained via the localized single-valued extension property. The third chapter also deals with another important class of operators, the class of Riesz operators. Many of the results related to these operators are here revisited through the single-valued extension property at a point.

The fourth chapter is devoted to the basic ingredients of the theory of multipliers of commutative semi-simple Banach algebras. Our intention is to make this book as much self-contained as possible, so we shall give in our exposition of the elementary theory of multipliers the proofs of all the results needed in the subsequent chapters. Of course, part of the material concerning the basic theory of multipliers may be found in some books on the subject, for instance in the monograph by Larsen [**199**], or the book by Laursen and Neumann [**214**]. However, some parts of this chapter seems to be new, and the basic material treated here is that strictly related to the study of spectral properties of multipliers. The theory of multipliers may also be developed for non-commutative Banach algebras. We shall not investigate this case extensively but instead shall restrict our study essentially to the commutative algebras. In this case the Gelfand theory permits to represent every multiplier on a commutative semi-simple Banach algebra as a bounded continuous complex-valued function, the Helgason–Wang function defined on the regular maximal ideal space of the algebra. By means of this representation it appears evident how the spectral properties of a multiplier are related, in a certain measure, to the range of the Helgason–Wang function. In this chapter we shall also give an account of the multiplier theory of various algebras: group algebras, Banach algebras with an orthogonal basis, commutative H^* algebras and commutative C^* Banach algebras. In particular, we describe the multiplier algebras and the ideal of all compact multipliers of these algebras. The characterizations of compact multipliers will permit to us to characterize the Fredholm operators which are multipliers for several concrete Banach algebras.

The initial part of the fifth chapter is devoted to the introduction of some basic ingredients of the abstract Fredholm theory on a not necessarily commutative Banach algebras. It should be mentioned that the interested reader may find a more complete treatment of this theory in the monograph by Barnes, Murphy, Smyth, and West [**62**], although our approach to the abstract Fredholm theory is somewhat different. The central notion of this chapter is that of inessential ideals. These ideals will be introduced by using some ideas of Aupetit [**53**].

An abstract Fredholm theory on a Banach algebra $\mathcal{A}$ is essentially the theory of all invertible elements of $\mathcal{A}$ modulo an essential ideal, i.e., an ideal of $\mathcal{A}$ for which the spectrum of every element is a finite or a denumerable set. The classical Fredholm theory for operators on Banach spaces corresponds to the Fredholm theory of the semi-simple Banach algebra $\mathcal{A} := L(X)$ of all bounded operators on a Banach space X with respect to the inessential ideal of all bounded finite rank operators. However, the abstract Fredholm theory in Banach algebras is not merely a generalization of the classical Fredholm theory for operators. An illuminating example which strengthens this assertion is given, in fact, by the theory of multipliers of semi-prime Banach algebra. Indeed, in a semi-prime Banach algebra, the Fredholm theory of multipliers is exactly the Fredholm theory of the semi-prime multiplier algebra with respect to the socle of the algebra, or, which is the same thing, with respect to the socle of the multiplier algebra. This result allows us to give a description of multipliers which are Fredholm operators by involving only objects of the same class, not involving operators which are not multipliers. As a consequence we can obtain a very clear description of convolution operators of group algebras which are Fredholm.

We shall focus our attention on the inessential ideal of the socle, which corresponds in the case of the Banach algebra of all bounded linear operators to the ideal of all bounded finite rank operators. Subsequently we shall consider the theory of Riesz algebras of Banach algebras, developed by Smyth [**304**]. We shall give further information on the Fredholm theory of multipliers of commutative regular Tauberian Banach algebras. These results deal to the complete description of Fredholm convolution operators acting on group algebras.

The content of the sixth chapter concerns some other important tools of local spectral theory. The first part of the chapter concerns the class of all decomposable operators, but we shall avoid the duality theory of these operators, since this theory requires two functional models for operators on Banach spaces, and to develop this argument in this book would lead too far afield. Another important part of Chapter 6 concerns the study of the property of decomposability within the theory of multipliers of commutative semi-simple Banach algebras. In this framework the Fredholm theory, the Gelfand theory, the local spectral theory, and harmonic analysis are closely intertwined. This part is strongly based on the work of Laursen, Neumann [**210**], [**243**], and Eschmeier, Laursen, and Neumann [**112**].

The concluding chapter of the book concerns some perturbation classes of operators which occur in Fredholm theory. In this theory we find two fundamentally different classes of operators: semi-groups, such as the class of Fredholm operators, the classes of upper and lower semi-Fredholm operators, and ideals, such as the classes of finite dimensional, compact operators. A perturbation class associated to one of these semi-groups is a class of operators T for which the sum of T with an operator of the semi-group is still an element of the semi-group. Here the concept of inessential operator reveals its significance in operator theory: the ideal of all inessential operators is the perturbation class of some important semi-groups, for instance the semi-group of Fredholm operators.

The notion of inessential operator is, in this chapter, extended to operators acting between different Banach spaces, and our presentation includes several examples of inessential operators acting between classical Banach spaces. These examples require the knowledge of the structures of the Banach spaces involved. For many pairs of classical Banach spaces the class of inessential operators is the space of *all* bounded operators. This property contrasts, curiously, with the historical denomination given at this class. The class of inessential operators, as well as the class of Riesz operators, presents also an elegant duality theory, and to make this clear we shall introduce two useful classes of operators, the classes $\Omega_+(X)$ and $\Omega_-(X)$, which are in a sense the dual of each other.

The third section of Chapter 7 addresses the study of two other important classes of operators, the class of all strictly singular operators and the class of all strictly cosingular operators. Both these classes of operators are contained, respectively, in the perturbation class of upper semi-Fredholm operators and the perturbation class of lower semi-Fredholm operators. For many classical Banach spaces these inclusions are actually equalities, but we shall also give a recent example of González [**131**] which shows that these inclusions are, in general, proper. This counterexample, which solves an old open problem in operator theory, is constructed by considering the Fredholm theory of a very special class of Banach spaces: the class of indecomposable Banach spaces.

An indecomposable Banach space is a Banach space which cannot be split into the direct sum of two infinite dimensional closed subspaces. The existence of indecomposable Banach spaces has been a long standing open problem and was raised by Banach in the early 1930s. This problem has been positively solved by Gowers and Maurey [**137**], who constructed an example of a reflexive hereditarily indecomposable Banach space.

A part of the chapter is also devoted to the improjective operators between Banach spaces. This class of improjective operators contains the class of inessential operators for all Banach spaces, and for several years it has been an open problem if the two classes coincide. We shall give the recent counterexample given by Aiena and González [**21**], which shows that if Z is an indecomposable Banach space which is neither hereditarily indecomposable nor quotient indecomposable then the ideal of inessential operators is properly contained in the set of improjective operators. In the last section of the chapter we shall briefly discuss two notions of incomparability of Banach spaces which originate from the class of all inessential and improjective operators. In particular, from the theory of indecomposable Banach spaces we shall construct a counterexample which shows that these two notions of incomparability do not coincide.

We conclude this preface by remarking that this monograph in intended for professional mathematicians and graduate students, especially those working in functional analysis. Of course, it is not possible to make a presentation such as this entirely self-contained, so a certain background in operator theory is required, specifically the classical Fredholm theory for operators. However, several ideas from Fredholm theory may be extracted from the chapter dedicated to the abstract Fredholm theory.

The present monograph is the result of intensive research work during the last ten years. There are several friends and colleagues to whom I am indebted for suggestions and ideas. In particular, I thank Manolo Gonzalez, Kield Laursen, Michael Neumann, and Mostafá Mbekhta. I also thank the participants of the Analysis Seminar of the Department of Mathematics of the Universidad de Oriente (Cumaná, Venezuela), and of the Department of Mathematics of the Universidad UCLA (Barquisimeto, Venezuela), where I gave several talks on some of the topics contained in this book. In particular I wish to thank Ennis Rosas for his numerous invitations and generous hospitality.

Finally, I would like to express my deep gratitude to my wife Maria and my sons Marco and Caterina for all the love, patience, and encouragement which they generously provided over the years. Without their presence writing this book would have been more difficult.

CHAPTER 1

The Kato decomposition property

The spectrum of a bounded linear operator can be divided into subsets in many different ways, depending on the purpose of the inquiry. In this chapter we shall look more closely to the spectrum of a bounded operator on a Banach space from the viewpoint of Fredholm theory, by introducing some special parts of the spectrum and, in the same time, acquiring gradually some basic tools and techniques needed for local spectral theory. In particular, we shall consider a part of the ordinary spectrum native from the Fredholm theory, the semi-regular spectrum, in literature also known as Kato spectrum, and some other spectra to it related.

The semi-regular spectrum was first introduced by Apostol [**48**] for operators on Hilbert spaces and successively studied by several authors in the more general context of operators acting on Banach spaces. The study of this spectrum was originated by the classical treatment of perturbation theory owed to Kato [**182**], which showed an important decomposition for semi-Fredholm operators.

Throughout this chapter all linear spaces or algebras will have the complex field $\mathbb{C}$ as their field of scalars. The first section of this chapter deals primarily with the basic properties of some important invariant subspaces associated with an operator. These subspaces, the hyper-range, the hyper-kernel and the algebraic core of an operator are introduced in the purely algebraic setting of a vector space. The importance of the role of these subspaces becomes more evident when one considers a special class of bounded operators on Banach spaces, the class of all semi-regular operators which will be introduced in the second section. In this section we shall also introduce the analytical core of a bounded operator on a Banach space X, a subspace of X which may be thought as the analytic counterpart of the algebraic core.

The concept of semi-regularity, amongst the various concepts of regularity originating from Fredholm theory, seems to be the most appropriate to investigate some important aspect of local spectral theory. This concept leads in a natural way to the above mentioned semi-regular spectrum $\sigma_{\rm se}(T)$, an important subset of the ordinary spectrum which is defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not semi-regular.

In the main results of the third section we establish many important properties of $\sigma_{\rm se}(T)$, as well as of its complement $\rho_{\rm se}(T)$. In particular we shall see that this spectrum is a non-empty compact subset of $\mathbb{C}$. We shall also establish non-elementary property of the Kato resolvent $\rho_{\mathrm{se}}(T)$: the analytical cores $K(\lambda I-T)$ are constant when λ ranges through a connected open component Ω of $\rho_{\mathrm{se}}(T)$. The proof of this property requires quite technical methods of gap theory. Another relevant result is that the semi-regular spectrum $\sigma_{\mathrm{se}}(T)$ contains the boundary $\partial\sigma(T)$ of the spectrum. Moreover, $\sigma_{\mathrm{se}}(T)$ obeys a spectral mapping theorem, in the sense that the semi-regular spectrum $\sigma_{\mathrm{se}}(f(T))$ coincides with $f(\sigma_{\mathrm{se}}(T))$, for every analytic function f defined on an open set containing the spectrum of T.

The fifth section addresses an important decomposition property, the generalized Kato decomposition for bounded operators on Banach spaces, originating from the work by Kato. A bounded operator T on a Banach space X has this property if X is the direct sum of two closed T invariant subspaces M and N, where T acts as a semi-regular operator on M and acts as a quasi-nilpotent operator on N. Further, we shall give a closer look at the more special cases in which the operator T is of Kato type, namely the restriction $T|N$ is nilpotent, or T is essentially semi-regular, that is $T|N$ is nilpotent and N is finite-dimensional. We shall establish some perturbation results, for instance, if T is of Kato type, then $\lambda I-T$ is semi-regular for all $\lambda\in\mathbb{C}$ which belong to a punctured open disc centered at 0.

Two important classes of operators in Fredholm theory are the classes of upper semi-Fredholm and lower semi-Fredholm operators. These operators are a natural generalization of Fredholm operators and one of the most important result of this chapter is that every semi-Fredholm operator is essentially semi-regular. To show this property we need to introduce the concept of the jump of a semi-Fredholm operator. We shall see that the semi-Fredholm operators which are semi-regular are exactly those having jump equal to zero.

All the classes of operators mentioned above motivate the study of some other distinguished parts of the spectrum, the semi-Fredholm spectra, the Kato type of spectrum $\sigma_{\mathrm{k}}(T)$, and the essentially semi-regular spectrum $\sigma_{\mathrm{es}}(T)$. We shall see how these spectra are related and show that a spectral mapping theorem holds for $\sigma_{\mathrm{es}}(T)$, i.e. the spectrum $\sigma_{\mathrm{es}}(T)$ behaves canonically under the Riesz functional calculus.

The last part of the chapter concerns another invariant subspace related to a bounded operator $T\in L(X)$ is the quasi-nilpotent part of T. Together with the basic properties of these subspaces we shall prove the local constancy of the closure of the quasi-nilpotent parts $H_0(\lambda I-T)$ on the connected components of $\rho_{\mathrm{se}}(T)$.

1. Hyper-kernel and hyper-range of an operator

The kernels and the ranges of the power T^n of a linear operator T on a vector space X form the following two sequences of subspaces:

$$\ker T^0=\{0\}\subseteq\ker T\subseteq\ker T^2\subseteq\cdots$$

and

$$T^0(X) = X \supseteq T(X) \supseteq T^2(X) \supseteq \cdots .$$

Generally all these inclusions are strict. In Chapter 5 we shall consider operators for which one or both of the two sequences becomes constant.

Definition 1.1. *Given a vector space X and a linear operator T on X, the* hyper-range *of T is the subspace*

$$T^\infty(X) = \bigcap_{n\in\mathbb{N}} T^n(X).$$

The hyper-kernel *of T is the subspace*

$$\mathcal{N}^\infty(T) := \bigcup_{n\in\mathbb{N}} \ker T^n.$$

It is easy to verify that both $T^\infty(X)$ and $\mathcal{N}^\infty(T)$ are T-invariant subspaces of X.

The following elementary lemma will be useful in the sequel.

Lemma 1.2. *Let X be a vector space and T a linear operator on X. If p_1 and p_2 are relatively prime polynomials then there exist polynomials q_1 and q_2 such that $p_1(T)q_1(T) + p_2(T)q_2(T) = I$.*

Proof If p_1 and q_1 are relatively prime polynomials then there are polynomials such that $p_1(\mu)q_1(\mu) + p_2(\mu)q_2(\mu) = 1$ for every $\mu \in \mathbb{C}$. ∎

The next result establishes some basic properties of the hyper-kernel and the hyper-range of an operator.

Theorem 1.3. *Let X be a vector space and T a linear operator on X. Then we have:*

(i) $(\lambda I + T)(\mathcal{N}^\infty(T)) = \mathcal{N}^\infty(T)$ *for every* $\lambda \neq 0$*;*

(ii) $\mathcal{N}^\infty(\lambda I + T) \subseteq (\mu I + T)^\infty(X)$ *for every* $\lambda \neq \mu$*.*

Proof (i) It suffices to prove that $(\lambda I + T)(\ker T^n) = \ker T^n$ for every $n \in \mathbb{N}$ and $\lambda \neq 0$. Clearly, $(\lambda I + T)(\ker T^n) \subseteq \ker T^n$ holds for all $n \in \mathbb{N}$. By Lemma 1.2 there exist polynomials p and q such that

$$(\lambda I + T)p(T) + q(T)T^n = I.$$

If $x \in \ker T^n$ then $(\lambda I + T)p(T)x = x$ and since $p(T)x \in \ker T^n$ this implies $\ker T^n \subseteq (\lambda I + T)(\ker T^n)$.

(ii) Put $S := \lambda I + T$ and write

$$\mu I + T = (\mu - \lambda)I + \lambda I + T = (\mu - \lambda)I + S.$$

By assumption $\mu - \lambda \neq 0$, so by part (i) we obtain that

$$(\mu I + T)(\mathcal{N}^\infty(\lambda I + T)) = ((\mu - \lambda)I + S)(\mathcal{N}^\infty(S)) = \mathcal{N}^\infty(\lambda I + T).$$

From this it easily follows that $(\mu I + T)^n(\mathcal{N}^\infty(\lambda I + T)) = \mathcal{N}^\infty(\lambda I + T)$ for all $n \in \mathbb{N}$, and consequently $\mathcal{N}^\infty(\lambda I + T) \subseteq (\mu I + T)^\infty(X)$. ∎

Lemma 1.4. *For every linear operator on a vector space X we have*

$$T^m(\ker T^{m+n}) = T^m(X) \cap \ker T^n \quad \text{for all } m, n \in \mathbb{N}.$$

Proof If $x \in \ker T^{m+n}$ then $T^m x \in T^m(X)$ and $T^n(T^m x) = 0$, so that $T^m(\ker T^{m+n}) \subseteq T^m(X) \cap \ker T^n$.

Conversely, if $y \in T^m(X) \cap \ker T^n$ then $y = T^m(x)$ and $x \in \ker T^{m+n}$, so the opposite inclusion is verified. ∎

The next result exhibits some useful connections between the kernels and the ranges of the iterates T^n of an operator T on a vector space X.

Theorem 1.5. *For a linear operator T on a vector space X the following statements are equivalent:*

(i) $\ker T \subseteq T^m(X)$ *for each* $m \in \mathbb{N}$*;*

(ii) $\ker T^n \subseteq T(X)$ *for each* $n \in \mathbb{N}$*;*

(iii) $\ker T^n \subset T^m(X)$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$*;*

(iv) $\ker T^n = T^m(\ker T^{m+n})$ *for each* $n \in \mathbb{N}$ *and each* $m \in \mathbb{N}$.

Proof The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (i) If we apply the inclusion (ii) to the operator T^m we obtain $\ker T^{mn} \subseteq T^m(X)$ and hence $\ker T \subseteq T^m(X)$, since $\ker T \subseteq \ker T^{mn}$.

(i) $\Rightarrow$ (iv) If we apply the inclusion (i) to the operator T^n we obtain $\ker T^n \subseteq (T^n)^m(X) \subseteq T^m(X)$. By Lemma 1.4 we then have

$$T^m(\ker T^{m+n}) = T^m(X) \cap \ker T^n = \ker T^n,$$

so the proof is complete. ∎

Corollary 1.6. *Let T be a linear operator on a vector space X. Then the statements of Theorem 1.5 are equivalent to each of the following inclusions:*

(i) $\ker T \subseteq T^\infty(X)$*;*

(ii) $\mathcal{N}^\infty(T) \subseteq T(X)$*;*

(iii) $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$.

The following subspace, introduced by Saphar [**284**] , is defined in purely algebraic terms.

Definition 1.7. *Let T be a linear operator on a vector space X. The* algebraic core $C(T)$ *is defined to be the greatest subspace M of X for which $T(M) = M$.*

Trivially, if $T \in L(X)$ is surjective then $C(T) = X$. Clearly, for every linear operator T we have $C(T) = T^n(C(T)) \subseteq T^n(X)$ for all $n \in \mathbb{N}$. From that it follows that $C(T) \subseteq T^\infty(X)$.

The next result gives a precise description of the subspace $C(T)$ in terms of sequences.

Theorem 1.8. *For a linear operator T on a vector space X the following statements are equivalent:*

(i) $x \in C(T)$;

(ii) *There exists a sequence $(u_n) \subset X$ such that $x = u_0$ and $Tu_{n+1} = u_n$ for every $n \in \mathbb{Z}_+$.*

Proof Let M denote the set of all $x \in X$ for which there exists a sequence $(u_n) \subset X$ such that $x = u_0$ and $Tu_{n+1} = u_n$ for all $n \in \mathbb{Z}_+$. We show first that $C(T) \subseteq M$.

Let $x \in C(T)$. From the equality $T(C(T)) = C(T)$, we obtain that there is an element $u_1 \in C(T)$ such that $x = Tu_1$. Since $u_1 \in C(T)$, the same equality implies that there exists $u_2 \in C(T)$ such that $u_1 = Tu_2$. By repeating this process we can find the desired sequence (u_n), with $n \in \mathbb{Z}_+$, for which $x = u_0$ and $Tu_{n+1} = u_n$. Therefore $C(T) \subseteq M$. Conversely, to show the inclusion $M \subseteq C(T)$ it suffices to prove, since M is a linear subspace of X, that $T(M) = M$.

Let $x \in M$ and let (u_n), $n \in \mathbb{Z}_+$, be a sequence for which $x = u_0$ and $Tu_{n+1} = u_n$. Define (w_n) by

$$w_0 := Tx \quad \text{and} \quad w_n := u_{n-1},\ n \in \mathbb{Z}_+.$$

Then

$$w_n = u_{n-1} = Tu_n = Tw_{n+1},$$

and hence the sequence satisfies the definition of M. Hence $w_0 = Tx \in M$, and therefore $T(M) \subseteq M$.

On the other hand, to prove the opposite inclusion, $M \subseteq T(M)$, let us consider an arbitrary element $x \in M$ and let $(u_n)_{n\in\mathbb{Z}_+}$ be a sequence such that the equalities $x = u_0$ and $Tu_{n+1} = u_n$ hold for every $(n \in \mathbb{Z}_+)$. Since $x = u_0 = Tu_1$ it suffices to verify that $u_1 \in M$. To see that let us consider the following sequence:

$$w_0 := u_1 \quad \text{and} \quad w_n := u_{n+1}.$$

Then

$$w_n = u_{n+1} = Tu_{n+2} = Tw_{n+1} \quad \text{for all} \ \ n \in \mathbb{Z}_+,$$

and hence u_1 belongs to M. Therefore $M \subseteq T(M)$, and hence $M = T(M)$. ■

The next result shows that under certain purely algebraic conditions the algebraic core and the hyper-range of an operator coincide.

Lemma 1.9. *Let T be a linear operator on a vector space X. Suppose that there exists $m \in \mathbb{N}$ such that*

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X) \quad \textit{for all integers } k \geq 0.$$

Then $C(T) = T^\infty(X)$.

Proof We have only to prove that $T^\infty(X) \subseteq C(T)$. We show that $T(T^\infty(X)) = T^\infty(X)$. Evidently the inclusion $T(T^\infty(X)) \subseteq T^\infty(X)$ holds for every linear operator, so we need only to prove the opposite inclusion.

Let $D := \ker T \cap T^m(X)$. Obviously we have

$$D = \ker T \cap T^m(X) = \ker T \cap T^\infty(X).$$

Let us now consider an element $y \in T^\infty(X)$. Then $y \in T^n(X)$ for each $n \in \mathbb{N}$, so there exists $x_k \in X$ such that $y = T^{m+k}x_k$ for every $k \in \mathbb{N}$. If we set

$$z_k := T^m x_1 - T^{m+k-1}x_k \quad (k \in \mathbb{N}),$$

then $z_k \in T^m(X)$ and since

$$Tz_k = T^{m+1}x_1 - T^{m+k}x_k = y - y = 0$$

we also have $z_k \in \ker T$. Thus $z_k \in D$, and from the inclusion

$$D = \ker T \cap T^{m+k}(X) \subseteq \ker T \cap T^{m+k-1}(X)$$

it follows that $z_k \in T^{m+k-1}(X)$. This implies that

$$T^m x_1 = z_k + T^{m+k-1}x_k \in T^{m+k-1}(X)$$

for each $k \in \mathbb{N}$, and therefore $T^m x_1 \in T^\infty(X)$. Finally, from

$$T(T^m x_1) = T^{m+1}x_1 = y$$

we may conclude that $y \in T(T^\infty(X))$. Therefore $T^\infty(X) \subseteq T(T^\infty(X))$, so the proof is complete. ■

Theorem 1.10. *Let T be a linear operator on a vector space X. Suppose that one of the following conditions holds:*

(i) *dim ker $T < \infty$;*

(ii) *codim $T(X) < \infty$;*

(iii) $\ker T \subseteq T^n(X)$ *for all $n \in \mathbb{N}$.*

Then $C(T) = T^\infty(X)$.

Proof (i) If ker T is finite-dimensional then there exists a positive integer m such that

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X)$$

for all integers $k \geq 0$. Hence it suffices to apply Lemma 1.9.

(ii) Suppose that $X = F \oplus T(X)$ with dim $F < \infty$. Clearly, if we let $D_n := \ker T \cap T^n(X)$ then we have $D_n \supseteq D_{n+1}$ for all $n \in \mathbb{N}$. Suppose that there exist k distinct subspaces D_n. There is no loss of generality in assuming $D_j \neq D_{j+1}$ for $j = 1, 2, \dots k$. Then for every one of these j we can find an element $w_j \in X$ such that $T^j w_j \in D_j$ and $T^j w_j \notin D_{j+1}$. By means of the decomposition $X = F \oplus T(X)$ we also find $u_j \in F$ and $v_j \in T(X)$ such that $w_j = u_j + v_j$. We claim that the vectors $u_1, \cdots, u_k$ are linearly independent.

To see this let us suppose $\sum_{j=1}^{k} \lambda_j u_j = 0$. Then

$$\sum_{j=1}^{k} \lambda_j w_j = \sum_{j=1}^{k} \lambda_j v_j$$

and therefore from the equalities $T^k w_1 = \cdots = T^k w_{k-1} = 0$ we deduce that

$$T^k(\sum_{j=1}^{k} \lambda_j w_j) = \lambda_k T^k w_k = T^k(\sum_{j=1}^{k} \lambda_j v_j) \in T^k(T(X)) = T^{k+1}(X).$$

From $T^k w_k \in \ker T$ we obtain $\lambda_k T^k w_k \in D_{k+1}$, and since $T^k w_k \notin D_{k+1}$ this is possible only if $\lambda_k = 0$. Analogously we have $\lambda_{k-1} = \cdots = \lambda_1 = 0$, so the vectors $u_1, \ldots, u_k$ are linearly independent. From this it follows that k is smaller than or equal to the dimension of F . But then for a sufficiently large m we obtain that

$$\ker T \cap T^m(X) = \ker T \cap T^{m+j}(X)$$

for all integers $j \geq 0$. So we are again in the situation of Lemma 1.9.

(iii) Obviously, if $\ker T \subseteq T^n(X)$ for all $n \in \mathbb{N}$, then

$$\ker T \cap T^n(X) = \ker T \cap T^{n+k}(X) = \ker T$$

for all integers $k \geq 0$. Hence also in this case we can apply Lemma 1.9. ∎

2. Semi-regular operators on Banach spaces

All the results of the previous section have been established in the purely algebraic setting of linear operators acting on vector spaces. The concept of semi-regular operators arises in a natural way when we consider operators acting on Banach spaces.

Definition 1.11. *Given a Banach space X, a bounded operator $T \in L(X)$ is said to be* semi-regular *if $T(X)$ is closed and T verifies one of the equivalent conditions of Theorem 1.5.*

Trivial examples of semi-regular operators are surjective operators as well as injective operators with closed range. Later other examples of semi-regular operators will be given amongst semi-Fredholm operators. Some other examples of semi-regular operators may be found in Mbekhta [**233**] and Labrousse [**193**].

A semi-regular operator T has a closed range. It is evident that it is useful to find conditions which ensure that $T(X)$ is closed. For a bounded operator $T \in L(X, Y)$, X and Y Banach spaces, the property of having T as a closed range may be characterized by means of the following quantity associated with the operator T.

Definition 1.12. *If $T \in L(X,Y)$, where X and Y are Banach spaces, the* reduced minimum modulus *of a non-zero operator T is defined to be*

$$\gamma(T) := \inf_{x \notin \ker\, T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker\, T)}.$$

If $T = 0$ then we take $\gamma(T) = \infty$.

Note that $\gamma(T) = \gamma(T^\star)$ for every $T \in L(X)$ (see Kato [**183**, Theorem 5.13]), where $T^\star$ denotes the dual of T.

Theorem 1.13. *Let $T \in L(X,Y)$, X and Y Banach spaces. Then:*
(i) $\gamma(T) > 0 \Leftrightarrow T(X)$ *is closed;*
(ii) $T(X)$ *is closed* $\Leftrightarrow T^\star(X^\star)$ *is closed.*

Proof (i) Let $\widetilde{X} := X/\ker\, T$ and let $\widetilde{T} : \widetilde{X} \to Y$ denote the continuous injection corresponding to T, defined by

$$\widetilde{T}\, \tilde{x} := Tx \quad \text{for every } x \in \tilde{x}.$$

Clearly $\widetilde{T}\widetilde{X} = T(X)$. From the open mapping theorem it follows that $\widetilde{T}\, \widetilde{X}$ is closed if and only if $\widetilde{T}$ admits a continuous inverse, there exists a constant $\delta > 0$ such that $\|\widetilde{T}\, \tilde{x}\| \geq \delta \|\tilde{x}\|$ for every $x \in X$. From the equality

$$\gamma(T) = \inf_{\tilde{x} \neq \tilde{0}} \frac{\|\widetilde{T}\, \tilde{x}\|}{\|\tilde{x}\|}$$

we then conclude that $\widetilde{T}\, \widetilde{X} = T(X)$ is closed if and only if $\gamma(T) > 0$.

(ii) It is obvious from the equality $\gamma(T) = \gamma(T^\star)$ observed above. ■

Theorem 1.14. *Let $T \in L(X)$, X a Banach space, and suppose that there exists a closed subspace Y of X such that $T(X) \cap Y = \{0\}$ and $T(X) \oplus Y$ is closed. Then $T(X)$ is also closed.*

Proof Consider the product space $X \times Y$ under the norm

$$\|(x,y)\| := \|x\| + \|y\| \quad (x \in X, y \in Y).$$

Then $X \times Y$ is a Banach space, and the continuous map $S : X \times Y \to X$ defined by $S(x,y) := Tx + y$ has range $S(X \times Y) = T(X) \oplus Y$, which is closed by assumption. Consequently by part (i) of Theorem 1.13 we have

$$\gamma(S) := \inf_{(x,y) \notin \ker\, S} \frac{\|S(x,y)\|}{\operatorname{dist}((x,y), \ker S)} > 0.$$

Clearly, $\ker S = \ker T \times \{0\}$, so that if $x \notin \ker T$ then $(x,0) \notin \ker S$. Moreover,

$$\operatorname{dist}((x,0), \ker\, S) = \operatorname{dist}(x, \ker\, T),$$

and therefore

$$\begin{aligned} \|Tx\| &= \|S(x,0)\| \geq \gamma(S)\, \operatorname{dist}((x,0), \ker\, S)) \\ &= \gamma(S)\, \operatorname{dist}(x, \ker\, T). \end{aligned}$$

This implies that $\gamma(T) \geq \gamma(S)$, and therefore T has closed range. ■

Corollary 1.15. *Let $T \in L(X)$, X a Banach space, and Y a finite-dimensional subspace of X such that $T(X) + Y$ is closed. Then $T(X)$ is closed. In particular, if codim $T(X) < \infty$ then $T(X)$ is closed.*

Proof Let Y_1 be any subspace of Y for which $Y_1 \cap T(X) = \{0\}$ and $T(X) + Y_1 = T(X) + Y$. From the assumption we infer that $T(X) \oplus Y_1$ is closed, so $T(X)$ is closed by Theorem 1.14. The second statements is clear, since every finite-dimensional subspace of a Banach space X is always closed. ■

Let M be a subset of a Banach space X. The *annihilator* of M is the closed subspace of $X^\star$ defined by

$$M^\perp := \{f \in X^\star : f(x) = 0 \text{ for every } x \in M\},$$

while the *pre-annihilator* of a subset W of $X^\star$ is the closed subspace of X defined by

$$^\perp W := \{x \in X : f(x) = 0 \text{ for every } f \in W\}.$$

Clearly $^\perp(M^\perp) = M$ if M is closed. Moreover, if M and N are closed linear subspaces of X then $(M + N)^\perp = M^\perp \cap N^\perp$. The dual relation $M^\perp + N^\perp = (M \cap N)^\perp$ is not always true, since $(M \cap N)^\perp$ is always closed but $M^\perp + N^\perp$ need not be closed. However, a classical theorem establishes that

$$M^\perp + N^\perp \text{ is closed in } X^\star \Leftrightarrow M + N \text{ is closed in } X,$$

see Kato [**183**, Theorem 4.8, Chapter IV].

The following standard duality relationships between the kernels and ranges of a bounded operator T on a Banach space and its dual $T^\star$ are well known, see Heuser [**159**, p.135]:

$$\text{(1)} \qquad \ker T =^\perp \overline{T^\star(X^\star)} \quad \text{and} \quad ^\perp\ker T^\star = \overline{T(X)},$$

and

$$\text{(2)} \qquad \overline{T(X)}^\perp = \ker T^\star \quad \text{and} \quad \overline{T^\star(X^\star)} \subseteq \ker T^\perp.$$

Note that the last inclusion is, in general, strict. However, a classical result states that the equality holds precisely when T has closed range, see Kato [**183**, Theorem 5.13, Chapter IV].

Theorem 1.16. *Suppose that $T \in L(X)$, X a Banach space, is semi-regular. Then $\gamma(T^n) \geq \gamma(T)^n$.*

Proof We proceed by induction. The case $n = 1$ is trivial.

Suppose that $\gamma(T^n) \geq \gamma(T)^n$. For every element $x \in X$, and $u \in \ker T^{n+1}$ we have

$$\begin{aligned} \operatorname{dist}(x, \ker T^{n+1}) &= \operatorname{dist}(x - u, \ker T^{n+1}) \\ &\leq \operatorname{dist}(x - u, \ker T). \end{aligned}$$

By assumption T is semi-regular, so by Theorem 1.5 $\ker T = T^n(\ker T^{n+1})$ and therefore

$$\begin{aligned} \operatorname{dist}(T^n x, \ker T) &= \operatorname{dist}(T^n x, T^n(\ker T^{n+1})) \\ &= \inf_{u \in \ker T^{n+1}} \|T^n(x-u)\| \\ &\geq \gamma(T^n) \cdot \inf_{u \in \ker T^{n+1}} \operatorname{dist}(x-u, \ker T^n) \\ &\geq \gamma(T^n) \operatorname{dist}(x, \ker T^{n+1}). \end{aligned}$$

From this estimate it follows that

$$\|T^{n+1}x\| \geq \gamma(T) \operatorname{dist}(T^n x, \ker T) \geq \gamma(T)\, \gamma(T^n) \cdot \operatorname{dist}(x, \ker T^{n+1}).$$

Consequently from our inductive assumption we obtain that

$$\gamma(T^{n+1}) \geq \gamma(T)\gamma(T)^n = \gamma(T)^{n+1},$$

which completes the proof. ■

Corollary 1.17. *If $T \in L(X)$, X a Banach space, is semi-regular then T^n is semi-regular for all $n \in \mathbb{N}$.*

Proof If T is semi-regular then by Theorem 1.16 $S = T^n$ has closed range. Furthermore, $S^\infty(X) = T^\infty(X)$ and, by Theorem 1.5, $\ker S \subseteq T^\infty(X) = S^\infty(X)$. From Corollary 1.6 we conclude that T^n is semi-regular. ■

Corollary 1.18. *Let $T \in L(X)$, where X is a Banach space. Then T is semi-regular if and only if $T^n(X)$ is closed for all $n \in \mathbb{N}$ and T verifies one of the equivalent conditions of Theorem 1.5.* ■

Theorem 1.19. *Let $T \in L(X)$, X a Banach space. Then T is semi-regular if and only if $T^\star$ is semi-regular.*

Proof Suppose that T is semi-regular. Then $T(X)$ is closed so that $\gamma(T) > 0$ by part (i) of Theorem 1.13. From Theorem 1.16 we then obtain that $\gamma(T^n) \geq \gamma(T)^n > 0$ and this implies, again by part (i) of Theorem 1.13, that $T^n(X)$ is closed for every $n \in \mathbb{N}$. The same argument also shows that $T^{n\star}(X^\star) = T^{\star n}(X^\star)$ is closed for every $n \in \mathbb{N}$, by part (ii) of Theorem 1.13. Therefore from the relationships (1) and (2) it follows that the equalities

$$\ker T^{n\perp} = T^{\star n}(X^\star) \quad \text{and} \quad {}^\perp\ker T^{\star n} = T^n(X). \tag{3}$$

hold for all $n \in \mathbb{N}$.

Now, since T is semi-regular then $\ker T \subseteq T^n(X)$ for every $n \in \mathbb{N}$ and therefore $T^n(X)^\perp \subseteq \ker T^\perp = T^\star(X^\star)$. Moreover, from the second equality of (3) we obtain $\ker T^{\star n} = T^n(X)^\perp$, so that $\ker T^{\star n} \subseteq T^\star(X^\star)$ holds for every $n \in \mathbb{N}$. This shows, since $T^\star(X^\star)$ is closed, that $T^\star$ is semi-regular.

A similar argument shows that if $T^\star$ is semi-regular then also T is semi-regular. ■

3. Analytical core of an operator

The following subspace has been introduced by Vrbová [**313**] and is, in a certain sense, the analytic counterpart of the algebraic core $C(T)$.

Definition 1.20. *Let X be a Banach space and $T \in L(X)$. The* analytical core *of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and a constant $\delta > 0$ such that:*

(1) $x = u_0$, *and* $Tu_{n+1} = u_n$ *for every* $n \in \mathbb{Z}_+$*;*

(2) $\|u_n\| \leq \delta^n \|x\|$ *for every* $n \in \mathbb{Z}_+$*.*

In the following theorem we collect some elementary properties of $K(T)$.

Theorem 1.21. *Let $T \in L(X)$, X a Banach space. Then:*

(i) $K(T)$ *is a linear subspace of* X*;*

(ii) $T(K(T)) = K(T)$*;*

(iii) $K(T) \subseteq C(T)$*.*

Proof (i) It is evident that if $x \in K(T)$ then $\lambda x \in K(T)$ for every $\lambda \in \mathbb{C}$. We show that if $x, y \in K(T)$ then $x + y \in K(T)$. If $x \in K(T)$ there exists $\delta_1 > 0$ and a sequence $(u_n) \subset X$ satisfying the condition (1) and which is such that $\|u_n\| \leq {\delta_1}^n \|x\|$ for all $n \in \mathbb{Z}_+$. Analogously, since $y \in K(T)$ there exists $\delta_2 > 0$ and a sequence $(v_n) \subset X$ satisfying the condition (1) of the definition of $K(T)$ and such that $\|v_n\| \leq \delta_2^n \|y\|$ for every $n \in \mathbb{N}$.

Let $\delta := \max\,\{\delta_1, \delta_2\}$. We have

$$\|u_n + v_n\| \leq \|u_n\| + \|v_n\| \leq \delta_1^n \|x\| + \delta_2^n \|y\| \leq \delta^n (\|x\| + \|y\|).$$

Trivially, if $x + y = 0$ there is nothing to prove since $0 \in K(T)$. Suppose then $x + y \neq 0$ and set

$$\mu := \frac{\|x\| + \|y\|}{\|x + y\|}$$

Clearly $\mu \geq 1$, so $\mu \leq \mu^n$ and therefore

$$\|u_n + v_n\| \leq (\delta)^n \mu \|x + y\| \leq (\delta\mu)^n \|x + y\| \quad \text{for all } n \in \mathbb{Z}_+,$$

which shows that also the property (2) of the definition of $K(T)$ is verified for every sum $x + y$, with x, $y \in K(T)$. Hence $x + y \in K(T)$, and consequently $K(T)$ is a linear subspace of X.

The proof (ii) of is analogous to that of Theorem 1.8, whilst (iii) is a trivial consequence of (ii) and the definition of $C(T)$. ■

Observe that in general neither $K(T)$ nor $C(T)$ are closed. The next result shows that under the assumption that $C(T)$ is closed then these two subspaces coincide.

Theorem 1.22. *Suppose X a Banach space and $T \in L(X)$.*

(i) *If F is a closed subspace of X such that $T(F) = F$ then $F \subseteq K(T)$.*

(ii) *If $C(T)$ is closed then $C(T) = K(T)$.*

Proof (i): Let $T_0 : F \to F$ denote the restriction of T on F. By assumption F is a Banach space and $T(F) = F$, so, by the open mapping theorem, T_0 is open. This means that there exists a constant $\delta > 0$ with the property that for every $x \in F$ there is $u \in F$ such that $Tu = x$ and $\|u\| \leq \delta\|x\|$. Now, if $x \in F$, define $u_0 := x$ and consider an element $u_1 \in F$ such that

$$Tu_1 = u_0 \quad \text{and} \quad \|u_1\| \leq \delta\|u_0\|.$$

By repeating this procedure, for every $n \in \mathbb{N}$ we find an element $u_n \in F$ such that

$$Tu_n = u_{n-1} \quad \text{and} \quad \|u_n\| \leq \delta\|u_{n-1}\|.$$

From the last inequality we obtain the estimate

$$\|u_n\| \leq \delta^n\|u_0\| = \delta^n\|x\| \quad \text{for every} \quad n \in \mathbb{N},$$

so $x \in K(T)$. Hence $F \subseteq K(T)$.

(ii) Suppose that $C(T)$ is closed. Since $C(T) = T(C(T))$ the first part of the theorem shows that $C(T) \subseteq K(T)$, and hence, since the reverse inclusion is always true, $C(T) = K(T)$. ∎

Theorem 1.23. *Let $T \in L(X)$ be a semi-regular operator on a Banach space X. If $x \in X$, then $Tx \in C(T)$ if and only if $x \in C(T)$.*

Proof Clearly the equality $T(C(T)) = C(T)$ implies that $Tx \in C(T)$ for every $x \in C(T)$. Conversely let $Tx \in C(T)$. By Theorem 1.10 we have that $C(T) = T^\infty(X)$, consequently for each $n \in \mathbb{N}$ there exists $y_n \in X$ such that $T^{n+1}y_n = Tx$, hence $z := x - T^n y_n \in \ker T \subseteq T^n(X)$. Then $x = z + T^n x \in T^n(X)$ for each $n \in \mathbb{N}$, and consequently $x \in C(T)$. ∎

Theorem 1.24. *Let $T \in L(X)$, X a Banach space, be semi-regular. Then $C(T)$ is closed and*

$$C(T) = K(T) = T^\infty(X).$$

Proof The semi-regularity of T gives, by definition, $\ker T \subseteq T^n(X)$ for all $n \in \mathbb{N}$. Hence by Theorem 1.10 we have $T^\infty(X) = C(T)$. By Corollary 1.17 T^n is semi-regular for all $n \in \mathbb{N}$, so $T^n(X)$ is closed for all $n \in \mathbb{N}$ and hence $T^\infty(X) = \bigcap_{n=1}^\infty T^n(X)$ is closed. By part (ii) of Theorem 1.22 then we conclude that $K(T)$ coincides with $C(T)$. ∎

Theorem 1.25. *Let $T \in L(X)$, where X is a Banach space. Then T is semi-regular if and only if there exists a closed subspace M of X such that $T(M) = M$ and the operator $\widetilde{T} : X/M \to X/M$, induced by T, is injective and has closed range.*

Proof If T is semi-regular then by Theorem 1.24 the subspace $M := T^\infty(X)$ has the required properties.

Conversely, let M be a closed subspace such that $T(M) = M$ and suppose that $\widetilde{T} : X/M \to X/M$ is injective and has closed range. Then

$M \subseteq C(T) \subseteq T^\infty(X)$. Moreover, if $x \in \ker T$ then $\widetilde{T}\tilde{x} = \tilde{0}$, and therefore the injectivity of $\widetilde{T}$ implies $x \in M$. Hence $\ker T \subseteq M \subseteq T^\infty(X)$.

To show that T is semi-regular it remains to prove that $T(X)$ is closed. Let $\pi : X \to X/M$ be the canonical quotient map. We show that $T(X) = \pi^{-1}(\widetilde{T}(X/M))$. If $y \in T(X)$ then $y = Tx$ for some $x \in X$, so $\pi y = \widetilde{Tx} = \widetilde{T}\tilde{x} \in \widetilde{T}(X/M)$. Hence $T(X) \subseteq \pi^{-1}(\widetilde{T}(X/M))$.

On the other hand, let $y \in X$ be such that $\pi y \in \widetilde{T}(X/M)$. Then $\tilde{y} = \widetilde{Tx}$ for some $x \in X$ and from the inclusion $M \subseteq T(X)$ we infer that $y \in T(X)$; which shows the reverse inclusion. Therefore $T(X) = \pi^{-1}(\widetilde{T}(X/M))$, and this set is obviously closed since $\widetilde{T}(X/M)$ is closed and π is continuous. ∎

Theorem 1.26. *Suppose that $T, S \in L(X)$, X a Banach space, commute and TS semi-regular. Then T and S are semi-regular.*

Proof Clearly we need only to show that one of the two operators, say T, is semi-regular. From the semi-regularity of TS we obtain

$$\ker T \subseteq \ker\,(TS) \subseteq \bigcap_{n=1}^{\infty} (T^n S^n)(X) \subseteq \bigcap_{n=1}^{\infty} T^n(X). \tag{4}$$

At this point we need only to show that $T(X)$ is closed. Let $(y_n) := (Tx_n)$ be a sequence of $T(X)$ which converges to some y_0. Then $Sy_n = STx_n = TSx_n \in (TS)(X)$ and (Sy_n) converges to Sy_0. By assumption $(TS)(X)$ is closed, thus $Sy_0 \in (TS)(X) = (ST)(X)$. Hence there exists an element $z_0 \in X$ such that $Sy_0 = STz_0$. Consequently $y_0 - Tz_0 \in \ker\, S \subseteq \ker\,(TS)$. From (4) we obtain

$$y_0 - Tz_0 \in \bigcap_{n=1}^{\infty} T^n(X) \subseteq T(X).$$

From this it follows that $y_0 \in T(X)$, so $T(X)$ is closed. Hence T is semi-regular. ∎

The following example, owed to Müller [**240**], shows that the product of two semi-regular operators, also commuting semi-regular operators, need not be semi-regular.

Example 1.27. Let H be a Hilbert space with an orthonormal basis $(e_{i,j})$ where i, j are integers for which $ij \le 0$. Let $T \in L(H)$ and $S \in L(H)$ are defined by the assignment:

$$Te_{i,j} := \begin{cases} 0 & \text{if } i = 0,\ j > 0 \\ e_{i+1,j} & \text{otherwise,} \end{cases}$$

and

$$Se_{i,j} := \begin{cases} 0 & \text{if } j = 0,\ i > 0 \\ e_{i,j+1} & \text{otherwise.} \end{cases}$$

Then

$$TSe_{i,j} = STe_{i,j} = \begin{cases} 0 & \text{if } i = 0,\ j \geq 0, \text{or } j = 0,\ i \geq 0, \\ e_{i+1,j+1} & \text{otherwise.} \end{cases}$$

Hence $TS = ST$ and, as it is easy to verify,

$$\ker T = \bigvee_{j>0} \{e_{i,0}\} \subset T^{\infty}(H),$$

where $\bigvee_{j>0}\{e_{0,j}\}$ denotes the linear subspace of H generated by the set $\{e_j : j > 0\}$. Analogously we have

$$\ker S = \bigvee_{i>0} \{e_{i,0}\} \subset S^{\infty}(H).$$

Moreover, since T and S have closed range it follows that T, S are semi-regular.

On the other hand, $e_{0,0} \in \ker TS$ and $e_{0,0} \notin (TS)(H)$, thus TS is not semi-regular.

The next example, owed also to Müller [**240**], shows that the set of all semi-regular operators need not be an open subset of $L(X)$.

Example 1.28. Let H be a Hilbert space with an orthonormal basis $(e_{i,j})$ where i, j are integers and $i \geq 1$. Let T be defined by:

$$Te_{i,j} := \begin{cases} e_{i,j+1} & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Clearly $T(H)$ is closed and

$$\ker T = \bigvee_{i>1} \{e_{0,j}\} \subset T^{\infty}(H),$$

so that T is semi-regular.

Now let $\varepsilon > 0$ be arbitrarily given and define $S \in L(H)$ by

$$Se_{i,j} := \begin{cases} \dfrac{\varepsilon}{i}\, e_{i,0} & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

It is easy to see that $\|S\| = \varepsilon$. Moreover, S is a compact operator having an infinite-dimensional range, thus $S(H)$ is not closed. Let M denote the subspace generated by the set $\{e_{i,0} : i \geq 1\}$. Then $T(H)$ is orthogonal to M and hence to $S(H)$, since $S(H) \subseteq M$. Moreover, $(T+S)(H) = T(H)+S(H)$, so that $(T + S)(H)$ is not closed and hence $T + S$ is not semi-regular.

4. The semi-regular spectrum of an operator

Among the many concepts dealt with in Kato's extensive treatment of perturbation theory [**182**] there is a very important part of the spectrum defined as follows:

Definition 1.29. *Given a Banach space X, the* semi-regular resolvent *of a bounded operator T is defined by*

$$\rho_{\mathrm{se}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-regular}\}.$$

The semi-regular spectrum, *known in the literature also as the* Kato spectrum, *of T is defined to be the set $\sigma_{\mathrm{se}}(T) := \mathbb{C} \setminus \rho_{\mathrm{se}}(T)$.*

Obviously

$$\sigma_{\mathrm{se}}(T) \subseteq \sigma(T) \quad \text{and} \quad \rho(T) \subseteq \rho_{\mathrm{se}}(T).$$

Later we shall see that $\sigma_{\mathrm{se}}(T)$ is a non-empty compact subset of $\mathbb{C}$ and that for some important classes of operators the inclusions above are equalities.

An operator $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. It is easy to show that T is bounded below if and only if there exists $K > 0$ such that

$$\|Tx\| \geq K\|x\| \quad \text{for all } x \in X. \tag{5}$$

Indeed, if $\|Tx\| \geq K\|x\|$ for some $K > 0$ and all $x \in X$ then T is injective. Moreover, if (x_n) is a sequence in X for which (Tx_n) converges to $y \in X$ then (x_n) is a Cauchy sequence and hence convergent to some $x \in X$. Since T is continuous then $Tx = y$ and therefore $T(X)$ is closed.

Conversely, if T is injective and $T(X)$ is closed then, from the open mapping theorem, it easily follows that there exists a $K > 0$ for which the inequality (5) holds.

Clearly, if T is bounded below or surjective then T is semi-regular. The next result shows that the properties to be bounded below or to be surjective are dual each other.

Lemma 1.30. *Let $T \in L(X)$, X a Banach space. Then:*

(i) *T is surjective (respectively, bounded below) if and only if $T^\star$ is bounded below (respectively, surjective);*

(ii) *If T is bounded below (respectively, surjective) then $\lambda I - T$ is bounded below (respectively, surjective) for all $|\lambda| < \gamma(T)$.*

Proof (i) Suppose that T is surjective. Trivially T has closed range and therefore also $T^\star$ has closed range. From the equality $\ker T^\star = T(X)^\perp = X^\perp = \{0\}$ we conclude that $T^\star$ is injective.

Conversely, suppose that $T^\star$ is bounded below, then $T^\star$ has closed range and hence by Theorem 1.13 also the operator T has closed range. From the equality $T(X) = {}^\perp \ker T^\star = {}^\perp \{0\} = X$ we then conclude that T is surjective. The proof of T being bounded below if and only if $T^\star$ is surjective is analogous.

(ii) Suppose that T is injective with closed range. Then $\gamma(T) > 0$ and from definition of $\gamma(T)$ we obtain

$$\gamma(T) \cdot \mathrm{dist}(x, \ker T) = \gamma(T)\|x\| \leq \|Tx\| \quad \text{for all } x \in X.$$

From that we obtain

$$\|(\lambda I - T)x\| \geq \|Tx\| - |\lambda|\|x\| \geq (\gamma(T) - |\lambda|)\|x\|,$$

thus for all $|\lambda| < \gamma(T)$, the operator $\lambda I - T$ is bounded below.

The case that T is surjective follows now easily by considering the adjoint $T^\star$. ∎

Theorem 1.31. *Let $T \in L(X)$, X a Banach space, be semi-regular. Then $\lambda I - T$ is semi-regular for all $|\lambda| < \gamma(T)$. Moreover, $\rho_{\mathrm{se}}(T)$ is open.*

Proof First we show that $C(T) \subseteq C(\lambda I - T)$ for all $|\lambda| < \gamma(T)$. Let $T_0 : C(T) \to C(T)$ denote the restriction of T to $C(T)$. By Theorem 1.24, $C(T)$ is closed and T_0 is surjective. Thus by Lemma 1.30 the equalities

$$(\lambda I - T_0)(C(T)) = (\lambda I - T)(C(T)) = C(T)$$

hold for all $|\lambda| < \gamma(T_0)$.

On the other hand, T is semi-regular, so that by Theorem 1.10 $\ker T \subseteq T^\infty(X) = C(T)$. This easily implies that $\gamma(T_0) \geq \gamma(T)$, and hence

$$(\lambda I - T)(C(T)) = C(T) \quad \text{for all} |\lambda| < \gamma(T).$$

Note that this last equality implies that

$$C(T) \subseteq C(\lambda I - T) \quad \text{for all } |\lambda| < \gamma(T). \tag{6}$$

Moreover, for every $\lambda \neq 0$ we have $T(\ker\,(\lambda I - T)) = \ker\,(\lambda I - T)$, so from Theorem 1.24 and Theorem 1.22 we obtain that

$$\ker\,(\lambda I - T) \subseteq C(T) \quad \text{for all } \lambda \neq 0.$$

From the inclusion (6) we now conclude that the inclusions

$$\ker\,(\lambda I - T) \subseteq C(\lambda I - T) \subseteq (\lambda I - T)^n(X) \tag{7}$$

hold for all $|\lambda| < \gamma(T)$, $\lambda \neq 0$ and $n \in \mathbb{N}$. Of course, this is still true for $\lambda = 0$ since T is semi-regular, so the inclusions (7) are valid for all $|\lambda| < \gamma(T)$.

To prove that $\lambda I - T$ is semi-regular for all $|\lambda| < \gamma(T)$, it only remains to show that $(\lambda I - T)(X)$ is closed for all $|\lambda| < \gamma(T)$. Observe that as a consequence of Lemma 1.30 we can limit ourselves to considering only the case $C(T) \neq \{0\}$ and $C(T) \neq X$. Indeed, if $C(T) = \{0\}$ then $\ker T \subseteq C(T) = \{0\}$, and hence T is bounded below, whilst in the other case $C(T) = X$ the operator T is surjective.

Let $\overline{X} := X/C(T)$, and let $\overline{T} : \overline{X} \to \overline{X}$ be the quotient map defined by $\overline{T}\,\overline{x} := \overline{Tx}$, where $x \in \overline{x}$. Clearly $\overline{T}$ is continuous. Moreover, $\overline{T}$ is injective since from $\overline{T}\,\overline{x} = \overline{Tx} = \overline{0}$ we have $Tx \in C(T)$, and this implies by Theorem 1.23 that $x \in C(T)$, which yields $\overline{x} = \overline{0}$.

Next we show that $\overline{T}$ is bounded below. We only need to prove that $\overline{T}$

has closed range. To see this we show the inequality $\gamma(\overline{T}) \geq \gamma(T)$. In fact, for each $x \in X$ and each $u \in C(T)$ we have, recalling that $\ker T \subseteq C(T)$,

$$\begin{aligned} \|\overline{x}\| &= \operatorname{dist}(x, C(T)) = \operatorname{dist}(x-u, C(T)) \\ &\leq \operatorname{dist}(x-u, \ker T) \leq \frac{1}{\gamma(T)}\|Tx - Tu\|. \end{aligned}$$

From the equality $C(T) = T(C(T))$ we obtain that

$$\|\overline{Tx}\| = \inf_{u \in C(T)} \|Tx - Tu\| \quad \text{for all } u \in C(T),$$

so that $\|\overline{x}\| \leq 1/\gamma(T)\|\overline{Tx}\|$ from which we obtain that $\gamma(\overline{T}) \geq \gamma(T)$.

$\|\overline{x}\| \leq \gamma(T)^{-1}\|\overline{T}\ \overline{x}\|$ and, consequently, $\gamma(\overline{T}) \geq \gamma(T)$. Hence $\overline{T}$ is bounded below. By Lemma 1.30 $\lambda\overline{I} - \overline{T}$ is then bounded below for all $|\lambda| < \gamma(\overline{T})$ and *a fortiori* for all $|\lambda| < \gamma(T)$.

Finally, to show that $(\lambda I - T)(X)$ is closed for all $|\lambda| < \gamma(T)$ let us consider a sequence (x_n) of $(\lambda I - T)(X)$ which converges to $x \in X$. Clearly, the sequence $(\overline{x}_n)$ converges to $\overline{x}$ and $\overline{x}_n \in (\lambda\overline{I} - \overline{T})(\overline{X})$. The last space is closed for all $|\lambda| < \gamma(T)$, and hence $\overline{x} \in (\lambda\overline{I} - \overline{T})(\overline{X})$. Let $\overline{x} = (\lambda\overline{I} - \overline{T})\overline{v}$ and $v \in \overline{v}$. Then

$$x - (\lambda I - T)v \in C(T) \subseteq (\lambda I - T)(C(T)) \quad \text{for all} |\lambda| < \gamma(T),$$

and so there exists $u \in C(T)$ for which $x = (\lambda I - T)(v + u)$, hence $x \in (\lambda I - T)(X)$ for all $|\lambda| < \gamma(T)$. Hence $(\lambda I - T)(X)$ is closed for all $|\lambda| < \gamma(T)$, and, consequently, $\lambda I - T$ is semi-regular for all $|\lambda| < \gamma(T)$. Therefore $\rho_{\rm se}(T)$ is an open subset of $\mathbb{C}$. ∎

The semi-regular resolvent $\rho_{\rm se}(T)$ is open therefore it can be decomposed into (maximal, open, connected, pairwise disjoint) non-empty components. Next we want to show that $C(\lambda I - T) = K(\lambda I - T)$ is locally constant on each component Ω of $\rho_{\rm se}(T)$. To show this property we first need to show some preliminary results on the gap between subspaces.

Let M, N be two closed linear subspaces of the Banach space X and set

$$\delta(M, N) := \sup\{\operatorname{dist}(u, N) : u \in M, \|u\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, N) = 0$ for any subspace N.

The *gap* between M and N is defined by

$$\widehat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}.$$

The function $\widehat{\delta}$ is a metric on the set $\mathcal{C}(X)$ of all linear closed subspaces of X, see [**182**, §2, Chapter IV] and the convergence $M_n \to M$ is obviously defined by $\widehat{\delta}(M_n, M) \to 0$ as $n \to \infty$.

Remark 1.32. Note that for two closed linear subspaces M and N of X we have

$$\delta(M, N) = \delta(N^\perp, M^\perp) \quad \text{and} \quad \widehat{\delta}(M, N) = \widehat{\delta}(N^\perp, M^\perp),$$

see Kato book [**182**, Theorem 2.9, Chapter IV]. From these equalities it easily follows, as $n \to \infty$, that $M_n \to M$ if and only if ${M_n}^\perp \to M^\perp$. Moreover,

$$\widehat{\delta}(M, N) < 1 \Rightarrow \text{ dim M } = \text{ dim N}, \tag{8}$$

see Corollary 2.6 of [**182**, §2, Chapter IV].

Lemma 1.33. *For every operator $T \in L(X)$, X a Banach space, and arbitrary λ, $\mu \in \mathbb{C}$, we have:*

(i) $\gamma(\lambda I - T) \cdot \delta(\ker(\mu I - T), \ker(\lambda I - T)) \le |\mu - \lambda|$;

(ii) $\min\{\gamma(\lambda I - T), \gamma(\mu I - T)\} \cdot \widehat{\delta}(\ker(\lambda I - T), \ker(\mu I - T)) \le |\mu - \lambda|$.

Proof The statement is trivial for $\lambda = \mu$. Suppose that $\lambda \neq \mu$ and consider an element $0 \neq x \in \ker(\mu I - T)$. Then $x \notin \ker(\lambda I - T)$ and hence

$$\begin{aligned} \gamma(\lambda I - T) \ \mathrm{dist}(x, \ker(\lambda I - T)) &\le \|(\lambda I - T)x\| \\ &= \|(\lambda I - T)x - (\mu I - T)x\| \\ &= |\lambda - \mu| \ \|x\|. \end{aligned}$$

From this estimate we obtain, if $B := \{x \in \ker(\mu I - T) : \|x\| \le 1\}$, that

$$\gamma(\lambda I - T) \cdot \sup_{x \in B} \ \mathrm{dist}\ (x, \ker(\lambda I - T)) \le |\lambda - \mu|,$$

and therefore

$$\gamma(\lambda I - T) \cdot \delta(\ker(\lambda I - T), \ker(\mu I - T)) \le |\mu - \lambda|.$$

(ii) Clearly, the inequality follows from (i) by interchanging λ and μ. ■

Lemma 1.34. *For every $x \in X$ and $0 < \varepsilon < 1$ there exists $x_0 \in X$ such that $x - x_0 \in M$ and*

$$\mathrm{dist}(x_0, N) \ge \left((1 - \varepsilon)\frac{1 - \delta(M, N)}{1 + \delta(M, N)}\right) \|x_0\|. \tag{9}$$

Proof If $x \in M$ it suffices to take $x_0 = 0$. Assume therefore that $x \notin M$. Let $\widehat{X} := X/M$ denote the quotient space and put $\widehat{x} := x + M$. Evidently, $\|\widehat{x}\| = \inf_{z \in \widehat{x}} \|z\| > 0$. We claim that there exists an element $x_0 \in X$ such that

$$\|\widehat{x_0}\| = \mathrm{dist}(x_0, M) \ge (1 - \varepsilon)\|x_0\|.$$

Indeed, were it not so then

$$\|\widehat{x}\| = \|\widehat{z}\| < (1 - \varepsilon)\|z\| \quad \text{for every } z \in \widehat{x}$$

and therefore

$$\|\widehat{x}\| \le (1 - \varepsilon) \inf_{z \in \widehat{x}} \|z\| = (1 - \varepsilon)\|\widehat{x}\|,$$

and this is impossible since $\|\widehat{x}\| > 0$.

Let $\mu := \mathrm{dist}(x_0, N) = \inf_{u \in N} \|x_0 - u\|$. We know that there exists $y \in N$

such that $\|x_0 - y\| \leq \mu + \varepsilon\|x_0\|$. From that we obtain $\|y\| \leq (1+\varepsilon)\|x_0\| + \mu$.

On the other hand, we have $\operatorname{dist}(y, M) \leq \delta(N, M) \cdot \|y\|$ and hence

$$\begin{aligned}(1-\varepsilon)\|\widehat{x_0}\| &\leq \operatorname{dist}(x_0, M) \leq \|x_0 - y\| + \operatorname{dist}(y, M) \\ &\leq \mu + \varepsilon\|x_0\| + \delta(N, M) \cdot \|y\| \\ &\leq \mu + \varepsilon\|x_0\| + \delta(N, M)[(1+\varepsilon)\|x_0\| + \mu].\end{aligned}$$

From this we obtain that

$$\mu \geq \left[\frac{1-\varepsilon-\delta(N, M)}{1+\delta(N, M)} - \varepsilon\right]\|x_0\|.$$

Since $\varepsilon > 0$ is arbitrary, this implies the inequality (9). ■

Lemma 1.35. *Suppose that $T \in L(X)$, X a Banach space, is semi-regular. Then*

$$\gamma(\lambda I - T) \geq \gamma(T) - 3|\lambda| \quad \textit{for every } \lambda \in \mathbb{C}. \tag{10}$$

Proof Obviously, for every $T \in L(X)$ and $|\lambda| \geq \gamma(T)$ we have

$$\gamma(\lambda I - T) \geq 0 \geq \gamma(T) - 3|\lambda|,$$

so we need to prove the inequality (10) only in the case that $\lambda < \gamma(T)$.

Suppose that T is semi-regular and hence, by Theorem 1.24, $C(T) = T^\infty(X)$. If $C(T) = \{0\}$ then $\ker T \subseteq T^\infty(X) = \{0\}$, so that T is injective, and since $T(X)$ is closed it follows that T is bounded below. From an inspection of the proof of Lemma 1.30 we then conclude that

$$\gamma(\lambda I - T) \geq \gamma(T) - |\lambda| \geq \gamma(T) - 3|\lambda|$$

for every $|\lambda| < \gamma(T)$.

Also the case $C(T) = X$ is trivial. Indeed in such a case T is surjective and hence $T^\star$ is bounded below, and consequently

$$\gamma(\lambda I - T) = \gamma(\lambda I^\star - T^\star) \geq \gamma(T^\star) - 3|\lambda| = \gamma(T) - 3|\lambda|.$$

It remains to prove the inequality (10) in the case that $C(T) \neq \{0\}$ and $C(T) \neq X$. Assume that $|\lambda| < \gamma(T)$ and let $x \in C(T) = T(C(T))$. Then there exists $u \in C(T)$ such that $x = Tu$ and therefore

$$\operatorname{dist}(u, \ker T) \leq (\gamma(T))^{-1}\|Tu\| = (\gamma(T))^{-1}\|x\|.$$

Let $\varepsilon > 0$ be arbitrary and choose $w \in \ker T$ such that

$$\|u - w\| \leq [(1-\varepsilon)\gamma(T)]^{-1}\|x\|.$$

Let $u_1 := u - w$ and $\mu := (1-\varepsilon)\gamma(T)$. Clearly, $u_1 \in C(T)$, $Tu_1 = x$ and $\|u_1\| \leq \mu^{-1}\|x\|$. Since $u_1 \in C(T)$, by repeating the same procedure we obtain a sequence $(u_n)_{n\in\mathbb{Z}_+}$, where $u_0 := x$, such that

$$u_n \in C(T),\ Tu_{n+1} = u_n \quad \text{and} \quad \|u_n\| \leq \mu^{-n}\|x\|.$$

Let $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < \mu\}$ and let us consider the function $f : \mathbb{D} \to X$

$$f(\lambda) := \sum_{n=0}^{\infty} \lambda^n u_n.$$

Clearly $f(0) = x$ and $f(\lambda) \in \ker\,(\lambda I - T)$ for all $|\lambda| < \mu$. Moreover,

$$\|x - f(\lambda)\| = \|\sum_{n=1}^{\infty} \lambda^n u_n\| \leq \frac{|\lambda|}{\mu - |\lambda|}.$$

From this we obtain

$$\operatorname{dist}(x, \ker\,(\lambda I - T)) \leq \frac{|\lambda|}{\mu - |\lambda|}$$

which yields

$$\delta(\ker T, \ker\,(\lambda I - T)) \leq \frac{|\lambda|}{\mu - |\lambda|} = \frac{|\lambda|}{(1-\varepsilon)\gamma(T) - |\lambda|}$$

for every $|\lambda| < \mu$. Since ε is arbitrary then we conclude that

$$\delta(\ker T, \ker\,(\lambda I - T)) \leq \frac{|\lambda|}{\gamma(T) - |\lambda|} \quad \text{for every} |\lambda| < \gamma(T). \tag{11}$$

If we let $\delta := \delta(\ker T, \ker\,(\lambda I - T))$, by Lemma 1.34 to the element u and $\varepsilon > 0$ there corresponds $v \in X$ such that $z := u - v \in \ker\,(\lambda I - T)$ and

$$\operatorname{dist}(v, \ker T) \geq \frac{1-\delta}{1+\delta}(1-\varepsilon)\|v\|.$$

From this it follows that

$$\begin{aligned} \|(\lambda I - T)u\| &= \|(\lambda I - T)v\| \geq \|Tv\| - |\lambda|\|v\| \\ &\geq \gamma(T) \cdot \operatorname{dist}(v, \ker T) - |\lambda|\|v\| \\ &\geq \gamma(T)\frac{1-\delta}{1+\delta}(1-\varepsilon)\|v\| - |\lambda|\|v\|. \end{aligned}$$

By using the inequality (11) we then obtain

$$\begin{aligned} \|(\lambda I - T)u\| &\geq [(1-\varepsilon)(\gamma(T) - 2|\lambda|) - |\lambda|]\|v\| \\ &\geq [(1-\varepsilon)(\gamma(T) - 2|\lambda|) - |\lambda|]\|u - z\| \\ &\geq [(1-\varepsilon)(\gamma(T) - 2|\lambda|) - |\lambda|] \cdot \operatorname{dist}(u, \ker(\lambda I - T)). \end{aligned}$$

From the last inequality it easily follows that

$$\gamma(\lambda I - T) \geq (1-\varepsilon)(\gamma(T) - 2|\lambda|) - |\lambda|,$$

and since ε is arbitrary we conclude that $\gamma(\lambda I - T) \geq \gamma(T) - 3|\lambda|$. ∎

In the following result we show that the subspaces $C(\lambda I - T)$ are constant as λ ranges through a component Ω of $\rho_{\mathrm{se}}(T)$. Note that $C(\lambda I - T) = K(\lambda I - T) = (\lambda I - T)^{\infty}(X)$ for all $\lambda \in \rho_{\mathrm{se}}(T)$.

Theorem 1.36. *Let $T \in L(X)$, X a Banach space, and consider a connected component Ω of $\rho_{\rm se}(T)$. If $\lambda_0 \in \Omega$ then $C(\lambda I - T) = C(\lambda_0 I - T)$ for every $\lambda \in \Omega$.*

Proof Observe first that by the first part of the proof of Theorem 1.31 we have $C(T) \subseteq C(\delta I - T)$ for every $|\delta| < \gamma(T)$. Now, take $|\delta| < \frac{1}{4}\gamma(T)$ and define $S := \delta I - T$. By Lemma 1.35 we obtain

$$\gamma(S) = \gamma(\delta I - T) \geq \gamma(T) - 3|\delta| > |\delta|$$

and hence, again by the observation above,

$$C(\delta I - T) = C(S) \subseteq C(\delta I - S) = C(T).$$

From this it follows that $C(\delta I - T) = C(T)$ for δ sufficiently small.

Consider now two arbitrary points $\lambda, \mu \in \Omega$. Then $\lambda I - T = (\lambda - \mu)I - (T - \mu I)$ and the previous argument shows that if we choice λ, μ sufficiently close to each other then

$$C(\lambda I - T) = C((\lambda - \mu)I - (T - \mu I)) = C(\mu I - T).$$

Finally, the next standard compactness argument shows that $C(\lambda I - T) = C(\mu I - T)$ for all $\lambda, \mu \in \Omega$.

Join a fixed point $\lambda_0 \in \Omega$ with an arbitrary point $\lambda_1 \in \Omega$ by a polygonal line $P \subset \Omega$ and associate with each $\mu \in P$ a disc in which $C(\mu I - T)$ is constant. By the classical Heine–Borel theorem already finitely many of these discs cover P, so $C(\lambda_0 I - T) = C(\lambda_1 I - T)$. Hence the subspaces $C(\lambda I - T)$ are constant on Ω. ∎

Lemma 1.37. *Let X be a Banach space and suppose that $T \in L(X)$ has a closed range. Let Y be a (not necessarily closed) subspace of X. If $Y + \ker(T)$ is closed then $T(Y)$ is closed.*

Proof Let us denote by $\overline{x}$ the equivalence class $x + \ker T$ in the quotient space $X/\ker T$ and by $\overline{T} : X/\ker T \to X$ the canonical injection defined by $\overline{T}(\overline{x}) := Tx$, where $x \in \overline{x}$. Since $T(X)$ is closed $\overline{T}$ has a bounded inverse $\overline{T}^{-1} : T(X) \to X/\ker T$. Let $\overline{Y} := \{\overline{y} : y \in Y\}$. Clearly $T(Y) = \overline{T}(\overline{Y})$ is the inverse image of $\overline{Y}$ under the continuous map $\overline{T}^{-1}$, so $T(Y)$ is closed if $\overline{Y}$ is closed .

It remains to show that $\overline{Y}$ is closed if $Y + \ker T$ is closed. Suppose that the sequence $(\overline{x}_n)$ of $\overline{Y}$ converges to $\overline{x} \in X/\ker T$. This implies that there exists a sequence (x_n) with $x_n \in \overline{x}_n$ such that $\mathrm{dist}\,(x_n - x, \ker(T))$, the distance of $x_n - x$ to $\ker T$, converges to zero, and so there exists a sequence $(z_n) \subset \ker T$ such that $x_n - x - z_n \to 0$. Then the sequence $(x_n - z_n) \subset Y + \ker T$ converges to x and since by assumption $Y + \ker T$ is closed, we have $x \in Y + \ker T$. This implies $\overline{x} \in \overline{Y}$; thus $\overline{Y}$ is closed . ∎

The next result shows that the semi-regularity of an operator may be characterized in terms of the continuity of certain maps.

Theorem 1.38. *For a bounded operator T on a Banach space X and $\lambda_0 \in \mathbb{C}$, the following statements are equivalent:*

(i) *$\lambda_0 I - T$ is semi-regular;*

(ii) *$\gamma(\lambda_0 I - T) > 0$ and the mapping $\lambda \to \gamma(\lambda I - T)$ is continuous at the point λ_0;*

(iii) *$\gamma(\lambda_0 I - T) > 0$ and the mapping $\lambda \to \ker(\lambda I - T)$ is continuous at λ_0 in the gap metric;*

(iv) *The range $(\lambda I - T)(X)$ is closed in a neighborhood of λ_0 and the mapping $\lambda \to (\lambda I - T)(X)$ is continuous at λ_0 in the gap metric.*

Proof There is no loss of generality if we assume that $\lambda_0 = 0$.

(i) $\Rightarrow$ (ii) By assumption T is semi-regular and hence has closed range, so that $\gamma(T) > 0$. Moreover, for every $|\lambda| < \gamma(T)$, the operator $\lambda I - T$ is semi-regular, by Theorem 1.31. Consider $|\lambda| < \gamma(T)$ and $|\mu| < \gamma(T)$. Then by Lemma 1.35 we have

$$|\gamma(\lambda I - T) - \gamma(\mu I - T)| \leq 3|\lambda - \mu|,$$

and this obviously implies the continuity of the mapping $\lambda \to \gamma(\lambda I - T)$ at the point 0.

(ii) $\Rightarrow$ (iii) The continuity of the mapping $\lambda \to \gamma(\lambda I - T)$ at 0 ensures that there exists a neighborhood $\mathcal{U}$ of 0 such that

$$\gamma(\lambda I - T) \geq \frac{\gamma(T)}{2} \quad \text{for all } \lambda \in \mathcal{U}.$$

From Lemma 1.33 we infer that

$$\widehat{\delta}(\ker(\mu I - T), \ker(\lambda I - T)) \leq \frac{2}{\gamma(T)}|\lambda - \mu| \quad \text{for all } \lambda, \mu \in \mathcal{U},$$

and in particular that

$$\widehat{\delta}(\ker T, \ker(\lambda I - T)) \leq \frac{2}{\gamma(T)}|\lambda| \quad \text{for all } \lambda \in \mathcal{U}.$$

From this estimate we conclude that $\ker(\lambda I - T))$ converges in the gap metric to $\ker T$, as $\lambda \to 0$. Hence the mapping $\lambda \to \ker(\lambda I - T)$ is continuous at 0.

(iii) $\Rightarrow$ (i) It is clear that $\ker(\lambda I - T) \subseteq T^n(X)$ for every n. For every $x \in \ker T$, $n \in \mathbb{N}$, and $\lambda \neq 0$ we then have

$$\text{dist}\ (x, T^n(X)) \leq\ \text{dist}(x, \ker(\lambda I - T)) \leq \delta(\ker T, \ker(\lambda I - T)) \cdot \|x\|.$$

From this estimate we deduce that

$$\text{dist}\ (x, T^n(X)) \leq \widehat{\delta}(\ker T, \ker(\lambda I - T)) \cdot \|x\|.$$

The continuity at 0 of the mapping $\lambda \to \ker(\lambda I - T)$ then implies that $x \in \overline{T^n(X)}$ for every n. Hence $\ker T \subseteq \overline{T^n(X)}$ for every $n = 1, \cdots$.

To establish the semi-regularity of T it suffices to prove that $T^n(X)$ is closed for $n \in \mathbb{N}$. We proceed by induction.

The case $n = 1$ is obvious, from the assumption. Assume that $T^n(X)$ is closed. Then $\ker T \subseteq \overline{T^n(X)} = T^n(X)$ and hence $\ker T + T^n(X) = T^n(X)$ is closed. By Lemma 1.37 we then conclude that $T(T^n(X)) = T^{n+1}(X)$ is closed. Hence (i), (ii) and (iii) are equivalent.

(i) $\Rightarrow$ (iv) If T is semi-regular and $\mathbb{D}(0, \gamma(T))$ is the open disc centered at 0 with radius $\gamma(T)$ then $(\lambda I - T)$ is semi-regular for all $\lambda \in \mathbb{D}(0, \gamma(T))$, by Theorem 1.31. In particular, $(\lambda I - T)(X)$ is closed and this implies that $\ker(\lambda I - T^\star)^\perp = (\lambda I - T)(X)$ for all $\lambda \in \mathbb{D}(0, \gamma(T))$.

Now, $T^\star$ is semi-regular by Theorem 1.19, and, by the first part of the proof, this is equivalent to the continuity at 0 of the mapping

$$\lambda \to \ker(\lambda I - T^\star) = (\lambda I - T)(X)^\perp.$$

But from Remark 1.32 we know that

$$\widehat{\delta}(T(X)^\perp, (\lambda I - T)(X)^\perp) = \widehat{\delta}(T(X), (\lambda I - T)(X)),$$

and consequently the mapping $\lambda \to (\lambda I - T)(X)$ is continuous at 0.

(iv) $\Rightarrow$ (iii) Let $\mathcal{U}$ a neighborhood of 0 such that $(\lambda I - T)(X)$ is closed for every $\lambda \in \mathcal{U}$. Then $\gamma(T) > 0$. Again, from Remark 1.32 we infer that

$$\begin{aligned}\widehat{\delta}(\ker T^\star, \ker(\lambda I - T^\star)) &= \widehat{\delta}(^\perp\ker T^\star, ^\perp\ker(\lambda I - T^\star)) \\ &= \widehat{\delta}(T(X), (\lambda I - T)(X)),\end{aligned}$$

and hence the mapping $\lambda \to \gamma(\lambda I - T^\star) = \gamma(\lambda I - T)$ is continuous at 0. ∎

Theorem 1.39. *Suppose that $\lambda_0 I - T \in L(X)$, X a Banach space, is semi-regular and Ω the component of $\rho_{\mathrm{se}}(T)$ containing λ_0. If $(\lambda_n)_{n\in\mathbb{Z}_+}$ is a sequence of distinct points of Ω which converges to λ_0, then*

$$K(T) = \bigcap_{n=0}^{\infty}(\lambda_n I - T)(X) = \bigcap_{n=1}^{\infty}(\lambda_n I - T)(X). \tag{12}$$

Proof We show first the second equality in (12). Trivially, the inclusion $\bigcap_{n=0}^{\infty}(\lambda_n I - T)(X) \subseteq \bigcap_{n=1}^{\infty}(\lambda_n I - T)(X)$ hold for every $T \in L(X)$.

To show the opposite inclusion, suppose that $x \in \bigcap_{n=1}^{\infty}(\lambda_n I - T)(X)$. Then

$$\mathrm{dist}\,(x, (\lambda_0 I - T)(X)) \leq \widehat{\delta}((\lambda_n I - T)(X), (\lambda_0 I - T)(X)) \cdot \|x\|$$

for every $n \in \mathbb{N}$. Because $\lambda_n \to \lambda_0$, from Theorem 1.38 it follows that $x \in \overline{(\lambda_0 I - T)(X)} = (\lambda_0 I - T)(X)$. Therefore the equality

$$\bigcap_{n=0}^{\infty}(\lambda_n I - T)(X) = \bigcap_{n=1}^{\infty}(\lambda_n I - T)(X) \tag{13}$$

is proved.

By Theorem 1.36 we now have

$$K(\lambda_0 I - T) = K(\lambda_n I - T) \subseteq (\lambda_n I - T)(X), \quad n \in \mathbb{N}.$$

Hence

$$K(\lambda_0 I - T) \subseteq \bigcap_{n=1}^{\infty} (\lambda_n I - T)(X).$$

Conversely, assume that $x \in \bigcap_{n=1}^{\infty} (\lambda_n I - T)(X)$. From the equality (13) we know that $x \in (\lambda_0 I - T)(X)$ so that there exists an element $u \in X$ such that $x = (\lambda_0 I - T)u$. Write $x = (\lambda_n I - T)u + (\lambda_0 - \lambda_n)u$, where $n \in \mathbb{N}$. Since $x \in (\lambda_n I - T)(X)$ for every $n \in \mathbb{N}$ then $(\lambda_0 - \lambda_n)u$ belongs to $\bigcap_{n=1}^{\infty} (\lambda_n I - T)(X)$. Now, by assumption, $\lambda_n \neq \lambda_0$ for every $n \in \mathbb{N}$, so that $u \in \bigcap_{n=1}^{\infty} (\lambda_n I - T)(X)$. This shows that

$$x = (\lambda_0 I - T)u \in (\lambda_0 I - T)(\bigcap_{n=1}^{\infty} (\lambda_n I - T)(X)),$$

and therefore the inclusion

$$\bigcap_{n=1}^{\infty} (\lambda_n I - T)(X) \subseteq (\lambda_0 I - T)(\bigcap_{n=1}^{\infty} (\lambda_n I - T)(X)).$$

is proved. The opposite inclusion is clearly satisfied. By Theorem 1.22 we then conclude that

$$\bigcap_{n=1}^{\infty} (\lambda_n I - T)(X) \subseteq K(\lambda_0 I - T);$$

which conclude the proof. ∎

5. The generalized Kato decomposition

In this section we introduce an important property of decomposition for bounded operators which involves the concept of semi-regularity.

Definition 1.40. *An operator $T \in L(X)$, X a Banach space, is said to admit a* generalized Kato decomposition, *abbreviated as GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent.*

Clearly, every semi-regular operator has a GKD $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD $M = \{0\}$ and $N = X$.

A relevant case is obtained if we assume in the definition above that $T|N$ is nilpotent, there exists $d \in \mathbb{N}$ for which $(T|N)^d = 0$. In this case T is said to be of *Kato type of operator of order d*.

An operator is said to be *essentially semi-regular* if it admits a GKD (M, N) such that N is finite-dimensional. Note that if T is essentially semi-regular then $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent.

Hence we have the following implications:

$$\begin{aligned} T \text{ semi-regular} \quad &\Rightarrow \quad T \text{ essentially semi-regular} \Rightarrow T \text{ of Kato type} \\ &\Rightarrow \quad T \text{ admits a GKD.} \end{aligned}$$

Some of the properties already observed for semi-regular operators may be extended to operators which admit a GKD.

Theorem 1.41. *Suppose that (M, N) is a GKD for $T \in L(X)$. Then we have:*

(i) $K(T) = K(T|M)$ *and* $K(T)$ *is closed;*

(ii) $\ker T|M = \ker T \cap M = K(T) \cap \ker T$.

Proof (i) To prove the equality $K(T) = K(T|M)$, we need only to show that $K(T) \subseteq M$. Let $x \in K(T)$ and, according the definition of $K(T)$, let (u_n), $n \in \mathbb{Z}_+$, be a sequence of X and $\delta > 0$ such that

$$x = u_0, \quad Tu_{n+1} = u_n, \quad \text{and} \quad \|u_n\| \leq \delta^n \|x\| \quad \text{for all } n \in \mathbb{N}.$$

Clearly $T^n u_n = x$ for all $n \in \mathbb{N}$. From the decomposition $X = M \oplus N$ we know that $x = y + z$, $u_n = y_n + z_n$, with $y, y_n \in M$ and $z, z_n \in N$. Then $x = T^n u_n = T^n y_n + T^n z_n$, hence, by the uniqueness of the decomposition, $y = T^n y_n$ and $z = T^n z_n$ for all n. Let P denote the projection of X onto N along M. From the estimate

$$\|((T|N)P)^n\|^{1/n} \leq \|(T|N)^n\|^{1/n} \|P^n\|^{1/n} = \|(T|N)^n\|^{1/n} \|P\|^{1/n},$$

we infer that also $(T|N)P$ is quasi-nilpotent, since, by assumption, $T|N$ is quasi-nilpotent. Therefore, if $\varepsilon > 0$, there is a positive integer n_0 such that $\|(TP)^n\|^{1/n} = \|((T|N)P)^n\|^{1/n} < \varepsilon$ for all $n > n_0$. Now we have

$$\|z\| = \|T^n z_n\| = \|T^n P u_n\| = \|(TP)^n u_n\| \leq \varepsilon^n \delta^n \|x\|, \tag{14}$$

for all $n > n_o$. Since ε is arbitrary the last term of (14) converges at 0, so $z = 0$ and this implies that $x = y \in M$.

The last assertion is a consequence of Theorem 1.24, since the restriction $T|M$ is semi-regular.

(ii) This equality is a consequence of (i). Indeed, $K(T) \subseteq M$ and, since $T|M$ is semi-regular, from Theorem 1.10 and part (i) we infer that

$$\ker (T|M) \subseteq (T|M)^\infty (M) = K(T|M) = K(T).$$

From this we conclude that

$$K(T) \cap \ker T = K(T) \cap M \cap \ker T = K(T) \cap \ker (T|M) = \ker (T|M),$$

so part (ii) also is proved. ∎

The property that the hyper-range and the analytical core coincide for every semi-regular operator may be extended to the more general situation of operators of Kato type.

Theorem 1.42. *Let $T \in L(X)$, X a Banach space, and assume that T is of Kato type of order d with a GKD (M, N). Then:*

(i) $K(T) = T^\infty(X)$;

(ii) $\ker (T|M) = \ker T \cap T^\infty(X) = \ker T \cap T^n(X)$ *for every natural* $n \geq d$;

(iii) *We have* $T(X) + \ker T^n = T(M) \oplus N$ *for every natural* $n \geq d$. *Moreover,* $T(X) + \ker T^n$ *is closed in* X.

Proof (i) We have $(T|N)^d = 0$. For $n \geq d$ we have

$$T^n(X) = T^n(M) \oplus T^n(N) = T^n(M) \tag{15}$$

and consequently $T^\infty(X) = (T|M)^\infty(M)$. By Theorem 1.24 the semi-regularity of $T|M$ gives $(T|M)^\infty(M) = K(T|M)$ and the last set, by Theorem 1.41, coincides with $K(T)$.

(ii) Let $n \geq d$. Clearly, $T^n(X) = T^n(M)$. From the equalities (15) and part (ii) of Theorem 1.41 we obtain

$$\begin{aligned} \ker(T|M) = \ker T \cap K(T) &\subseteq \ker T \cap T^n(X) = \ker T \cap T^n(M) \\ &\subseteq \ker T \cap M = \ker(T|M). \end{aligned}$$

Hence for all $n \geq d$, $\ker\,(T|M) = \ker T \cap T^n(X)$.

(iii) It is obvious that if $n \geq d$ then $N \subseteq \ker T^n$, so $T(M) \oplus N \subseteq T(X) + \ker T^n$. Conversely, if $n \geq d$ then

$$\ker T^n = \ker(T|M)^n \oplus \ker(T|N)^n = \ker(T|M)^n \oplus N$$

and from the semi-regularity of $T|M$ it follows that $\ker T^n \subseteq T(M) \oplus N$.

Since $T(X) = T(M) \oplus T(N) \subseteq T(M) \oplus N$ we then conclude that $T(X) + \ker T^n \subseteq T(M) \oplus N$. Hence $T(X) + \ker T^n = T(M) \oplus N$.

To complete the proof we show that $T(M) \oplus N$ is closed. Let $M \times N$ be provided with the canonical norm

$$\|(x, y)\| := \|x\| + \|y\| \quad (x \in M, y \in N),$$

Clearly, $M \times N$ with respect to this norm is complete. Let $\Psi : M \times N \to M \oplus N = X$ denote the topological isomorphism defined by

$$\Psi(x, y) := x + y \quad \text{for every } x \in M, y \in N.$$

We have $\Psi(T(M), N) = T(M) \oplus N$ and hence, since $(T(M), N)$ is closed in $M \times N$, the subspace $T(M) \oplus N$ is closed in X. ■

Note that the equality $K(T) = T^\infty(X)$ need not be verified if we assume only that T admits a GKD. In fact, as we will see later, for every quasi-nilpotent operator we have $K(T) = \{0\}$, whilst there are examples of quasi-nilpotent operators for which $T^\infty(X) \neq \{0\}$, see Example 2.80.

Theorem 1.43. *Assume that* $T \in L(X)$, X *a Banach space, admits a GKD* (M, N). *Then* $(N^\perp, M^\perp)$ *is a GKD for* $T^\star$. *Furthermore, if* T *is of Kato type then* $T^\star$ *is of Kato type .*

Proof Suppose that T has a GKD (M, N). Clearly both subspaces $N^\perp$ and $M^\perp$ are invariant under $T^\star$. Let P_M denote the projection of X onto M along N. Trivially, ${P_M}^\star$ is idempotent and from the equalities $M = P_M(X)$, $N = \ker\, P_M$ we obtain that

$${P_M}^\star(X^\star) = (\ker P_M)^\perp = N^\perp \quad \text{and} \quad \ker {P_M}^\star = [P_M(X)]^\perp = M^\perp.$$

Hence
$$X^\star = P_M{}^\star(X^\star) \oplus \ker P_M{}^\star = N^\perp \oplus M^\perp.$$

Now, if $P_N := I - P_M$ then $TP_N = P_N T$ is quasi-nilpotent and therefore also $T^\star P_N{}^\star = P_N{}^\star T^\star$ is quasi-nilpotent, from which we conclude that the restriction $T^\star|M^\perp$ is quasi-nilpotent.

To end the proof of the first assertion we need only to show that $T^\star|N^\perp$ is semi-regular, that is $T^\star(N^\perp)$ is closed and $\ker(T^\star|N^\perp)^n \subseteq T^\star(N^\perp)$ for all positive integer $n \in \mathbb{N}$.

From assumption $T(M) = TP_M(X)$ is closed, and therefore, by Theorem 1.13, $(P_M{}^\star T^\star)(X^\star)$ is closed. From the equality $T^\star P_M{}^\star = P_M{}^\star T^\star$ it then follows that
$$(TP_M)^\star(X^\star) = (T^\star P_M{}^\star)(X^\star) = T^\star(N^\perp)$$
is closed. Furthermore, for all $n \in \mathbb{N}$ we have
$$\ker(T^\star|N^\perp)^n = \ker(T^\star)^n \cap N^\perp = T^n(X)^\perp \cap N^\perp = [T^n(X) + N]^\perp .$$
From the equalities
$$\ker(TP_M) = \ker T|M + N \subseteq T^n(M) + N \subseteq T^n(X) + N ,$$
we then conclude that
$$\ker(T^\star|N^\perp)^n = [T^n(X) + N]^\perp \subseteq [\ker(TP_M)]^\perp = T^\star P_M{}^\star(X^\star) = T^\star(N^\perp)$$
for all $n \in \mathbb{N}$, thus $T^\star|N^\perp$ is semi-regular. This shows that if T has a GKD (M, N) then $T^\star$ has the GKD $(N^\perp, M^\perp)$. Evidently, if additionally $T|N$ is nilpotent then $T|M^\perp$ is nilpotent, so that $T^\star$ is of Kato type. ∎

Theorem 1.44. *Suppose that $T \in L(X)$, X a Banach space, is of Kato type. Then there exists an open disc $\mathbb{D}(0, \varepsilon)$ for which $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}$.*

Proof Let (M, N) be a GKD for T such that $T|N$ is nilpotent.

First we show that $(\lambda I - T)(X)$ is closed for all $0 < |\lambda| < \gamma(T|M)$, where $\gamma(T|M)$ denotes the minimal modulus of $T|M$. From the nilpotency of $T|N$ we know that the restriction $\lambda I - T|N$ is bijective for every $\lambda \neq 0$, thus $N = (\lambda I - T)(N)$ for every $\lambda \neq 0$, and therefore
$$(\lambda I - T)(X) = (\lambda I - T)(M) \oplus (\lambda I - T)(N) = (\lambda I - T)(M) \oplus N$$
for every $\lambda \neq 0$. By assumption $T|M$ is semi-regular, so by Theorem 1.31 $(\lambda I - T)|M$ is semi-regular for every $|\lambda| < \gamma(T|M)$, and hence for these values of λ $(\lambda I - T)(M)$ is a closed subspace of M.

We show now that $(\lambda I - T)(X)$ is closed for every $0 < |\lambda| < \gamma(T|M)$. Consider the Banach space $M \times N$ provided with the canonical norm
$$\|(x, y)\| := \|x\| + \|y\|, \quad x \in M,\ y \in N,$$
and let $\Psi : M \times N \to M \oplus N = X$ denote the topological isomorphism defined by $\Psi(x, y) := x + y$ for every $x \in M$ and $y \in N$. Then for every

$0 < |\lambda| < \gamma(T|M)$ the set

$$\Psi[(\lambda I - T)(M) \times N] = (\lambda I - T)(M) \oplus N = (\lambda I - T)(X)$$

is closed since the product $(\lambda I - T)(M) \times N$ is closed in $M \times N$.

We show now that there exists an open disc $\mathbb{D}(0, \varepsilon)$ such that

$$\mathcal{N}^\infty(\lambda I - T) \subseteq (\lambda I - T)^\infty(X) \quad \text{for all } \lambda \in \mathbb{D}(0, \varepsilon) \setminus \{0\}.$$

Since T is of Kato type, the hyper-range is closed and coincides with $K(T)$, by Theorem 1.41 and Theorem 1.42. Consequently $T(T^\infty(X)) = T^\infty(X)$. Let $T_0 := T|T^\infty(X)$. The operator T_0 is onto and hence, by part (ii) of Lemma 1.30, also $\lambda I - T_0$ is onto for all $|\lambda| < \gamma(T_0)$. Therefore $(\lambda I - T)(T^\infty(X)) = T^\infty(X)$ for $|\lambda| < \gamma(T_0)$, and hence, since by Theorem 1.22 the hyper-range $T^\infty(X)$ is closed, we infer that

$$T^\infty(X) \subseteq K(\lambda I - T) \subseteq (\lambda I - T)^\infty(X) \quad \text{for all } |\lambda| < \gamma(T_0).$$

From Theorem 1.3, part (ii) we then conclude that

$$\mathcal{N}^\infty(\lambda I - T) \subseteq T^\infty(X) \subseteq (\lambda I - T)^\infty(X) \quad \text{for all } \ 0 < |\lambda| < \gamma(T_0). \tag{16}$$

The inclusions (16), together with $(\lambda I - T)(X)$ being closed for all $0 < |\lambda| < \gamma(T|M)$, then imply the semi-regularity of $\lambda I - T$ for all $0 < |\lambda| < \varepsilon$, where $\varepsilon :=$ min $\{\gamma(T_0), \gamma(T|M)\}$. ∎

For every bounded operator $T \in L(X)$, X a Banach space, let us denote by

$$\sigma_{\rm k}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \ \text{ s not of Kato type}\}$$

the *Kato type of spectrum*, and by

$$\sigma_{\rm es}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular}\}$$

the *essentially semi-regular spectrum*. The spectrum $\sigma_{\rm es}(T)$ is an essential version of the semi-regular spectrum $\sigma_{\rm se}(T)$ and, as we shall see, is very closely related to the semi-Fredholm spectra which will be studied later. Evidently

$$\sigma_{\rm k}(T) \subseteq \sigma_{\rm es}(T) \subseteq \sigma_{\rm se}(T).$$

As a straightforward consequence of Theorem 1.44 we easily obtain that these spectra differ of at most countably many points.

Corollary 1.45. *If $T \in L(X)$, X a Banach space, then $\sigma_{\rm k}(T)$ and $\sigma_{\rm es}(T)$ are compact subsets of $\mathbb{C}$. Moreover, the sets $\sigma_{\rm se}(T) \setminus \sigma_{\rm k}(T)$ and $\sigma_{\rm es}(T) \setminus \sigma_{\rm k}(T)$ consist of at most countably many isolated points.*

Proof Clearly $\rho_{\rm k}(T) := \mathbb{C} \setminus \sigma_{\rm k}(T)$ and $\rho_{\rm se}(T) := \mathbb{C} \setminus \sigma_{\rm se}(T)$ are open, by Theorem 1.44, so $\rho_{\rm k}(T)$ and $\sigma_{\rm se}(T)$ are compact. Furthermore, if $\lambda_0 \in \sigma_{\rm se}(T) \setminus \sigma_{\rm k}(T)$ then $\lambda I - T$ is semi-regular as λ belongs to a suitable punctured disc centered at λ_0. Hence λ_o is an isolated point of $\sigma_{\rm se}(T)$. From this it follows that $\sigma_{\rm se}(T) \setminus \sigma_{\rm k}(T)$ consists of at most countably many isolated points. ∎

If M_1 and M_2 are subspaces of X then we shall say that M_1 is *essentially contained* in M_2, shortly $M_1 \subset_{\rm e} M_2$ if there exists a finite-dimensional subspace F of X for which $M_1 \subseteq M_2 + F$. Obviously

$$M_1 \subset_{\rm e} M_2 \Leftrightarrow \quad \dim M_1/(M_1 \cap M_2) < \infty.$$

Lemma 1.46. *Let $T \in L(X)$, X a Banach space, be an operator with closed range. Suppose that for every $n \in \mathbb{N}$ there is a finite-dimensional subspace $F_n \subseteq \ker T$ such that $\ker T \subseteq \overline{T^n(X)} + F_n$. Then $T^n(X)$ is closed for all $n \in \mathbb{N}$.*

Proof We proceed by induction on n. For $n = 1$ the statement is true, by assumption. Suppose that $T^n(X) = \overline{T^n(X)}$. To show that $T^{n+1}(X)$ is closed let $y \in \overline{T^{n+1}(X)}$ be arbitrarily given. Then there is a sequence $(x_j) \subset X$ such that $T^{n+1}x_j \to y$ as $j \to \infty$. From the inclusion $T^{n+1}(X) \subseteq T^n(X)$ we infer that $y \in \overline{T^n(X)} = T^n(X)$, so we can find some $x \in X$ for which $y = T^n x$. Thus $T(T^n x_j - T^{n-1}x) \to 0$ as $j \to \infty$.

Consider the canonical injection $\widetilde{T} : \widetilde{X} := X/\ker\, T \to X$ induced by T. The operator $\widetilde{T}$ is bounded below because it is injective and has range equal to $T(X)$. Furthermore, $\tilde{T}(T^n x_j - T^{n-1}x + \ker T) \to 0$ as $j \to 0$, so that in the quotient space $\widetilde{X}$ the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$, whose terms are defined by $\tilde{u}_n := T^n x_j - T^{n-1}x + \ker T$, converges to 0 as $j \to 0$. Hence there exists a sequence $(z_j)_{j\in\mathbb{N}} \in \ker\, T$ such that $T^n x_j + z_j \to T^{n-1}x$ as $j \to \infty$. Since $z_j \in \ker\, T \subseteq T^n(X) + F_n$ and $T^n(X) + F_n$ is closed it follows that

$$T^{n-1}x = T^n u + f \quad \text{for some } u \in X \quad \text{and } f \in F_n \subseteq \ker T.$$

Therefore $y = T^n x = T^{n+1}u \in T^{n+1}(X)$, so $T^{n+1}(X)$ is closed. ∎

Lemma 1.47. *Let $T \in L(X)$, X a Banach space. Suppose that either of the two following conditions hold:*

(i) $\mathcal{N}^\infty(T) \subset_{\rm e} T^\infty(X)$;

(ii) $\mathcal{N}^\infty(T) \subset_{\rm e} T(X)$.

Then for every $n, m \in \mathbb{N}$ we have $\ker T^n \subset_{\rm e} T^m(X)$.

Proof Suppose that the inclusion (i) holds, namely, there is a finite-dimensional space F for which $\mathcal{N}^\infty(T) \subseteq T^\infty(X) + F$. Since $\ker T^n \subseteq \mathcal{N}^\infty(T)$ and $T^m(X) \supseteq T^\infty(X)$, for all $n, m \in \mathbb{N}$, we easily obtain that $\ker T^n \subseteq T^m(X) + F$, for every $n, m \in \mathbb{N}$.

Assume that the inclusion (ii) is fulfilled. Here we proceed by induction on m. The statement is immediate for $m = 1$. Suppose that the inclusions $\ker T^n \subseteq T^j(X) + F_{n,j}$ hold for every n and every $j \leq m - 1$, where $F_{n,j}$ is finite-dimensional and $F_{n,j} \subseteq \ker T^n$. Then

$$\ker T^{n+1} \subseteq T^{m-1}(X) + F_{n+1,m-1}$$

and hence

$$T(\ker T^{n+1}) \subseteq T^m(X) + T(F_{n+1,m-1}).$$

Moreover, $T(\ker T^{n+1}) = \ker T^n \cap T(X)$ and $\ker T^n \subseteq T(X) + F_{n,1}$, where $F_{n,1} \subseteq \ker T^n$. Consequently

$$\begin{aligned}\ker T^n &\subseteq [T(X) \cap \ker T^n] + F_{n,1} = T(\ker T^{n+1}) + F_{n,1} \\ &\subseteq T^m(X) + T(F_{n+1,m-1}) + F_{n,1} = T^m(X) + F_{n,m},\end{aligned}$$

where $F_{n,m} \subseteq T(F_{n+1,m}) + F_{n,1} \subseteq \ker T^n$ is finite-dimensional. Hence the assertion is true also for n, thus the proof is complete. ■

The following result shows that T is essentially semi-regular when $T(X)$ is closed and the inclusions of Theorem 1.5 and Corollary 1.6 hold in the essential sense.

Theorem 1.48. *Let T be a bounded operator on a Banach space X. Then the following conditions are equivalent:*

(i) *T is essentially semi-regular;*

(ii) *$\mathcal{N}^\infty(T) \subset_{\mathrm{e}} T^\infty(X)$ and $T(X)$ is closed;*

(iii) *$\mathcal{N}^\infty(T) \subset_{\mathrm{e}} T(X)$ and $T(X)$ is closed;*

(iv) *$\ker T \subset_{\mathrm{e}} T^\infty(X)$ and $T(X)$ is closed.*

Proof (i) $\Rightarrow$ (ii) Note first that if T is essentially semi-regular then $T(X)$ is closed. Indeed, if (M, N) is a GKD for T for which N is finite-dimensional then $T(X) = T(M) \oplus T(N)$. But $T(M)$ is closed, since $T|M$ is semi-regular, and $T(N)$ is finite-dimensional, so the sum $T(X)$ is closed.

Now, as already observed in the proof of part (i) of Theorem 1.42, we have $T^\infty(X) = (T|M)^\infty(M)$. The semi-regularity of $T|M$ yields that $\ker (T|M)^n \subseteq (T|M)^\infty(M) = T^\infty(X)$ for every $n \in \mathbb{N}$. By assumption there is $d \in \mathbb{N}$ such that $(T|N)^d = 0$ and hence for every natural $n \geq d$ we obtain

$$\ker T^n = \ker (T|M)^n \oplus N \subseteq T^\infty(X) \oplus N.$$

This shows that $\mathcal{N}^\infty(T) \subset_{\mathrm{e}} T^\infty(X)$, because N is finite-dimensional.

The implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are clear, since $T^\infty(X) \subseteq T(X)$ and $\ker T \subseteq \mathcal{N}^\infty(T)$.

(iii) $\Rightarrow$ (i) By Lemma 1.47, for all $n, m \in \mathbb{N}$ there exist some finite-dimensional subspaces $F_{n,m} \subseteq \ker T^n$ such that $\ker T^n \subseteq T^m(X) + F_{n,m}$ and hence $\ker T^n \subset_{\mathrm{e}} T^m(X)$. Note that by Lemma 1.46 $T^n(X)$ is closed for every $n \in \mathbb{N}$.

To show that T is essentially semi-regular, observe first that if $\ker T^n \subseteq T^\infty(X)$ for all $n \in \mathbb{N}$ then T is semi-regular since $T(X)$ is closed, and hence essentially semi-regular. Therefore to prove the assertion (i) we may only to consider the case that $\ker T^n$ is not contained in $T^\infty(X)$ for some n. Let n be exactly the smallest natural with this property. Let Y_1 denote a subspace of X such that

$$\ker T = Y_1 \oplus [\ker T \cap T^n(X)].$$

If k is the dimension of Y_1 then $k \geq 1$. Since $Y_1 \subseteq \ker T \subseteq T^{n-1}(X)$ we can find a subspace Y_n having dimension k and such that $T^{n-1}(Y_n) = Y_1$.

Let $Y_j := T^{n-j}(Y_n)$ for every $j = 1, 2, \cdots, n$. Clearly, for all j we have

$$Y_j \subseteq T^{n-j}(X) \quad \text{and } Y_j \cap T^{n-j+1}(X) = \{0\}.$$

From this it follows that the subspaces $Y_1, Y_2, \cdots, Y_n$ and $T^n(X)$ are linearly independent in the following sense: if $x + y_1 + \cdots + y_n = 0$, $y_j \in Y_j$, $1 \le j \le n$ and $x \in T^n(X)$ then $x = y_1 = \cdots = y_n = 0$.

Consider a basis $\{e_1, \cdots, e_k\}$ in Y_1. Since $e_1, \cdots, e_k$ are linearly independent modulo $T^n(X) + Y_2 + \cdots + Y_n$, we can find linear functionals $f_1, \cdots, f_k \in [T^n(X) + Y_2 + \cdots Y_n]^\perp$ for which $f_i(e_j) = \delta_{ji}$, $1 \le j, i \le k$. Set

$$M_1 := \bigvee_{i=0}^{n-1} \bigcup_{j=1}^{k} \ker \ (T^{\star i} f_j) \quad \text{and} \quad N_1 := \bigcup_{j=1}^{n} Y_j.$$

Clearly N_1 is T-invariant, finite-dimensional and $(T|N_1)^n = 0$. Furthermore, if $x \in M_1$ is arbitrarily given then

$$(T^{\star i} f_j)(Tx) = (T^{\star i+1} f_j)(x) = 0 \quad \text{for } 0 \le j \le n-2 \ .$$

and $(T^{\star n-1} f_j)(Tx) = f_j(T^n x) = 0$, thus $T(M_1) \subseteq M_1$.

Now choose $z_1, z_2, \cdots, z_k \in Y_n$ such that $x_j = T^{n-1} z_j$, where $1 \le j \le k$. Then the set

$$\{T^i z_j : 0 \le i \le n-1, 1 \le j \le k\}$$

forms a basis of N_1. Moreover, these elements together with the elements

$$\{T^{\star i} f_j : 0 \le i \le n-1, 1 \le j \le k\}$$

form a bi-orthogonal system. From this it easily follows that $X = M_1 \oplus N_1$. Set $T_1 := T|M_1$ and $T_2 := T|N_1$. Of course we have $\ker T = \ker T_1 \oplus N_1$ and $T^\infty(X) = {T_1}^\infty(X)$.

From the inclusion $\mathcal{N}^\infty(T) \subset_{\mathrm{e}} T(X)$ we know that

$$\dim \frac{\mathcal{N}^\infty(T)}{\mathcal{N}^\infty(T) \cap T^\infty(X)} = r < \infty \ ,$$

thus

$$\begin{aligned} \dim \frac{\mathcal{N}^\infty(T_2)}{\mathcal{N}^\infty(T_2) \cap T_2(X)} \quad &= \quad r - \ \dim \frac{\mathcal{N}^\infty(T_1)}{\mathcal{N}^\infty(T_1) \cap {T_1}^\infty(X)} \\ &= \quad r - \ \dim \frac{N_1}{\bigvee_{j=1}^{n-1} Y_j} = r - k < r \ , \end{aligned}$$

and we can repeat the same construction for T_2. After a finite number of steps we obtain the required a GKD (M, N), with N finite-dimensional and T nilpotent, so T is essentially semi-regular.

(iv) $\Rightarrow$ (i) Again, by Lemma 1.47 there exist, for every $n, m \in \mathbb{N}$, finite-dimensional subspaces $F_{n,m} \subseteq \ker \ T^n$ such that $\ker T^n \subseteq T^m(X) + F_{n,m}$, hence $\ker T^n \subset_{\mathrm{e}} T^m(X)$. Repeating the same construction as above we find two closed T-invariant subspaces M_1, N_1 such that $X = M_1 \oplus N_1$ and such $T_2 := T|N_1$ nilpotent. Set, as above, $T_1 := T|M_1$. By assumption

$$\dim \frac{\ker\ T}{\ker\ T \cap T^\infty(X)} < \infty\ ,$$

and hence

$$\begin{aligned}\dim \frac{\ker\ T_2}{\ker\ T_2 \cap T_2(X)} &= \dim\frac{\ker\ T}{\ker\ T \cap T^\infty(X)} - k \\ &< \dim\frac{\ker\ T}{\ker\ T \cap T^\infty(X)} < \infty.\end{aligned}$$

As above, after a finite number of steps we obtain the desired decomposition, thus T is essentially semi-regular. ∎

Corollary 1.49. *If $T \in L(X)$, X a Banach space, then T is essentially semi-regular if and only if $T^\star$ is essentially semi-regular.*

Proof If T is essentially semi-regular and (M, N) is a GKD for which $T|N$ is nilpotent and N is finite-dimensional then, see Theorem 1.43, $T^\star|M^\perp$ is nilpotent and $\dim\ M^\perp =$ codim $M = \dim\ N < \infty$, so $T^\star$ is essentially semi-regular.

Conversely, if $T^\star$ is essentially semi-regular then $T^\star(X^\star)$ and $T^{\star n}(X^\star)$ are closed for every $n \in \mathbb{N}$ and therefore also $T(X)$ and $T^n(X)$ are closed, by part (ii) of Theorem 1.13. Moreover, $T^{\star\star}$ is essentially semi-regular, by the first part of the proof, so $\ker T^{\star\star} \subset_{\rm e} (T^{\star\star})^\infty(X^{\star\star})$. From the equalities $\ker T = \ker T^{\star\star} \cap X$ and $T^n(X) = (T^{\star\star})^n(X^{\star\star}) \cap X$ we then conclude that $T^\infty(X) = (T^{\star\star})^\infty(X^{\star\star}) \cap X$ and hence $\ker T \subset_{\rm e} T^\infty(X)$. ∎

Theorem 1.50. *Suppose that T and S are commuting bounded operators on the Banach space X for which TS is essentially semi-regular. Then both T and S are essentially semi-regular.*

Proof We show that T is essentially semi-regular. Clearly

$$\ker T \subseteq \ker TS \subset_{\rm e} (TS)^\infty(X) \subseteq T^\infty(X).$$

It remains only to prove that $T(X)$ is closed. By assumption we know that there exists a finite-dimensional subspace F such that $\ker TS \subseteq (TS)(X)+F$. By Corollary 1.15 it suffices to prove that $T(X)+F$ is closed. Let (x_n), (y_n) be two sequences of X for which $Tx_n + y_n \to x$ as $n \to \infty$. Then $STx_n + Sy_n \to Sx$. Clearly $(TS)(X)+S(F)$ is closed, since ST is essentially semi-regular and $S(F)$ is finite-dimensional, so that $Sx \in (TS)(X) + S(F)$. Hence $Sx = TSz + Sy$ for some $z \in X$ and $y \in F$, and consequently

$$Tz + y - x \in \ker S \subseteq \ker TS \subseteq (TS)(X) + F \subseteq T(X) + F.$$

Hence $x \in T(X) + F$ and therefore $T(X) + F$ is closed. ∎

Corollary 1.51. *If $T \in L(X)$, X a Banach space, and $n \in \mathbb{N}$ then T is essentially semi-regular if and only if T^n is essentially semi-regular.*

Proof Suppose that T is essentially semi-regular. If (M, N) is a GKD for which $T|N$ is nilpotent and N is finite-dimensional, then the same decomposition satisfies all required conditions for T^n. Conversely, if T^n is essentially semi-regular , from Theorem 1.50 we easily conclude that T is essentially semi-regular. ∎

6. Semi-Fredholm operators

We now introduce some important classes of operators in Fredholm theory. Let X and Y are Banach spaces. In the sequel, for every bounded operator $T \in L(X, Y)$, we shall denote by $\alpha(T)$ the *nullity* of T, defined as $\alpha(T) := \dim \ker T$, whilst the *deficiency* $\beta(T)$ of T is defined $\beta(T) := \operatorname{codim} T(X)$.

Definition 1.52. *Given two Banach spaces X and Y, the set of all* upper semi-Fredholm operators *is defined by*

$$\Phi_+(X, Y) := \{T \in L(X, Y) : \alpha(T) < \infty \quad \textit{and } T(X) \quad \textit{closed}\},$$

whilst the set of all lower semi-Fredholm operators *is defined by*

$$\Phi_-(X, Y) := \{T \in L(X, Y) : \beta(T) < \infty\}.$$

The set of all semi-Fredholm operators *is defined by*

$$\Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y).$$

The class $\Phi(X, Y)$ of all Fredholm operators *is defined by*

$$\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y).$$

We shall set

$$\Phi_+(X) := \Phi_+(X, X) \quad \text{and} \quad \Phi_-(X) := \Phi_-(X, X),$$

whist

$$\Phi(X) := \Phi(X, X) \quad \text{and} \quad \Phi_\pm(X) := \Phi_\pm(X, X).$$

Note that $T \in L(X, Y)$ if and only if $\alpha(T)$ and $\beta(T)$ are both finite.

The following theorem establishes an important characterization of Fredholm operators.

Theorem 1.53. (Atkinson characterization of Fredholm operators) *If $T \in L(X, Y)$ then $T \in \Phi(X, Y)$ if and only there exist $U_1, U_2 \in L(Y, X)$ and finite-dimensional operators $K_1 \in F(X)$, $K_2 \in F(Y)$ such that*

$$U_1 T = I_X - K_1 \quad \textit{and} \quad T U_2 = I_Y - K_2.$$

In particular, $T \in \Phi(X)$ if and only if T is invertible in $L(X)$ modulo the ideal of finite-dimensional operators $F(X)$. ∎

For a proof of this classical result we refer to Proposition 25.2 and Proposition 51.6 of Heuser [**159**] in the case $X = Y$, or to Theorem 2.1 of Schechter [**287**] for the more general case. It should be noted that in the characterization above the ideal $F(X)$ may be replaced by the ideal $K(X)$ of all compact operators, see also Chapter 5.

The *index* of a semi-Fredholm operator $T \in \Phi_\pm(X,Y)$ is defined by

$$\text{ind } T := \alpha(T) - \beta(T).$$

Clearly, ind T is an integer or $\pm\infty$.

Remark 1.54. We shall make frequent use of the following basic and well known properties of the classes of semi-Fredholm operators.

(a) If $T \in \Phi_\pm(X,Y)$ then $T(X)$ is closed, see Corollary 1.15.

(b) Upper and lower semi-Fredholm operators are dual each other,

$$T \in \Phi_+(X,Y) \Leftrightarrow T^\star \in \Phi_-(Y^\star, X^\star),$$

and

$$T \in \Phi_-(X,Y) \Leftrightarrow T^\star \in \Phi_+(Y^\star, X^\star).$$

Moreover,

$$\alpha(T) = \beta(T^\star) \quad \text{and} \quad \beta(T) = \alpha(T^\star),$$

see Caradus, Pfaffenberger and Yood [**76**, §1.3, p.8] or Schechter [**287**, Chap. V, Theorem 4.1].

(c) If $T \in \Phi_+(X,Y)$ and $S \in \Phi_+(Y,Z)$, then $ST \in \Phi_+(X,Z)$. Analogously, if $T \in \Phi_-(X,Y)$ and $S \in \Phi_-(Y,Z)$ then $ST \in \Phi_-(X,Z)$. In particular if $T \in \Phi_+(X)$ (respectively, $T \in \Phi_-(X)$ then $T^n \in \Phi_+(X)$ for every $n \in \mathbb{N}$ (respectively, $T^n \in \Phi_-(X)$), see [**76**, Corollary 1.3.3] and [**287**, Chap. V, Theorem 6.6].

Again, if $T \in L(X,Y)$, $S \in L(Y,Z)$, and $ST \in \Phi(X,Z)$ then $T \in \Phi(X,Y)$ if and only if $S \in \Phi(Y,Z)$, see Theorem 13.1 of Lay and Taylor [**217**]. From this it follows that if $T, S \in L(X)$ and $TS \in \Phi(X)$, then either both T and S belong to $\Phi(X)$ or neither of T, S belongs to $\Phi(X)$. Analogously, if $T \in L(X,Y)$, $S \in L(Y,Z)$, and $ST \in \Phi_+(X,Z)$ (respectively, $ST \in \Phi_+ - (X,Z)$) then $T \in \Phi_+(X,Y)$ (respectively, $S \in \Phi_-(Y,X)$). The sets $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$ are semi-groups in $L(X)$.

(d) Let $T \in \Phi_+(X,Y)$. Then there exists $\varepsilon := \varepsilon(T) > 0$ such that $S \in L(X,Y)$ and $\|S\| < \varepsilon$ implies $T + S \in \Phi_+(X,Y)$. Moreover,

$$\alpha(T+S) \leq \alpha(T) \quad \text{and} \quad \text{ind } (T+S) = \text{ ind } T.$$

Analogously, if $T \in \Phi_-(X,Y)$ then there exists $\varepsilon := \varepsilon(T) > 0$ such that for every $S \in L(X,Y)$ with $\|S\| < \varepsilon$ we have $T + S \in \Phi_-(X,Y)$ and

$$\beta(T+S) \leq \beta(T) \quad \text{and} \quad \text{ind } (T+S) = \text{ ind } T.$$

The proof of this important result may be found in Caradus, Pfaffenberger, and Yood [**76**, p. 61], Heuser [**160**, p. 418]) or Schechter [**287**, Chap. V, Theorem 6.3 and Theorem 6.8]. From this it follows that $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are open subsets of $L(X,Y)$ and the index function

$$\text{ind} : T \in \Phi_\pm(X) \to \text{ ind } T \in \mathbb{Z} \cup \{\pm\infty\}$$

is continuous and therefore constant on the connected components of the open set $\Phi_\pm(X,Y)$. Clearly $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_+(X,Y)$ are open subsets of $L(X,Y)$.

(e) If the perturbation S above is caused by a multiple of the identity we have the *punctured neighbourhood* theorem, see Kato [**183**, Theorem 5.31]: if $T \in \Phi_+(X)$ then there exists $\varepsilon > 0$ such that $\lambda I + T \in \Phi_+(X)$ and $\alpha(\lambda I + T)$ is constant on the punctured neighbourhood $0 < |\lambda| < \varepsilon$. Moreover,

$$\alpha(\lambda I + T) \le \alpha(T) \quad \text{for all } |\lambda| < \varepsilon \;, \tag{17}$$

and

$$\text{ind}\,(\lambda I + T) = \text{ind}\, T \quad \text{for all } |\lambda| < \varepsilon.$$

Analogously, if $T \in \Phi_-(X)$ then there exists $\varepsilon > 0$ such that $\lambda I + T \in \Phi_-(X)$, $\beta(\lambda I + T)$ is constant on the punctured neighbourhood $0 < |\lambda| < \varepsilon$. Moreover,

$$\beta(\lambda I + T) \le \beta(T) \quad \text{for all } |\lambda| < \varepsilon, \tag{18}$$

and

$$\text{ind}\,(\lambda I + T) = \text{ ind}\, T \quad \text{for all } |\lambda| < \varepsilon.$$

(f) If $T \in \Phi_+(X,Y)$ (respectively, $T \in \Phi_-(X,Y)$) and $K \in L(X,Y)$ is a finite-dimensional or a compact operator, then $T + K \in \Phi_+(X,Y)$ (respectively , $T + K \in \Phi_-(X,Y)$) see Heuser [**160**, Satz 82.5]. Moreover ind $(T+K) =$ ind (T) for every $T \in \Phi_\pm(X,Y)$ and $K \in L(X,Y)$ compact.

(g) If X is an infinite-dimensional complex Banach space then $\lambda I - T \notin \Phi(X)$ for some $\lambda \in \mathbb{C}$. This follows from the classical result that the spectrum of an arbitrary element of a complex infinite-dimensional Banach algebra is always non-empty. In fact, by the Atkinson characterization of Fredholm operators, $\lambda I - T \notin \Phi(X)$ if and only if $\widehat{T} := T + K(X)$ is non-invertible in the Calkin algebra $L(X)/K(X)$. An analogous result hold for semi-Fredholm operators on infinite-dimensional Banach spaces: if X is an infinite-dimensional Banach space and $T \in L(X)$ then $\lambda I - T \notin \Phi_+(X)$ (respectively, $\lambda I - T \notin \Phi_-(X)$) for some $\lambda \in \mathbb{C}$.

Definition 1.55. *Let $T \in \Phi_\pm(X)$, X a Banach space. Let $\varepsilon > 0$ as in (17) or (18). If $T \in \Phi_+(X)$ the* jump $j(T)$ *is defined by*

$$j(T) := \alpha(T) - \alpha(\lambda I + T), \quad 0 < |\lambda| < \varepsilon,$$

while, if $T \in \Phi_-(X)$, the jump $j(T)$ is defined by

$$j(T) := \beta(T) - \beta(\lambda I + T), \quad 0 < |\lambda| < \varepsilon.$$

Clearly $j(T) \geq 0$ and the continuity of the index ensures that both definitions of $j(T)$ coincide whenever $T \in \Phi(X)$, so $j(T)$ is unambiguously defined. An immediate consequence of part (b) and part (e) of Remark 1.54 is that if $T \in \Phi_\pm(X)$ then $j(T) = j(T^\star)$.

In the sequel we shall denote by T_∞ the restriction $T|T^\infty(X)$ of T to the invariant subspace $T^\infty(X)$ of a linear space X. Let $\widehat{x} := x + T^\infty(X)$ be the coset corresponding to x in the quotient space $\widehat{X} := X/T^\infty(X)$. If Y is a subset of X, we set $\widehat{Y} := \{\widehat{y} : y \in Y\}$. Obviously $\widehat{Y}$ coincides with the quotient $[Y + T^\infty(X)]/T^\infty(X)$.

Let $\widehat{T_\infty} : \widehat{X} \to \widehat{X}$ denote the quotient operator defined by

$$\widehat{T_\infty}\, \widehat{x} := \widehat{Tx}, \qquad x \in X.$$

It is easily seen that $\widehat{T_\infty}$ is well defined. In the next lemma we collect some elementary properties of $\widehat{T_\infty}$.

Lemma 1.56. *Let T be a linear operator on a vector space X, and assume that $\alpha(T) < \infty$ or $\beta(T) < \infty$. Then:*

(i) $\mathcal{N}^\infty(\widehat{T_\infty}) = \widehat{\mathcal{N}^\infty(T)}$;

(ii) $\widehat{T_\infty}^{\,\infty}(\widehat{X}) = \{\widehat{0}\}$.

Proof (i) From Theorem 1.10 we know that $T(T^\infty(X)) = T^\infty(X)$. Let $\widehat{x} \in \ker \widehat{T_\infty}$. Then $Tx \in T^\infty(X) = T(T^\infty(X))$. Consider an element $u \in T^\infty(X)$ such that $Tx = Tu$. Clearly $x - u \in \ker T$, so $x = u + v$ for some $v \in \ker T$, $x \in \ker T + T^\infty(X)$ and hence $\widehat{x} \in \widehat{\ker T} + \widehat{T^\infty(X)} = \widehat{\ker T}$. This shows the inclusion $\ker \widehat{T_\infty} \subseteq \widehat{\ker T}$. The opposite inclusion is obvious, so $\ker \widehat{T_\infty} = \widehat{\ker T}$. Similarly $\ker (\widehat{T_\infty})^n = \widehat{\ker T^n}$ for every $n \in \mathbb{N}$, and from this the equality (i) easily follows.

(ii) It is easy to check that $\widehat{T_\infty}^{\,n}(\widehat{X}) = \widehat{T^n(X)}$ for all $n \in \mathbb{N}$, and from we obtain that $\widehat{T_\infty}^{\,\infty}(\widehat{X}) = \widehat{T^\infty(X)} = \{\widehat{0}\}$. ∎

Note that if $T \in \Phi_\pm(X)$ then $T^\infty(X)$ is closed since $T^n \in \Phi_\pm(X)$ for every $n \in \mathbb{N}$, by part (c) of Remark 1.54.
The following result reduces properties of semi-Fredholm operators to equivalent properties of Fredholm operators.

Lemma 1.57. *Let $T \in \Phi_+(X)$, X a Banach space. Then:*

(i) T_∞ *is a Fredholm operator;*

(ii) $\widehat{T_\infty}$ *is an upper semi-Fredholm operator.*

Proof (i) Since $\alpha(T) < \infty$, from Theorem 1.10 we have $\beta(T_\infty) = 0$, and from the inclusion $\ker T_\infty \subseteq \ker T$ we conclude that $\ker T$ is finite-dimensional, hence T_∞ is a Fredholm operator.

(ii) From Lemma 1.56 we have $\ker \widehat{T_\infty} = \widehat{\ker T}$ and hence $\alpha(\widehat{T_\infty}) < \infty$. Moreover, it is easy to see that $\widehat{T_\infty}(\widehat{X}) = \widehat{T(X)}$ is a closed subspace of $\widehat{X}$, thus $\widehat{T_\infty} \in \Phi_+(\widehat{X})$. ∎

The next result gives a characterization of the semi-Fredholm operators which are semi-regular.

Theorem 1.58. *Let $T \in \Phi_\pm(X)$, X a Banach space. Then $j(T) = 0$ if and only if is semi-regular.*

Proof Since $T(X)$ is closed it suffices to show the equivalence

$$j(T) = 0 \Leftrightarrow \mathcal{N}^\infty(T) \subseteq T^\infty(X)\ .$$

Assume first $T \in \Phi_+(X)$ and $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$. Observe first that

$$\alpha(\lambda I + T) = \alpha(\lambda I + T_\infty) \quad \text{for all } \lambda \in \mathbb{C}.$$

For $\lambda = 0$ this is clear, since $\ker T \subseteq \mathcal{N}^\infty(T) \subseteq T^\infty(X)$ implies that $\ker T = \ker T_\infty$. For $\lambda \neq 0$ we have, by part (ii) of Theorem 1.3,

$$\ker T \subseteq \mathcal{N}^\infty(\lambda I + T) \subseteq T^\infty(X),$$

so that $\ker(\lambda I + T = \ker(\lambda I + T_\infty)$.

Now, from Theorem 1.10 we know that $\beta(T_\infty) = 0$ and hence there exists $\varepsilon > 0$ such that $\beta(\lambda I + T_\infty) = 0$ for all $|\lambda| < \varepsilon$, see Lemma 1.30. From Lemma 1.57 we know that T_∞ is Fredholm, so we can assume ε such that

$$\text{ind } (\lambda I + T_\infty) = \text{ ind } (T_\infty) \quad \text{for all} |\lambda| < \varepsilon.$$

Therefore $\alpha(\lambda I + T_\infty) = \alpha(T_\infty)$ for all $|\lambda| < \varepsilon$ and hence $\alpha(\lambda I + T) = \alpha(T)$ for all $|\lambda| < \varepsilon$, so that $j(T) = 0$.

Conversely, suppose that $T \in \Phi_+(X)$ and $j(T) = 0$, namely there exists $\varepsilon > 0$ such $\alpha(\lambda I + T)$ is constant for $|\lambda| < \varepsilon$. Then

$$\alpha(T_\infty) \leq \alpha(T) = \alpha(\lambda I + T) = \alpha(\lambda I + T_\infty) \quad \text{for all } 0 < |\lambda| < \varepsilon.$$

But T_∞ is Fredholm by Lemma 1.57, and hence, see Remark 1.54, part (e), we can choose $\varepsilon > 0$ such that $\alpha(\lambda I + T_\infty) \leq \alpha(T_\infty)$ for all $|\lambda| < \varepsilon$. This shows that $\alpha(T_\infty) = \alpha(T)$ and consequently, $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$.

Consider now the case that $T \in \Phi_-(X)$ and $j(T) = 0$. Then $T^\star \in \Phi_+(X^\star)$ and $j(T) = j(T^\star) = 0$. From the first part of the proof we deduce that $\mathcal{N}^\infty(T^\star) \subseteq T^{\star\infty}(X^\star)$. From Corollary 1.6 it follows that $\ker T^{\star n} \subseteq T^\star(X^\star)$ for all $n \in \mathbb{N}$, or equivalently $T^n(X)^\perp \subseteq \ker T^\perp$ for all $n \in \mathbb{N}$. Since all these subspaces are closed then $T^n(X) \supseteq \ker T$ for all $n \in \mathbb{N}$, so by Corollary 1.6 we conclude that $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$. ∎

Now we are interested in the case that $\mathcal{N}^\infty(T)$ is not contained in $T^\infty(X)$. In this case, from Theorem 1.5 there exists a smallest integer $n \in \mathbb{N}$ such that

$$\ker T^{n-1} \subseteq T(X) \quad \text{but} \quad \ker T^n \not\subseteq T(X).$$

Hence there is an element $y \in X$ and a smallest $n \in \mathbb{N}$ such that

$$y \in \ker T^n \quad \text{but } y \notin T(X). \tag{19}$$

In the next result we examine the elements $T^k y$, $k = 0, 1, \ldots, n-1$.

Lemma 1.59. *Suppose that a linear operator T on a vector space X verifies $\mathcal{N}^\infty(T) \not\subseteq T^\infty(X)$. Let $y \in X$ and $n \in \mathbb{N}$ as in (19). Then*

$$T^k y \in \ker T^{n-k} \setminus T^{k+1}(X) \quad \textit{for all } k = 0, 1, \ldots, n-1.$$

Furthermore, the elements $y, Ty, \ldots, T^{n-1}y$ are linearly independent modulo the subspace $T^n(X)$.

Proof The case $n = 1$ is obvious. Suppose that $n \geq 2$ and let $y \in X$, and that $n \in \mathbb{N}$ verifies the conditions (19). Then $Ty \in T(\ker T^n) \subseteq \ker T^{n-1}$. Suppose that $Ty \in T^2(X)$. Let $v \in X$ such that $Ty = T(Tv)$. Obviously, if $w := Tv \in T(X)$ then $w - y \in \ker T$, so there exists $u \in \ker T$ such that $w = y + u$. But we also have $u \in \ker T \subseteq \ker T^{n-1} \subseteq T(X)$ and this implies $y \in T(X)$, which is false, since by assumption $\ker T^n \setminus T(X)$. Therefore $Ty \in \ker T^{n-1} \setminus T^2(X)$ and the process can be continued for the elements $T^2 y, \ldots, T^{n-1}y$.

In order to show the linear independence modulo $T^k(X)$, assume that there exist $\lambda_k \in \mathbb{C}$, $k = 0, 1, \ldots, n-1$, such that $\sum_{k=0}^{1} \lambda_k T^k y \in T^n(X)$. Applying T^{n-1} to this sum we obtain

$$\lambda_0 T^{n-1} y \in T^{2n-1}(X) \subseteq T^n(X),$$

which gives $\lambda_0 = 0$. A similar argument shows that $\lambda_k = 0$ for every $k = 1, \ldots, n-1$, so the proof is complete. ■

Lemma 1.60. *Assume that $T \in L(X)$, X a Banach space, verifies $\mathcal{N}^\infty(T) \not\subseteq T^\infty(X)$. Let $y \in X$ and $n \in \mathbb{N}$ as in (19). Then there exists $f \in \ker T^{\star n}$ such that*

$$T^{\star n} f(T^{n-j-1}y) = \delta_{ij} \quad \textit{for every } 0 \leq i, j \leq n-1.$$

In particular, $T^{\star n} f = 0$ and

$$T^{\star k} f \in \ker T^{\star n-k} \setminus T^{\star k+1}(X) \quad \textit{for all } k = 0, 1, \ldots, n-1.$$

Proof Since $y, Ty, \ldots, T^{n-1}y$ are linearly independent modulo $T^n(X)$, the Hahn- Banach theorem ensures that there exists $f \in T^n(X)^\perp = \ker T^{\star n}$ such that

$$f(T^{n-1}y) = 1 \quad \text{and} \quad f(T^j y) = 0 \quad \text{for all } 0 \leq j \leq n-2.$$

Clearly, $T^\star f(T^{n-1}y) = T^{\star n} f(y) = 0$ and

$$T^\star f(T^j y) = f(T^{j+1}y) = 0 \quad \text{for all } 0 \leq j \leq n-2,$$

so

$$T^\star f(T^{n-2}y) = 1 \quad \text{and} \quad T^\star f(T^j y) = 0 \quad \text{for all } 0 \leq j \leq n-3.$$

Continuing the process proves the lemma. The last assertion is obvious. ■

Lemma 1.61. *Let $T \in L(X)$, X a Banach space, and suppose that $\mathcal{N}^\infty(T) \nsubseteq T^\infty(X)$. Let y be chosen as in (19). Then*

$$P := \sum_{j=0}^{n-1} T^{\star j} f \otimes T^{n-j-1} y$$

is a bounded projection which commutes with T. Furthermore, the range of P is the subspace Y spanned by the elements $y, Ty, \cdots T^{n-1}y$, the restriction $T|Y$ is nilpotent and $j(T|Y) = 1$.

Proof P is idempotent by Lemma 1.60. Moreover,

$$TP = \sum_{j=0}^{n-1} T^{\star j} f \otimes T^{n-1} y = \sum_{j=1}^{n-1} T^{\star j} f \otimes T^{n-1} y,$$

and

$$PT = \sum_{j=0}^{n-1} T^{\star j+1} f \otimes T^{n-j-1} y = \sum_{j=0}^{n-2} T^{\star j+1} f \otimes T^{n-j-1} y,$$

hence $PT = TP$. Clearly, $T|Y$ is nilpotent, $\alpha(T|Y) = 1$, and $j(T|Y) = 1$, since Y is finite-dimensional. ∎

Theorem 1.62. *If $T \in \Phi_\pm(X)$ then T is essentially semi-regular .*

Proof Let $T \in \Phi_\pm(X)$. If T is semi-regular then the pair (M, N), with $M = X$ and $N = 0$, is a Kato decomposition which verifies the desired properties. If T is not semi-regular then $j(T) > 0$, by Theorem 1.58 and hence $\mathcal{N}^\infty(T) \subseteq T^\infty(X)$. Let P be the non-zero finite-rank projection of Lemma 1.61. P commutes with T. The restriction $T|\ker P$ is semi-Fredholm and $j(T|\ker P) = j(T) - 1$. Continuing this process a finite number of times reduces the jump of the residual operator to zero. ∎

Remark 1.63. We have already noted that if $T \in \Phi_\pm(X)$ then $\lambda I - T$ is still semi-Fredholm near 0. By Theorem 1.62 every semi-Fredholm operator is of Kato type and therefore, see Theorem 1.44, there exists a punctured open disc $\mathbb{D}_\varepsilon$ centered at 0 for which $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}_\varepsilon$. From Theorem 1.58 we then conclude that if a semi-Fredholm operator has jump $j(T) > 0$ then there is an open disc $\mathbb{D}_\varepsilon$ centered at 0 for which $j(T) = 0$ for all $\lambda \in \mathbb{D}_\varepsilon\{0\}$.

In the same vein the following result establishes that if T is essentially semi-regular, but not semi-Fredholm, then $\lambda I - T$ is essentially semi-regular and not semi-Fredholm in an open punctured neighbourhood of 0.

Theorem 1.64. *Let $T \in L(X)$, X a Banach space, be of Kato type of order d. Then there exists an open disc $\mathbb{D}(0, \varepsilon)$ centered at 0 such that the following properties hold:*

(i) *The dimension of* $\ker(\lambda I - T)$ *is constant for* λ *ranging through* $\mathbb{D}(0,\varepsilon) \setminus \{0\}$. *Precisely,*

$$\dim \ker(\lambda I - T) = \dim \ker T \cap T^d(X) \quad \textit{for all } \lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\};$$

(ii) *The codimension of* $(\lambda I - T)(X)$ *is constant for* λ *ranging through* $\mathbb{D}(0,\varepsilon) \setminus \{0\}$. *Precisely,*

$$\operatorname{codim}(\lambda I - T)(X) = \operatorname{codim}(T(X) + \ker T^d) \textit{ for all } \lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}.$$

Moreover, if T *is essentially semi-regular and* $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$ *then*

$$\lambda I - T \in \Phi_+(X) \Leftrightarrow T \in \Phi_+(X), \tag{20}$$

and

$$\lambda I - T \in \Phi_-(X) \Leftrightarrow T \in \Phi_-(X). \tag{21}$$

In particular,

$$\lambda I - T \in \Phi(X) \Leftrightarrow T \in \Phi(X). \tag{22}$$

Proof (i) Take ε as in the proof of Theorem 1.44. Assume that $\lambda I - T \in \Phi_+(X)$ for $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$. If $T_0 := T|T^\infty(X)$ then, as observed in the proof of Theorem 1.44, T_0 and $\lambda I - T_0$ are both onto and hence upper semi-Fredholm for all $|\lambda| < \varepsilon \leq \gamma(T_0)$. Moreover, $\ker(\lambda I - T) \subseteq \mathcal{N}^\infty(\lambda I - T) \subseteq T^\infty(X)$ for all $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$, by Theorem 1.3. From the continuity of the index and part (ii) of Theorem 1.42 we then infer that

$$\begin{aligned} \alpha(\lambda I - T) &= \operatorname{ind}(\lambda I - T_0) = \operatorname{ind}(T_0) \\ &= \alpha(T_0) = \dim\,[\ker T \cap M] \\ &= \dim\,[\ker T \cap T^d(X)] \end{aligned}$$

for all $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$.

(ii) From the proof of Theorem 1.44 we know that $(\lambda I - T)(X)$ is closed, and hence $(\lambda I - T)(X) = \ker(\lambda I - T^\star)^\perp$, for all $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$. Therefore by part (i) we have

$$\begin{aligned} \beta(\lambda I - T) = \alpha(\lambda I - T^\star) &= \dim\,[T^{\star d}(X^\star) \cap \ker T^\star] \\ &= \operatorname{codim}\,[T^{\star d}(X^\star) \cap \ker T^\star]^\perp \\ &= \operatorname{codim}[\,T^{\star d}(X^\star)^\perp + (\ker T^\star)^\perp] \\ &= \operatorname{codim}[\ker T^d + T(X)], \end{aligned}$$

for all $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$.

Now, assume that T is essentially semi-regular. If $\lambda I - T \in \Phi_+(X)$ for $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$, then $\dim \ker(\lambda I - T) = \dim[\ker T \cap T^d(X)] < \infty$. From Theorem 1.42 we obtain

$$\ker T = \ker T|M \oplus \ker T|N = [\ker T \cap T^d(X)] \oplus \ker T|N,$$

and this implies, since by assumption $\ker T|N$ is finite-dimensional, that $\ker T$ is finite-dimensional. Moreover, $T(X) = T(M) \oplus T(N)$ is closed since $T|M$ is semi-regular and $T(N)$ is finite-dimensional. Hence $T \in \Phi_+(X)$.

The opposite implication is a consequence of the fact that $\Phi_+(X)$ is open in $L(X)$, so (20) is proved.

To conclude the proof consider the case $\lambda I - T \in \Phi_-(X)$ for some $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$. From part (ii) we know that

$$T(X) + \ker T^d(X) = T(M) \oplus N$$

is finitely-codimensional and from this, and from N being finite-dimensional, we conclude that $T(M)$ is finitely-codimensional. This also shows that $T(X)$ has finite-codimension, so that $T \in \Phi_-(X)$. The opposite implication is clear, since $\Phi_-(X)$ is open, so also the equivalence (21) is proved.

The equivalence (22) is an obvious consequence of (20) and (21). ■

The two classes of semi-Fredholm operator lead to the definition of the *upper semi-Fredholm spectrum* of a bounded operator T on a Banach space X, defined by

$$\sigma_{\mathrm{uf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\},$$

and the *lower semi-Fredholm spectrum* of T defined by

$$\sigma_{\mathrm{lf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \ \notin \Phi_-(X)\}.$$

The *semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{sf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \ \notin \Phi_\pm(X)\},$$

while the *Fredholm spectrum* is defined by

$$\sigma_{\mathrm{f}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \ \notin \Phi(X)\},$$

Clearly,

$$\sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{uf}}(T) \cap \sigma_{\mathrm{lf}}(T) \quad \text{and} \quad \sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{uf}}(T) \cup \sigma_{\mathrm{lf}}(T).$$

Furthermore, from part (b) of Remark 1.54 we obtain that

$$\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{lf}}(T^\star) \quad \text{and} \quad \sigma_{\mathrm{lf}}(T) = \sigma_{\mathrm{uf}}(T^\star).$$

Again, from Theorem 1.62 we have

$$\sigma_{\mathrm{es}}(T) \subseteq \sigma_{\mathrm{sf}}(T) \subseteq \sigma_{\mathrm{f}}(T).$$

The spectrum $\sigma_{\mathrm{f}}(T)$ in the literature is often called the *essential spectrum* of T. The two spectra $\sigma_{\mathrm{uf}}(T)$ and $\sigma_{\mathrm{lf}}(T)$ are also known as the *left essential spectrum* and the *right essential spectrum*. These denominations are originated from the property that in a Hilbert space H, $\lambda I - T \in \Phi_+(H)$ exactly when $\lambda I - T$ has a left inverse in $L(H)$ modulo the ideal $K(H)$ of all compact operators, and, symmetrically, $\lambda I - T \in \Phi_-(H)$ when $\lambda I - T$ has a right inverse in $L(H)$ modulo $K(H)$.

It is easy to find an example of operator for which $\sigma_{\mathrm{uf}}(T) \neq \sigma_{\mathrm{lf}}(T)$. Let T be defined on $\ell_2(\mathbb{N})$ by

$$T(x) := (x_1, 0, x_2, 0, x_3, 0, \dots) \quad \text{for all } x = (x_n) \in \ell_2(\mathbb{N}).$$

Obviously, T is injective with closed range of infinite-codimension, so that $0 \in \sigma_{\rm lf}(T)$, but $0 \notin \sigma_{\rm uf}(T)$.

Note that from Remark 1.54, part (e) the resolvents

$$\rho_{\rm sf}(T) := \mathbb{C} \setminus \sigma_{\rm sf}(T),$$

and

$$\rho_{\rm uf}(T) := \mathbb{C} \setminus \sigma_{\rm uf}(T), \qquad \rho_{\rm lf}(T) := \mathbb{C} \setminus \sigma_{\rm lf}(T),$$

are open subsets of $\mathbb{C}$ and hence $\sigma_{\rm uf}(T)$, $\sigma_{\rm lf}(T)$ and $\sigma_{\rm sf}(T)$ are compact subset of $\mathbb{C}$. Moreover, as observed in part (g) of Remark 1.54, if X is infinite-dimensional then these spectra are non-empty.

Theorem 1.65. *For a bounded operator T on a Banach space X the following properties hold:*

(i) *If $\lambda_0 \in \partial\sigma_{\rm f}(T)$ is non-isolated point of $\sigma_{\rm f}(T)$ then $\lambda_0 \in \sigma_{\rm k}(T)$;*

(ii) *$\partial\sigma_{\rm f}(T) \subseteq \sigma_{\rm es}(T)$. Consequently, $\sigma_{\rm es}(T)$ is a non-empty compact subset of $\mathbb{C}$;*

(iii) *The set $\sigma_{\rm f}(T) \setminus \sigma_{\rm se}(T)$ consists of at most countably many isolated points.*

Moreover, similar statements hold if, instead of boundary points of $\sigma_{\rm f}(T)$, we consider boundary points of $\sigma_{\rm uf}(T)$, $\sigma_{\rm lf}(T)$ and $\sigma_{\rm sf}(T)$.

Proof (i) Let $\lambda_0 \in \partial\sigma_{\rm f}(T)$ be a non-isolated point of $\sigma_{\rm f}(T)$. Assume that $\lambda_0 I - T$ is of Kato type. Then by Theorem 1.64 there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ centered at λ_0 such that the dimension and the codimension of $\lambda I - T$ are constant as λ ranges throughout $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{0\}$. But $\lambda_0 \in \partial\sigma_{\rm f}(T)$, so that $\lambda I - T$ is Fredholm for some $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$ and consequently $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{0\}$. This contradicts our assumption that λ_0 is non-isolated in $\sigma_{\rm f}(T)$.

(ii) Suppose that $\lambda_0 \in \partial\sigma_{\rm f}(T)$ and $\lambda_0 \notin \sigma_{\rm es}(T)$. Then $\lambda_0 I - T$ is essentially semi-regular so that $(\lambda_0 I - T)(X)$ is closed and there is a decomposition $X = M \oplus N$, M and N closed T-invariant subspaces, N finite-dimensional, such that $(\lambda_0 I - T)|M$ is semi-regular and $(\lambda_0 I - T)|N$ is nilpotent. Let (λ_n) be a sequence which converges at λ_0 such that $\lambda_n I - T \in \Phi(X)$. Clearly,

$$\alpha((\lambda_n I - T)|M)) \leq \alpha(\lambda_n I - T) < \infty.$$

From part (iii) of Theorem 1.38, since $(\lambda_0 I - T)|M$ is semi-regular we know that $\ker((\lambda_n I - T)|M)$ converges in the gap metric to $\ker((\lambda_0 I - T)|M)$. Hence, see the implication (8) of Remark 1.32, $\dim \ker((\lambda_0 I - T)|M) < \infty$.

On the other hand, since N is finite-dimensional, $\ker(\lambda_0 I - T)|N$ is also finite-dimensional, so

$$\alpha(\lambda_0 I - T) = \alpha((\lambda_0 I - T)|M) + \alpha((\lambda_0 I - T)|N) < \infty.$$

A similar argument shows that $\beta(\lambda_0 I - T) < \infty$, therefore $\lambda_0 I - T \in \Phi(X)$ and this is a contradiction, since by assumption $\lambda_0 \in \sigma_{\rm f}(T)$.

(iii) This is clear, again by Theorem 1.64.

The last assertion is proved in a similar way. ∎

Note that if T is semi-Fredholm operator and $\gamma(T)$ denotes the minimal modulus then the limit $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ does exist and is equal to semi-Fredholm radius of T, i.e.:

$$\begin{aligned}\lim_{n\to\infty}\gamma(T^n)^{1/n} &= \text{dist }\{0,\sigma_{\rm sf}(T)\}\\ &= \sup\{r\in\mathbb{C}:\lambda I-T\in\Phi_\pm(X)\ \text{ for all}|\lambda|<r\}.\end{aligned}$$

This classical result is owed to Förster and Kaashoek [**117**], see also Zemánek [**332**]. We do not present it here, but just mention that similar formulas hold for semi-regular operators and essentially semi-regular operators. Precisely, if T is semi-regular then limit $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ does exist and

$$\lim_{n\to\infty}\gamma(T^n)^{1/n} = \text{ dist }\{0,\sigma_{\rm es}(T)\}, \tag{23}$$

whilst, if T is essentially semi-regular then

$$\begin{aligned}\lim_{n\to\infty}\gamma(T^n)^{1/n} &= \text{dist }\{0,\sigma_{\rm es}(T)\setminus\{0\}\}\\ &= \sup\{r\in\mathbb{C}:\lambda I-T \text{ is semi-regular for all } 0<|\lambda|<r\}.\end{aligned}$$

Formula (23) has been proved by Apostol [**48**] and Mbekhta [**227**] for Hilbert space operators. The formulas for semi-regular operators and essentially semi-regular operators on Banach space operators are due to Kordula and Müller [**190**].

7. Quasi-nilpotent part of an operator

Another important invariant subspace for a bounded operator $T\in L(X)$, X a Banach space, is defined as follows :

Definition 1.66. *Let $T\in L(X)$, X a Banach space. The* quasi-nilpotent part *of T is defined to be the set*

$$H_0(T) := \{x\in X:\lim_{n\to\infty}\|T^nx\|^{1/n}\}=0.$$

As usual, $T\in L(X)$ is said to be quasi-nilpotent *if its* spectral radius

$$r(T) := \inf_{n\in\mathbb{N}}\|T^n\|^{1/n} = \lim_{n\to\infty}\|T^n\|^{1/n}$$

is zero.

Clearly $H_0(T)$ is a linear subspace of X, generally not closed. In the following theorem we collect some elementary properties of $H_0(T)$.

Lemma 1.67. *For every $T\in L(X)$, X a Banach space, we have:*

(i) ker $(T^m)\subseteq\mathcal{N}^\infty(T)\subseteq H_0(T)$ *for every $m\in\mathbb{N}$;*

(ii) $x\in H_0(T)\Leftrightarrow Tx\in H_0(T)$*;*

(iii) ker $(\lambda I-T)\cap H_0(T)=\{0\}$ *for every $\lambda\neq 0$.*

Proof (i) If $T^m x = 0$ then $T^n x = 0$ for every $n \geq m$.

(ii) If $x_0 \in H_0(T)$ from the inequality $\|T^n Tx\| \leq \|T\| \|T^n x\|$ it easily follows that $Tx \in H_0(T)$. Conversely, if $Tx \in H_0(T)$ from

$$\|T^{n-1}Tx\|^{1/n-1} = (\|T^n x\|^{1/n})^{n/n-1}$$

we conclude that $x \in H_0(T)$.

(iii) If $x \neq 0$ is an element of ker $(\lambda I - T)$ then $T^n x = \lambda^n x$, so

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = \lim_{n\to\infty} |\lambda| \|x\|^{1/n} = |\lambda|$$

and therefore $x \notin H_0(T)$. ∎

Theorem 1.68. *Let X be a Banach space. Then $T \in L(X)$ is quasi-nilpotent if and only if $H_0(T) = X$.*

Proof If T is quasi-nilpotent then $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, so that from $\|T^n x\| \leq \|T^n\| \|x\|$ we obtain that $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$ for every $x \in X$. Conversely, assume that $H_0(T) = X$. By the n-th root test the series

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{|\lambda|^{n+1}},$$

converges for each $x \in X$ and $\lambda \neq 0$. Define

$$y := \sum_{n=0}^{\infty} \frac{T^n x}{\lambda^{n+1}}.$$

It is easy to verify that $(\lambda I - T)y = x$, thus $(\lambda I - T)$ is surjective for all $\lambda \neq 0$. On the other hand, for every $\lambda \neq 0$ we have that

$$\{0\} = \ker(\lambda I - T) \cap H_0(T) = \ker(\lambda I - T) \cap X = \ker(\lambda I - T),$$

which shows that $\lambda I - T$ is invertible and therefore $\sigma(T) = \{0\}$.

∎

The next result describes the quasi-nilpotent part of an operator T which admits a GKD.

Corollary 1.69. *Assume that $T \in L(X)$, X a Banach space, admits a GKD (M, N). Then*

$$H_0(T) = H_0(T|M) \oplus H_0(T|N) = H_0(T|M) \oplus N.$$

Proof From Theorem 1.68 we know that $N = H_0(T|N)$. The inclusion $H_0(T) \supseteq H_0(T|M) + H_0(T|N)$ is clear. In order to show the opposite inclusion, consider an arbitrary element $x \in H_0(T)$ and let $x = u + v$, with $u \in M$ and $v \in N$. Evidently $N = H_0(T|N) \subseteq H_0(T)$. Consequently $u = x - v \in H_0(T) \cap M = H_0(T|M)$ and hence $H_0(T) \subseteq H_0(T|M) + H_0(T|N)$. Clearly the sum $H_0(T|M) + N$ is direct since $M \cap N = \{0\}$. ∎

The next result shows that for an essentially semi-regular operator T, the closure of the quasi-nilpotent part of T and of the hyper-kernel of T coincide.

Theorem 1.70. *For every bounded operator $T \in L(X)$, X a Banach space, we have:*

(i) *$H_0(T) \subseteq^{\perp} K(T^\star)$ and $K(T) \subseteq^{\perp} H_0(T^\star)$;*

(ii) *If T is essentially semi-regular, then*

$$\overline{H_0(T)} = \overline{\mathcal{N}^\infty(T)} =^{\perp} K(T^\star) \quad \textit{and } K(T) =^{\perp} H_0(T^\star). \tag{24}$$

In particular, these equalities hold for every $T \in \Phi_\pm(X)$, or for every semi-regular operator.

(iii) *If T is semi-regular then $\overline{H_0(T)} \subseteq K(T)$.*

Proof (i) Consider an element $u \in H_0(T)$ and $f \in K(T^\star)$. From the definition of $K(T^\star)$ we know that there exists $\delta > 0$ and a sequence (g_n), $n \in \mathbb{Z}_+$ of $X^\star$ such that

$$g_0 = f, \quad T^\star g_{n+1} = g_n \quad \text{and} \quad \|g_n\| \le \delta^n \|f\|$$

for every $n \in \mathbb{Z}_+$. These equalities entail that $f = (T^\star)^n g_n$ for every $n \in \mathbb{Z}_+$, so that

$$f(u) = (T^\star)^n g_n(u) = g_n(T^n u) \quad \text{for every} \quad n \in \mathbb{Z}_+.$$

From that it follows that $|f(u)| \le \|T^n u\| \|g_n\|$ for every $n \in \mathbb{Z}_+$ and therefore

$$|f(u)| \le \delta^n \|f\| \|T^n u\| \quad \text{for every } n \in \mathbb{Z}_+. \tag{25}$$

From $u \in H_0(T)$ we now obtain that $\lim_{n\to\infty} \|T^n u\|^{1/n} = 0$ and hence by taking the n-th root in (25) we conclude that $f(u) = 0$. Therefore $H_0(T) \subseteq^{\perp} K(T^\star)$.

The inclusion $K(T) \subseteq^{\perp} H_0(T^\star)$ is proved in a similar way.

(ii) Assume that T is essentially semi-regular and hence $T^\star$ essentially semi-regular, by Corollary 3.11. Every essentially semi-regular operator has closed range, thus by Corollary 1.51 $T^{\star n}(X^\star)$ is closed for all $n \in \mathbb{N}$. From the first part we also know that

$$\overline{\mathcal{N}^\infty(T)} \subseteq \overline{H_0(T)} \subseteq^{\perp} \overline{K(T^\star)} =^{\perp} K(T^\star),$$

since $^{\perp}K(T^\star)$ is closed.

To show the first two equalities of (24) we need only to show the inclusion $^{\perp}K(T^\star) \subseteq \overline{\mathcal{N}^\infty(T)}$. For every $T \in L(X)$ and every $n \in \mathbb{N}$ we have $\ker T^n \subseteq \mathcal{N}^\infty(T)$, and hence

$$\mathcal{N}^\infty(T)^\perp \subseteq \ker T^{n\perp} = T^{\star n}(X^\star)$$

because the last subspaces are closed for all $n \in \mathbb{N}$.

From this we easily obtain that $\overline{\mathcal{N}^\infty(T)}^\perp \subseteq T^{\star\infty}(X^\star) = K(T^\star)$, where the last equality follows from Theorem 1.42. Consequently $^{\perp}K(T^\star) \subseteq \overline{\mathcal{N}^\infty(T)}$, thus the equalities (24) are proved.

The equality $K(T) =^{\perp} H_0(T^\star)$ is proved in a similar way.

(iii) The semi-regularity of T entails that $\mathcal{N}^\infty(T) \subseteq T^\infty(X) = K(T)$, where the last equality follows from Theorem 1.24. Consequently from part (ii) it follows that

$$\overline{H_0(T)} = \overline{\mathcal{N}^\infty}(T) \subseteq \overline{K(T)} = K(T),$$

since $K(T)$ is closed, by Theorem 1.24. ∎

Corollary 1.71. *Let $T \in L(X)$, X a Banach space, be semi-regular. Then $T(H_0(T)) = H_0(T)$.*

Proof Clearly by (ii) of Lemma 1.67 it suffices to show the inclusion$H_0(T) \subseteq T(H_0(T))$. Let $x \in H_0(T)$. From part (iii) of Theorem 1.70 then $x \in K(T) = T(K(T))$, so $x = Ty$ for some $y \in X$ and from part (ii) of Lemma 1.67 we conclude that $y \in H_0(T)$. Hence $H_0(T) \subseteq T(H_0(T))$. ∎

Theorem 1.72. *Let $T \in L(X)$, X a Banach space, and let $\Omega \subset \mathbb{C}$ be a connected component of $\rho_{\rm se}(T)$. If $\lambda_0 \in \Omega$ then*

$$\overline{H_0(\lambda I - T)} = \overline{H_0(\lambda_0 I - T)} \quad \textit{for all } \lambda \in \Omega,$$

i.e., the subspaces $\overline{H_0(\lambda I - T)}$ are constant as λ ranges through on Ω.

Proof By Theorem 1.19 we know that $\rho_{\rm se}(T) = \rho_{\rm se}(T^\star)$. Further, Theorem 1.36 shows that $K(\lambda I^\star - T^\star) = K(\lambda_0 I^\star - T^\star)$ for all $\lambda \in \Omega$. From Theorem 1.70 we then conclude that

$$\overline{H_0(\lambda I - T)} =^\perp K(\lambda I^\star - T^\star) =^\perp K(\lambda_0 I^\star - T^\star) = \overline{H_0(\lambda_0 I - T)},$$

for all $\lambda \in \Omega$. ∎

In the sequel, given a subset $M \subseteq X$, by $\operatorname{span} M$ we shall denote the linear subspace generated by M.

Theorem 1.73. *Let $T \in L(X)$, X a Banach space, and let $\Omega_0 \subseteq \mathbb{C}$ be the connected component of $\rho_{\rm se}(T)$ that contains λ_0. Then*

$$\overline{H_0(\lambda_0 I - T)} = \overline{\operatorname{span}\{x \in \ker(\lambda I - T) : \lambda \in \Omega_0\}}.$$

Proof We can assume $\lambda_0 = 0$. The inclusion $\ker(\lambda I - T) \subseteq H_0(\lambda I - T)$ is obvious for every $\lambda \in \mathbb{C}$, so that from Theorem 1.72 we infer that

$$\overline{\operatorname{span}\{x \in \ker(\lambda I - T) : \lambda \in \Omega_0\}} \subseteq \overline{H_0(\lambda I - T)} = \overline{H_0(T)}.$$

Conversely, let $f \in [\,\overline{\operatorname{span}\{\ker(\lambda I - T) : \lambda \in \Omega_0\}}]^\perp$, in particular assume that $f \in \ker(\lambda_n I - T)^\perp$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of distinct points which converges to 0. We have $\ker(\lambda_n I - T)^\perp = (\lambda_n I - T^\star)(X^\star)$ for every $n \in \mathbb{N}$, since the last sets are closed by semi-regularity, see Theorem 1.19. Therefore, from Theorem 1.39 and Theorem 1.70, we obtain that

$$f \in \bigcap_{n=1}^\infty (\lambda_n I - T^\star)(X^\star) = K(T^\star) = \overline{H_0(T)}^\perp.$$

Hence

$$[\,\overline{\text{span}\,\{\ker(\lambda I-T):\lambda\in\Omega_o\}}]^\perp \subseteq H_0(T)^\perp = \overline{H_0(T)}^\perp,$$

and consequently

$$\overline{H_0(T)} \subseteq \overline{\text{ span }\{\in \ker(\lambda I-T):\lambda\in\Omega_0\}},$$

which completes the proof. ∎

Theorem 1.74. *Let $T\in L(X)$ be of Kato type. Then:*
(i) $\mathcal{N}^\infty(T)+T^\infty(X)=H_0(T)+K(T)$;
(ii) $\overline{\mathcal{N}^\infty(T)}\cap T^\infty(X)=\overline{H_0(T)}\cap K(T)$.

Proof (i) Let (M,N) be a GKD for T such that $(T|N)^d=0$ for some integer $d\in\mathbb{N}$. By part (i) of Theorem 1.41 we know that $K(T)=K(T|M)=K(T)\cap M$. Moreover, by part (iii) of Theorem 1.70 the semi-regularity of $T|M$ implies that $H_0(T|M)\subseteq K(T|M)=K(T)$. From this we obtain

$$\begin{aligned} H_0(T)\cap K(T) &= H_0(T)\cap(K(T)\cap M)=(H_0(T)\cap M)\cap K(T)\\ &= H_0(T|M)\cap K(T)=H_0(T|M).\end{aligned}$$

Therefore $H_0(T)\cap K(T)=H_0(T|M)$.

We claim that $H_0(T)+K(T)=N\oplus K(T)$. From $N\subseteq\ker T^d\subseteq H_0(T)$ we obtain that $N\oplus K(T)\subseteq H_0(T)+K(T)$. Conversely, from Corollary 1.69 we have

$$H_0(T)=N\oplus H_0(T|M)=N\oplus(H_0(T)\cap K(T))\subseteq N\oplus K(T),$$

so that

$$H_0(T)+K(T)\subseteq(N\oplus K(T))+K(T)\subseteq N\oplus K(T),$$

so our claim is proved.

Finally, from the inclusion $N\subseteq\ker T^d\subseteq\mathcal{N}^\infty(T)$, and, since $K(T)=T^\infty(X)$ for every operator of Kato type, we obtain that

$$H_0(T)+K(T)=N\oplus K(T)\subseteq\mathcal{N}^\infty(T)+T^\infty(X)\subseteq H_0(T)+K(T),$$

so the equality $\mathcal{N}^\infty(T)+T^\infty(X)=H_0(T)+K(T)$ is proved.

(ii) Let (M,N) be a GKD for T such that for some $d\in\mathbb{N}$ we have $(T|N)^d=0$. Then $\ker T^n=\ker(T|M)^n$ for every natural $n\geq d$. Since $\ker T^n\subseteq\ker T^{n+1}$ for all $n\in\mathbb{N}$ we then have

$$\mathcal{N}^\infty(T)=\bigcup_{n\geq d}^{\infty}\ker T^n=\bigcup_{n\geq d}^{\infty}\ker(T|M)^n=\mathcal{N}^\infty(T|M).$$

The semi-regularity of $T|M$ then implies by part (ii) of Theorem 1.70 that

$$\overline{\mathcal{N}^\infty(T)}=\overline{\mathcal{N}^\infty(T|M)}=\overline{H_0(T|M)}=\overline{H_0(T)\cap M}. \tag{26}$$

Next we show that the equality $\overline{H_0(T)\cap M}=\overline{H_0(T)}\cap M$ holds.
The inclusion $\overline{H_0(T)\cap M}\subseteq\overline{H_0(T)}\cap M$ is evident. Conversely, suppose that $x\in\overline{H_0(T)}\cap M$. Then there is a sequence $(x_n)\subset H_0(T)$ such that

$x_n \to x$ as $n \to \infty$. Let P be the projection of X onto M along N. Then $Px_n \to Px = x$ and $Px_n \in H_0(T) \cap P(H_0(T))$. From Corollary 1.69 we have $H_0(T) = (H_0(T) \cap M) \oplus N$, so

$$P(H_0(T)) = P(H_0(T) \cap M) = H_0(T) \cap M,$$

and hence $Px_n \in H_0(T) \cap M$, from which we deduce that $x \in \overline{H_0(T) \cap M}$. Therefore, $\overline{H_0(T) \cap M} = H_0(T) \cap M$. Finally, from the equality (26) and taking into account that by Theorem 1.41 and Theorem 1.42 we have $T^\infty(X) = K(T) \subseteq M$, we conclude that

$$\begin{aligned} \overline{\mathcal{N}^\infty(T)} \cap T^\infty(X) &= (\overline{H_0(T) \cap M}) \cap K(T) = (\overline{H_0(T)} \cap M) \cap K(T) \\ &= \overline{H_0(T)} \cap (M \cap K(T)) = \overline{H_0(T)} \cap K(T), \end{aligned}$$

so the proof is complete. ∎

8. Two spectral mapping theorems

In the first part of this section we prove that the semi-regular spectrum $\sigma_{\mathrm{se}}(T)$ of an operator $T \in L(X)$ on a non-trivial Banach space X is always non-empty. Indeed, we show that $\sigma_{\mathrm{se}}(T)$ contains the boundary points of $\sigma(T)$.

Theorem 1.75. *Let $T \in L(X)$, $X \neq \{0\}$ a Banach space. Then semi-regular spectrum $\sigma_{\mathrm{se}}(T)$ is a non-empty compact subset of $\mathbb{C}$ containing $\partial\sigma(T)$.*

Proof Let $\lambda_0 \in \partial\sigma(T)$ and suppose $\lambda_0 \in \rho_{\mathrm{se}}(T) = \mathbb{C} \setminus \sigma_{\mathrm{se}}(T)$. Since $\rho_{\mathrm{se}}(T)$ is open we can then consider a connected component Ω of $\rho_{\mathrm{se}}(T)$ containing λ_0. The set Ω is open so there exists a neighborhood $\mathcal{U}$ of λ_0 contained in Ω, and since $\lambda_0 \in \partial\sigma(T)$ $\mathcal{U}$ also contains points of $\rho(T)$. Hence $\Omega \cap \rho(T) \neq \varnothing$.

Consider a point $\lambda_1 \in \Omega \cap \rho(T)$. Clearly, $\ker(\lambda_1 I - T)^n = \{0\}$ for every $n \in \mathbb{N}$, so that $\mathcal{N}^\infty(\lambda_1 I - T) = \{0\}$. From Theorem 1.72 and Theorem 1.70 we have

$$\overline{H_0(\lambda_0 I - T)} = \overline{H_0(\lambda_1 I - T)} = \overline{\mathcal{N}^\infty(\lambda_1 I - T)} = \{0\}.$$

From Lemma 1.67 we then conclude that $\ker(\lambda_0 I - T) = \{0\}$, so $\lambda_0 I - T$ is injective. On the other hand, $\lambda_1 \in \rho(T)$ and hence from Theorem 1.36 we infer that

$$K(\lambda_0 I - T) = K(\lambda_1 I - T) = X,$$

so $\lambda_0 I - T$ is surjective. Hence $\lambda_0 \in \rho(T)$ and this is a contradiction, since $\lambda_0 \in \sigma(T)$. Therefore $\lambda_0 \in \sigma_{\mathrm{se}}(T)$ and $\partial\sigma(T) \subseteq \sigma_{\mathrm{se}}(T)$, so the last set is a compact non-empty subset of $\mathbb{C}$. ∎

Lemma 1.76. *Let $T \in L(X)$, X a Banach space, and let $\{\lambda_1, \cdots, \lambda_k\}$ be a finite subset of $\mathbb{C}$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Assume that $\{n_1, \cdots, n_k\} \subset$*

$\mathbb{N}$ *and set*

$$p(\lambda) := \prod_{i=1}^{k} (\lambda_i - \lambda)^{n_i} \quad \textit{and} \quad p(T) := \prod_{i=1}^{k} (\lambda_i I - T)^{n_i}.$$

Then

$$\text{(27)} \quad \ker\, p(T) = \bigoplus_{i=1}^{k} \ker\, (\lambda_i I - T)^{n_i} \quad \textit{and} \quad p(T)(X) = \bigcap_{i=1}^{k} (\lambda_i I - T)^{n_i}(X).$$

Proof We shall show the first equality of (27) for $k = 2$ and the general case then follows by induction.

Clearly $\ker\, (\lambda_i I - T)^{n_i} \subseteq \ker\, p(T)$ for $i = 1, 2$, so that if $p_i(T) := (\lambda_i I - T)^{n_i}$ then $\ker\, p_1(T) + \ker\, p_2(T) \subseteq \ker\, p(T)$.

In order to show the converse inclusion, observe that p_1, p_2 are relatively prime, hence by Lemma 1.2 there exist two polynomials q_1, q_2 such that

$$q_1(T)p_1(T) + q_2(T)p_2(T) = I,$$

so that every $x \in X$ admits the decomposition

$$x = q_1(T)p_1(T)x + q_2(T)p_2(T)x. \tag{28}$$

Now if $x \in \ker\, p(T)$ then

$$0 = p(T)x = p_1(T)p_2(T)x = p_2(T)p_1(T)x = p(T)x,$$

from which we deduce that $p_2(T)x \in \ker\, p_1(T)$ and $p_1(T)x \in \ker\, p_2(T)$. Moreover, since every polynomial in T maps the subspaces $\ker\, (\lambda_i I - T)^{n_i}$ into themselves we have $x_1 := q_2(T)p_2(T)x \in \ker p_1(T)$ and $x_2 := q_1(T)p_1(T)x \in \ker\, p_2(T)$. From(28) we have $x = x_1 + x_2$, so

$$\ker\, p(T) \subseteq \ker\, p_1(T) + \ker\, p_2(T).$$

Therefore $\ker\, p(T) = \ker\, p_1(T) + \ker\, p_2(T)$. It remains to prove that $\ker\, p_1(T) \cap \ker\, p_2(T) = \{0\}$. This is an immediate consequence of the identity (28).

As above we shall prove the second equality of (27) only for $n = 2$. Evidently $p(T)(X) = p_1(T)p_2(T)(X) = p_2(T)p_1(T)(X)$ is a subset of $p_1(T)(X)$ as well as a subset of $p_2(T)(X)$.

Conversely, suppose that $x \in p_1(T)(X) \cap p_2(T)(X)$ and let $y \in X$ such that $x = p_2(T)y$. Then $p_1(T)x = p(T)y \in p(T)(X)$. Analogously $p_2(T)x \in p(T)(X)$. Let q_1, q_2 be two polynomials for which the equality (28) holds. Then

$$\begin{aligned} x &= q_1(T)p_1(T)x + q_2(T)p_2(T)x \in q_1(T)p(T)(X) + q_2(T)p(T)(X) \\ &= p(T)q_1(T)(X) + p(T)q_2(T)(X) \subseteq p(T)(X) + p(T)(X) \subseteq p(T)(X). \end{aligned}$$

Hence $p_1(T)(X) \cap p_2(T)(X) \subseteq p(T)(X)$, so $p_1(T)(X) \cap p_2(T)(X) = p(T)(X)$, as desired. ∎

If $T \in L(X)$ let $\mathcal{H}(T)$ denote the set of all functions $f : \Delta(f) \to \mathbb{C}$ which are holomorphic on an open set $\Delta(f) \supset \sigma(T)$. From the well known

Riesz–Dunford functional calculus the operator $f(T)$ is defined for every $f \in \mathcal{H}(T)$, (see Heuser [**159**, §48]). The classical spectral theorem asserts that

$$\sigma(f(T)) = f(\sigma(T)) \quad \text{for all } f \in \mathcal{H}(T).$$

It is meaningful to note that a similar mapping property holds for the semi-regular spectrum $\sigma_{\rm se}(T)$, as well as for the essential semi-regular spectrum $\sigma_{\rm es}(T)$.

Theorem 1.77. *Let $T \in L(X)$, X a Banach space. Then*

$$\sigma_{\rm se}(f(T)) = f(\sigma_{\rm se}(T)) \quad \textit{for every } f \in \mathcal{H}(T).$$

Proof We show first the inclusion $f(\sigma_{\rm se}(T)) \subseteq \sigma_{\rm se}(f(T))$. To see this, let $\lambda_0 \notin \sigma_{\rm se}(f(T))$ and suppose that $\lambda_0 \in f(\sigma_{\rm se}(T))$. Let $\mu_0 \in \sigma_{\rm se}(T)$ be such that $f(\mu_0) = \lambda_0$. Define $g(\lambda) := \lambda_0 - f(\lambda)$. Then $g \in \mathcal{H}(T)$ and $g(\mu_0) = 0$, so there exists a function $h \in \mathcal{H}(T)$ such that $g(\lambda) = (\mu_0 - \lambda)h(\lambda)$. Hence

$$g(T) = \lambda_0 I - f(T) = (\mu_0 I - T)h(T).$$

On the other hand, from $\lambda_0 \notin \sigma_{\rm se}(f(T))$ we obtain that $\lambda_0 I - f(T) = (\mu_0 I - T)h(T)$ is semi-regular, and hence by Theorem 1.26 $\mu_0 I - T$ is semi-regular, $\mu_0 \notin \sigma_{\rm se}(T)$, which contradicts the assumption $\mu_0 \in \sigma_{\rm se}(T)$. Therefore the inclusion $f(\sigma_{\rm se}(T) \subseteq \sigma_{\rm se}(f(T))$ is proved.

In order to show the reverse inclusion $\sigma_{\rm se}(f(T)) \subseteq f(\sigma_{\rm se}(T))$, suppose that $\lambda_0 \notin f(\sigma_{\rm se}(T))$ and, as above, define $g(\lambda) := \lambda_0 - f(\lambda)$. Then $g(\lambda) \neq 0$ for every $\lambda \in \sigma_{\rm se}(T)$.

Consider first the case that $g(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$. In this case $g(T) = \lambda_0 I - f(T)$ is invertible, $\lambda_0 \in \rho(f(T)) \subseteq \rho_{\rm se}(f(T))$, and therefore $\lambda_0 \notin \sigma_{\rm se}(f(T))$. Hence, it remains to prove the inclusion $\sigma_{\rm se}(f(T)) \subseteq f(\sigma_{\rm se}(T))$ in the case that g vanishes at some points of $\sigma(T)$.

Now, g may admit only a finite number of zeros in $\sigma(T)$, say $\{\lambda_1, \dots, \lambda_k\}$, where $\lambda_i \neq \lambda_j$ for $j \neq k$. Since $g(\lambda_i) = 0$ we have $\lambda_i \notin \sigma_{\rm se}(T)$ for all $i = 1, 2, \dots, k$, i.e. the operators $\lambda_i - T$ are semi-regular for every $i = 1, 2, \dots, k$. Let $n_i \in \mathbb{N}$ denote the multiplicity of λ_i, write

$$p(\lambda) := \Pi_{i=1}^{k}(\lambda_i - \lambda)^{n_i}$$

and $g(\lambda) = p(\lambda)h(\lambda)$, where $h(\lambda) \in \mathcal{H}(T)$ has no zeros in $\sigma(T)$.

We show that $g(T)$ is semi-regular. Since $h(T)$ is invertible, from Lemma 1.76 we obtain

$$\ker g(T) = \ker p(T) = \bigoplus_{i=1}^{k} \ker\, (\lambda_i I - T)^{n_i}, \tag{29}$$

and for every $m \in \mathbb{N}$

$$g(T)^m(X) = p(T)^m(X) = \bigcap_{i=1}^{k} (\lambda_i I - T)^{m n_i}(X). \tag{30}$$

From the equality (30) we easily obtain that

$$g(T)^\infty(X) = \bigcap_{i=1}^{k} (\lambda_i I - T)^\infty(X).$$

Let $x \in \ker g(T)$. By (29) there exists for each $i = 1, \dots, k$ an element $x_i \in \ker(\lambda_i T - T)^{n_i}$ such that $x = \sum_{i=1}^{n} x_i$. Since, as already observed, the operators $\lambda_i - T$ are semi-regular for every i then $x_i \in (\lambda_i I - T)^\infty(X)$.

On the other hand $\lambda_i \neq \lambda_j$, by part (ii) of Theorem 1.3 we also have $\ker(\lambda_i - T) \subseteq (\lambda_j I - T)^\infty(X)$, so that $x_i \in (\lambda_j I - T)^\infty(X)$ for all $i, j = 1, 2, \dots, k$. Therefore

$$x = \sum_{i=1}^{k} x_i \in \bigcap_{i=1}^{k} (\lambda_i I - T)^\infty(X) = g(T)^\infty(X).$$

Note that, the subspaces $(\lambda_i I - T)(X)$ are closed for all $i = 1, 2, \dots, k$, by Corollary 1.17. Therefore also

$$g(T)(X) = p(T)(X) = \bigcap_{i=1}^{k} (\lambda_i I - T)^{n_i}(X)$$

is closed, so $g(T)$ is semi-regular. Since $g(T) = \lambda_0 I - f(T)$ we then conclude that $\lambda_0 \notin \sigma_{\rm se}(f(T))$. Hence $\sigma_{\rm se}(f(T)) \subseteq f(\sigma_{\rm se}(T))$, which completes the proof. ∎

We conclude this section by showing the essential version of Theorem 1.77, that $\sigma_{\rm es}(T)$ also behaves canonically under the Riesz functional calculus.

Theorem 1.78. *Let $T \in L(X)$, X a Banach space, and suppose that f is an analytic function on a neighbourhood of $\sigma(T)$. Then $f(\sigma_{\rm es}(T)) = \sigma_{\rm es}(f(T))$.*

Proof The inclusion $f(\sigma_{\rm es}(T)) \subseteq \sigma_{\rm es}(f(T))$ may be proved by using the same arguments of the first part of the proof of Theorem 1.77, replacing Theorem 1.26 with Theorem 1.50.

To show the opposite inclusion, suppose that $\lambda_0 \notin f(\sigma_{\rm es}(T))$ and define $g(\lambda) := \lambda_0 - f(\lambda)$. Then $g(\lambda) \neq 0$ for every $\lambda \in \sigma_{\rm es}(T)$. If $g(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$, proceeding as in the proof of Theorem 1.77, we then obtain that $\lambda_0 \notin \sigma_{\rm es}(f(T))$. Hence it remains to prove the inclusion $\sigma_{\rm es}(f(T)) \subseteq f(\sigma_{\rm es}(T))$ in the case where g vanishes at some points of $\sigma(T)$. Since g admits only a finite number of zeros $\lambda_1, \lambda_2, \dots, \lambda_k$ in $\sigma(T)$, we can write

$$g(\lambda) = h(\lambda) \prod_{i=1}^{k} (\lambda_i - \lambda)^{n_i}$$

where n_i denotes the multiplicity of λ_i, $h(\lambda)$ has no zero in $\sigma(T)$, and $\lambda_i I - T$ are essentially semi-regular for all $i = 1, 2, \dots, k$. As in the proof of Theorem

1.77, the operator $g(T)$ has closed range,

$$g(T)^\infty(X) = \bigcap_{i=1}^{k} (\lambda_i I - T)^\infty(X),$$

and $\ker(\lambda_i I - T) \subseteq (\lambda_j I - T)^\infty(X)$ for all $i \neq j$. By Theorem 1.48 we have $\ker\,(\lambda_i I - T) \subseteq (\lambda_i I - T)^\infty(X) + F_i$ for some finite-dimensional subspace F_i of X. Therefore

$$\ker\, g(T) \subseteq \bigcap_{i=1}^{k} (\lambda_i I - T)^\infty(X) + \sum_{i=1}^{\infty} F_i = g(T)^\infty(X) + F,$$

where $F := \sum_{i=1}^{\infty} F_i$ is finite-dimensional. Hence $g(T) = \lambda_0 I - f(T)$ is essentially semi-regular, so that $\lambda_0 \notin \sigma_{\mathrm{es}}(f(T))$. This shows that $\sigma_{\mathrm{es}}(f(T)) \subseteq f(\sigma_{\mathrm{es}}(T))$, so the proof is complete. ∎

8.1. Comments. The concept of the algebraic core of an operator has been introduced by Saphar [**284**], whilst the analytic core has been introduced by Vrbová [**313**] and Mbekhta [**230**]. The basic Lemma 1.9 is taken from the book of Heuser [**160**] (in particular, the result of Lemma 1.9, although not explicitly stated, is essentially contained in [**160**, Hilfsatz 72.7]), whilst Theorem 1.10 is modeled after Aiena and Monsalve [**31**].

The concept of semi-regularity of an operator $T \in L(X)$, X a Banach space, was originated by Kato's classical treatment [**182**] of perturbation theory, even if originally these operators were not named in this way. Later this class of operators was studied by several other authors, see for instance Mbekhta [**226**], [**227**], [**230**], [**231**], Mbekhta and Ouahab [**233**], Schmoeger [**291**]. Originally the semi-regular spectrum was defined for operators acting on Hilbert spaces by Apostol [**48**], and for this reason it is called by some authors the *Apostol spectrum*. This spectrum was defined for Hilbert space operators as the set of all complex λ such that either $\lambda I - T$ is not closed or λ is a discontinuity point for the function $\lambda \to (\lambda I - T)^{-1}$, see Theorem 1.38. Later the results of Apostol were generalized by Mbekhta [**226**], [**227**], Mbekhta and Ouahab [**233**], see also Harte [**149**] for operators defined on Banach spaces.

The methods and the proofs adopted in this book are strongly inspired by the paper of Mbekhta and Ouahab [**233**] and Schmoeger [**290**]. In particular, Theorem 1.22 and Theorem 1.24 were established by Schmoeger [**290**], whilst the subsequent part, except Lemma 1.34 owed to Kato [**182**], can be found in Mbekhta and Ouahad [**233**]. Example 1.27 and Example 1.27 are from Müller [**240**].

The proof of the local constancy of the hyper-range on the components of the Kato resolvent here given is taken from Mbekhta and Ouahab [**233**]. This important property, and other related results, has also been previously shown by others in a somewhat different language; see, for instance, Goldman and Kračkovskiĭ [**143**], Förster [**116**], Ó Searcóid and West [**253**].

Further investigation on semi-regular spectrum may be found in the paper by Kordula and Müller [**190**].

The generalized Kato decomposition has been studied in several papers by Mbekhta [**227**], [**229**] and [**228**]. The particular case of essentially semi-regular operators has been systematically investigated by Müller [**240**] and Rakočević [**274**]. The material presented here is completely inspired by Müller in [**240**]. Further information on the Kato decomposition may be found in Aiena and Mbekhta [**29**]. In particular, the proof of Theorem 1.44 and Theorem 1.64 are adaptations to our theory of more general results established in Aiena and Mbekhta [**29**], see also Rakočević [**274**].

The fundamental result that a semi-Fredholm operator is essentially semi-regular has been proved by Kato [**182**]. The clearer proof of this result, presented here, is completely modeled after West [**322**], see also [**324**].

The section on the quasi-nilpotent part of an operator is totally inspired by Mbekhta's thesis [**225**], Mbekhta [**230**], and Mbekhta and Ouahab [**233**], except Theorem 1.72 which is owed to Förster [**116**], whilst Theorem 1.74 is taken from Aiena and Villafãne [**34**]. Finally, the spectral mapping theorem for the semi-regular spectrum has been proved by Mbekhta [**227**] for Hilbert space operators and by Schmoeger [**290**] in the more general context of Banach spaces, whilst Theorem 1.78, which shows the spectral mapping theorem for the essentially semi-regular spectrum, was first proved by Müller [**240**]. It should be noted that an axiomatic approach to spectral mapping theorems for many parts of the spectrum may be found in Kordula and Müller [**190**] and Mbekhta and Müller [**232**]. Furthermore, Kordula [**189**] has shown that if $T \in L(X)$ is essentially semi-regular then every finite-dimensional perturbation of T is still essentially semi-regular.

CHAPTER 2

The single-valued extension property

In this chapter we shall introduce an important property for bounded operators on complex Banach spaces, the so called *single-valued extension property*. This property dates back to the early days of local spectral theory and appeared first in Dunford [**94**] and [**95**]. Subsequently this property has received a more systematic treatment in the classical texts by Dunford and Schwartz [**97**], as well as those by Colojoară and Foiaş [**83**], by Vasilescu [**309**] and, more recently, by Laursen and Neumann [**214**].

The single-valued extension property has a basic importance in local spectral theory since it is satisfied by a wide variety of linear bounded operators in the spectral decomposition problem. An important class of operators which enjoy this property is the class of all decomposable operators on Banach spaces that will be studied in Chapter 6, but this is also shared by many other operators which need not be decomposable. In fact, another class of operators which enjoy the single valued extension property is the class all multipliers of a semi-prime Banach algebra, and later it will be shown that there exist multipliers which are not decomposable.

In this chapter we shall deal with a localized version of the single-valued extension property and we shall employ the basic tools of local spectral theory to establish a variety of characterizations that ensure the single-valued extension property at a point λ_0. These characterizations involve the kernel type and range type of subspaces introduced in Chapter 3, as well as the quasi-nilpotent part and the analytic core of $\lambda_0 I - T$.

We shall introduce two important classes of subspaces which have a central role in local spectral theory: the class of local spectral subspaces $X_T(\Omega)$ associated with subsets Ω of $\mathbb{C}$. It will be also introduced a certain variant of these subspaces, the glocal spectral subspaces $\mathcal{X}_T(\Omega)$, which is better suited for operators without the single-valued extension property. There are important connections between the local and the glocal spectral subspaces and the invariant subspaces introduced in the previous chapter. In fact, the analytical cores and the quasi-nilpotent parts of operators are exactly the local spectral subspaces and glocal spectral subspaces, respectively, associated with certain subsets of $\mathbb{C}$.

From these characterizations we readily obtain in the second section that the single-valued extension property of T at a point λ_0 is satisfied if the mentioned kernel type and range type of subspaces have intersection equal to $\{0\}$. Dually, $T^\star$ has the single-valued extension property at λ_0 if the sum of these subspaces is the whole space X. We shall also produce several examples which prove that the converse of these implications are in general not true.

The third section concerns a spectral mapping theorem which shows that the single-valued extension property at a point behaves canonically under the Riesz functional calculus. In the fourth section we shall give a further look at some distinguished part of the spectrum of an operator T, as the semi-regular spectrum $\sigma_{\mathrm{se}}(T)$, the approximate point spectrum $\sigma_{\mathrm{ap}}(T)$ and the surjectivity spectrum $\sigma_{\mathrm{su}}(T)$, in the case that the T has the single-valued extension property.

These results are used successively to identify, for some general and important concrete cases, the set of points at which the single-valued extension property occur. As an application this property will be settled in the case of isometries, analytic Toeplitz operators, invertible composition operators on Hardy spaces, and unilateral or bilateral weighted shifts.

The fourth section of this chapter concerns basic properties of another class of T-invariant subspaces, the class of algebraic spectral subspaces $E_T(\Omega)$, $\Omega \subseteq \mathbb{C}$, associated with $T \in L(X)$. Subsequently we shall introduce the so called Dunford property (C), a stronger property than the single-valued extension property, which will have an important role in the subsequent chapters. The last section addresses to some local spectral properties of weighted shift operators on $\ell^p(\mathbb{N})$, with $1 \le p < \infty$. Most of these properties are established in the more general situation of operators T defined on Banach spaces for which the hyper-range $T^\infty(X) = \{0\}$.

1. Local spectrum and SVEP

The basic importance of the single-valued extension property arises in connection with some basic notions of local spectral theory. Before introducing the typical tools of this theory, and in order to give a first motivation, let us present some considerations on spectral theory.

It is well known that the resolvent function $R(\lambda, T) := (\lambda I - T)^{-1}$ of $T \in L(X)$, X a Banach space, is an analytic operator-valued function defined on the resolvent set $\rho(T)$. Setting

$$f_x(\lambda) := R(\lambda, T)x \quad \text{for any } x \in X,$$

the vector-valued analytic function $f_x : \rho(T) \to X$ satisfies the equation

$$(\lambda I - T)f_x(\lambda) = x \ \text{ for all } \lambda \in \rho(T). \tag{31}$$

It should be noted that it is possible to find analytic solutions of the equation $(\lambda I - T)f_x(\lambda) = x$ for some (sometimes even for all) values of λ that are in the spectrum of T. For instance, let $T \in L(X)$ be a bounded operator on a Banach space X such that the spectrum $\sigma(T)$ has a non-empty spectral subset $\sigma \neq \sigma(T)$. If $P_\sigma := P(\sigma, T)$ denotes the spectral projection of T associated with σ we know that $\sigma(T\,|P_\sigma(X)) = \sigma$ so the restriction

$(\lambda I - T) \,|P_\sigma(X)$ is invertible for all $\lambda \notin \sigma$.

Let $x \in P_\sigma(X)$. Then the equation (31) has the analytic solution

$$g_x(\lambda) := (\lambda I - T \,|P_\sigma(X))^{-1}x \quad \text{for all } \lambda \in \mathbb{C} \setminus \sigma.$$

This property suggests the following concepts:

Definition 2.1. *Given an arbitrary operator $T \in L(X)$, X a Banach space, let $\rho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $\mathcal{U}_\lambda$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U}_\lambda \to X$ such that the equation*

$$(\mu I - T)f(\mu) = x \quad \textit{holds for all } \mu \in \mathcal{U}_\lambda. \tag{32}$$

If the function f is defined on the set $\rho_T(x)$ then it is called a local resolvent function *of T at x. The set $\rho_T(x)$ is called the* local resolvent *of T at x. The* local spectrum *$\sigma_T(x)$ of T at the point $x \in X$ is defined to be the set*

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

Evidently $\rho_T(x)$ is the open subset of $\mathbb{C}$ given by the union of the domains of all the local resolvent functions. Moreover,

$$\rho(T) \subseteq \rho_T(x) \quad \text{and} \quad \sigma_T(x) \subseteq \sigma(T).$$

It is immediate to check the following elementary properties of $\sigma_T(x)$:

(a) $\sigma_T(0) = \varnothing$;

(b) $\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y)$ for all $x, y \in X$;

(c) $\sigma_{(\lambda I - T)}(x) \subseteq \{0\}$ if and only if $\sigma_T(x) \subseteq \{\lambda\}$.

Furthermore,

(d) $\sigma_T(Sx) \subseteq \sigma_T(x)$ for every $S \in L(X)$ which commutes with T. In fact, let $f : \mathcal{U}_\lambda \to X$ be an analytic function on the open set $\mathcal{U}_\lambda \subseteq \mathbb{C}$ for which $(\mu I - T)f(\mu) = x$ holds for all $\mu \in \mathcal{U}_\lambda$. If $TS = ST$ then the function $S \circ f : \mathcal{U}_\lambda \to X$ is analytic and satisfies the equation

$$(\mu I - T)S \circ f(\mu) = S((\mu I - T)f(\mu)) = Sx \quad \text{for all } \mu \in \mathcal{U}_\lambda,$$

Therefore $\rho_T(x) \subseteq \rho_T(Sx)$ and hence $\sigma_T(Sx) \subseteq \sigma_T(x)$.

A very important example of local spectrum is given in the case of multiplication operators on the Banach algebra $C(\Omega)$ of all continuous complex-valued functions on a compact Hausdorff space Ω, endowed with pointwise operations and supremum norm. Indeed, if T_f is the operator of multiplication on $C(\Omega)$ by an arbitrary function $f \in C(\Omega)$ then

$$\sigma_{T_f}(g) = f(\text{ supp } g),$$

where the *support* of g is defined by

$$\text{supp } g := \overline{\{\lambda \in \Omega : f(\lambda) \neq 0\}},$$

see Laursen and Neumann [**214**, Example 1.2.11].

The local spectrum at a point may be precisely characterized also in the case of a spectral operator in the sense of Dunford [**96**]. It is well-known that the spectrum $\sigma(T)$ of a *spectral operator* $T \in L(X)$ is the support of the spectral measure E for T, in the sense that $\sigma(T)$ is the smallest closed subset $\Omega \subseteq \mathbb{C}$ such that $E(\Omega) = I$. The local spectrum of a spectral operator T at x plays a similar role for the localized spectral measure $E(\cdot)x$, in the sense that $\sigma_T(x)$ is the smallest closed subset $\Omega \subseteq \mathbb{C}$ such that $E(\Omega)x = x$, see Corollary 1.2.25 of Laursen and Neumann [**214**].

Theorem 2.2. *Let $T \in L(X)$, X a Banach space, $x \in X$ and $\mathcal{U}$ an open subset of $\mathbb{C}$. Suppose that $f : \mathcal{U} \to X$ is an analytic function for which $(\mu I - T)f(\mu) = x$ for all $\mu \in \mathcal{U}$. Then $\mathcal{U} \subseteq \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$. Moreover,*

$$\sigma_T(x) = \sigma_T(f(\lambda)) \quad \textit{for all } \lambda \in \mathcal{U}. \tag{33}$$

Proof Let λ be arbitrarily chosen in $\mathcal{U}$. Define

$$h(\mu) := \begin{cases} \dfrac{f(\lambda) - f(\mu)}{\mu - \lambda} & \text{if } \mu \neq \lambda, \\ -f'(\lambda) & \text{if } \mu = \lambda, \end{cases}$$

for all $\mu \in \mathcal{U}$. Clearly h is analytic and it is easily seen that $(\mu I - T)h(\mu) = f(\lambda)$ for all $\mu \in \mathcal{U} \setminus \{\lambda\}$. By continuity the last equality is also true for $\mu = \lambda$, so

$$(\lambda I - T)h(\lambda) = f(\lambda) \quad \text{for all } \mu \in \mathcal{U}.$$

This shows that $\lambda \in \rho_T(f(\lambda))$ and since λ is arbitrary in $\mathcal{U}$ then $\mathcal{U} \subseteq \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$.

To prove the identity (33) we first show the inclusion $\sigma_T(f(\lambda)) \subseteq \sigma_T(x)$, or equivalently, $\rho_T(x) \subseteq \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$. If $\omega \in \mathcal{U}$ then $\omega \in \rho_T(f(\lambda))$ for all $\lambda \in \mathcal{U}$, by the first part of the proof. Suppose that $\omega \in \rho_T(x) \setminus \mathcal{U}$. Since $w \in \rho_T(x)$ there exists an open neighbourhood $\mathcal{W}$ of w such that $\lambda \notin \mathcal{W}$ and an analytic function $g : \mathcal{W} \to X$ such that $(\mu I - T)g(\mu) = x$ for all $\mu \in \mathcal{W}$. If we define

$$k(\mu) := \frac{f(\lambda) - g(\mu)}{\mu - \lambda} \quad \text{for all } \mu \in \mathcal{W},$$

then, as is easy to verify, $(\mu I - T)k(\mu) = f(\lambda)$ holds for all $\mu \in \mathcal{W}$. This shows that $\omega \in \rho_T(x)$, and hence $\sigma_T(x) \subseteq \sigma_T(f(\lambda))$.

It remains to prove the opposite inclusion $\sigma_T(f(\lambda)) \subseteq \sigma_T(x)$.

Let $\eta \notin \sigma_T(f(\lambda))$ and hence $\eta \in \rho_T(f(\lambda)$. Let $h : \mathcal{V} \to X$ be an analytic function defined on the open neighbourhood $\mathcal{V}$ of η for which the identity $(\mu I - T)h(\mu) = f(\lambda)$ holds for all $\mu \in \mathcal{V}$. Then

$$(\mu I - T)(\lambda I - T)h(\mu) = (\lambda I - T)(\mu I - T)h(\mu) = (\lambda I - T)f(\lambda) = x,$$

for all $\mu \in \mathcal{V}$, so that $\eta \in \rho_T(x)$ and hence $\eta \notin \sigma_T(x)$, so the proof is complete. ∎

Definition 2.3. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single-valued extension property *at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighbourhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

Remark 2.4. In the sequel we collect some basic properties of the SVEP.

(a) The SVEP ensures the consistency of the local solutions of equation (32), in the sense that if $x \in X$ and T has the SVEP at $\lambda_0 \in \rho_T(x)$ then there exists a neighborhood $\mathcal{U}$ of λ_0 and an *unique* analytic function $f : \mathcal{U} \to X$ satisfying the equation $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$.

Another important consequence of the SVEP is the existence of a maximal analytic extension $\tilde{f}$ of $R(\lambda, T)x := (\lambda I - T)^{-1}x$ to the set $\rho_T(x)$ for every $x \in X$. This function identically verifies the equation

$$(\mu I - T)\tilde{f}(\mu) = x \quad \text{for every } \mu \in \rho_T(x)$$

and, obviously,

$$\tilde{f}(\mu) = (\mu I - T)^{-1}x \quad \text{for every } \mu \in \rho(T).$$

(b) It is immediate to verify that the SVEP is inherited by the restrictions on invariant subspaces, i.e., if $T \in L(X)$ has the SVEP at λ_0 and M is a closed T-invariant subspace, then $T|M$ has the SVEP at λ_0. Moreover,

$$\sigma_T(x) \subseteq \sigma_{T|M}(x) \quad \text{for every } x \in M.$$

(c) Let $\sigma_{\mathrm{p}}(T)$ denote the *point spectrum* of $T \in L(X)$, i.e.,

$$\sigma_{\mathrm{p}}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

It is easy to see the implication:

$$\sigma_{\mathrm{p}}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ has the SVEP } \lambda_0.$$

Indeed, if $\sigma_p(T)$ does not cluster at λ_0 then there is an neighbourhood $\mathcal{U}$ of λ_0 such that $\lambda I - T$ is injective for every $\lambda \in \mathcal{U}$, $\lambda \neq \lambda_0$.

Let $f : \mathcal{V} \to X$ be an analytic function defined on another neighbourhood $\mathcal{V}$ of λ_0 for which the equation $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathcal{V}$. Obviously we may assume that $\mathcal{V} \subseteq \mathcal{U}$. Then $f(\lambda) \in \ker(\lambda I - T) = \{0\}$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$, and hence $f(\lambda) = 0$ for every $\lambda \in \mathcal{V}$, $\lambda \neq \lambda_0$. From the continuity of f at λ_0 we conclude that $f(\lambda_0) = 0$. Hence $f \equiv 0$ in $\mathcal{V}$ and therefore T has the SVEP at λ_0.

(d) From part (c) every operator T has the SVEP at an isolated point of the spectrum. Obviously T has the SVEP at every $\lambda \in \rho(T)$. From these facts it follows that every quasi-nilpotent operator T has the SVEP. More generally, if $\sigma_{\mathrm{p}}(T)$ has empty interior then T has the SVEP. In particular, any operator with a real spectrum has the SVEP.

Later we shall give an example of an operator having the SVEP and such that $\sigma_{\mathrm{p}}(T) \neq \varnothing$ (Example 3.11).

Definition 2.5. *For every subset Ω of $\mathbb{C}$ the* local spectral subspace *of T associated with Ω is the set*

$$X_T(\Omega) := \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

Obviously, if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$.
In the next theorem we collect some of the basic properties of the subspaces $X_T(\Omega)$.

Theorem 2.6. *Let $T \in L(X)$, X a Banach space, and Ω every subset of $\mathbb{C}$. Then the following properties hold:*

(i) *$X_T(\Omega)$ is a linear T-hyper-invariant subspace of X, i.e., for every bounded operator S that commutes with T we have $S(X_T(\Omega)) \subseteq X_T(\Omega)$;*

(ii) *$X_T(\Omega) = X_T(\Omega \cap \sigma(T))$;*

(iii) *If $\lambda \notin \Omega$, $\Omega \subseteq \mathbb{C}$, then $(\lambda I - T)(X_T(\Omega)) = X_T(\Omega)$;*

(iv) *Suppose that $\lambda \in \Omega$ and $(\lambda I - T)x \in X_T(\Omega)$ for some $x \in X$. Then $x \in X_T(\Omega)$;*

(v) *For every family $(\Omega_j)_{j \in J}$ of subsets of $\mathbb{C}$ we have*

$$X_T(\bigcap_{j\in J} \Omega_j) = \bigcap_{j\in J} X_T(\Omega_j);$$

(vi) *If Y is a T-invariant closed subspace of X for which $\sigma(T \mid Y) \subseteq \Omega$, then $Y \subseteq X_T(\Omega)$. In particular, $Y \subseteq X_T(\sigma(T \mid Y))$ holds for every closed T-invariant closed subspace of X.*

Proof (i) Evidently the set $X_T(\Omega)$ is a linear subspace of X, since the inclusion $\sigma_T(\alpha x + \beta y) \subseteq \sigma_T(x) \cup \sigma_T(y)$ holds for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$. Suppose now that $x \in X_T(\Omega)$, that is $\sigma_T(x) \subseteq \Omega$. If $TS = ST$ then $\sigma_T(Sx) \subseteq \sigma_T(x) \subseteq \Omega$, so $Sx \in X_T(\Omega)$.

(ii) Clearly $X_T(\Omega \cap \sigma(T)) \subseteq X_T(\Omega)$. Conversely, if $x \in X_T(\Omega)$ then $\sigma_T(x) \subseteq \Omega \cap \sigma(T)$, and hence $x \in X_T(\Omega \cap \sigma(T))$.

(iii) The operators $\lambda I - T$ and T commute, so from part (i) it follows that $(\lambda I - T)(X_T(\Omega)) \subseteq X_T(\Omega)$ for all $\lambda \in \mathbb{C}$. Let $\lambda \notin \Omega$ and consider an element $x \in X_T(\Omega)$, namely $\sigma_T(x) \subseteq \Omega$. Then $\lambda \in \rho_T(\Omega)$, so there is an open neighbourhood $\mathcal{U}$ of λ and an analytic function $f : \mathcal{U} \to X$ for which $(\mu I - T)f(\mu) = x$ for all $\mu \in \mathcal{U}$. In particular, $(\lambda I - T)f(\lambda) = x$. By Theorem 2.2 we obtain $\sigma_T(f(\lambda)) = \sigma_T(x) \subseteq \Omega$, and hence $f(\lambda) \in X_T(\Omega)$, from which we conclude that $x = (\lambda I - T)f(\lambda) \in (\lambda I - T)(X_T(\Omega))$.

(iv) Suppose that $(\lambda I - T)x \in X_T(\Omega)$, $\lambda \in \Omega$. We need to show that $\sigma_T(x) \subseteq \Omega$, or equivalently, $\mathbb{C} \setminus \Omega \subseteq \rho_T(x)$. Take $\eta \in \mathbb{C} \setminus \Omega$. By assumption $\mathbb{C} \setminus \Omega \subseteq \rho_T((\lambda I - T)x)$, so there is an analytic function $f : \mathcal{U}_\eta \to X$ defined on some open neighbourhood $\mathcal{U}_\eta$ of η such that $\lambda \notin \mathcal{U}$ and $(\mu I - T)f(\mu) = (\lambda I - T)x$ for all $\mu \in \mathcal{U}_\eta$. Define $g : \mathcal{U}_\eta \to X$ by

$$g(\mu) := \frac{x - f(\mu)}{\mu - \lambda} \quad \text{for all } \mu \in \mathcal{U}_\eta.$$

Clearly the analytic function g satisfies the equality $(\mu I - T)g(\mu) = x$ for all $\mu \in \mathcal{U}_\eta$, so that $\eta \in \rho_T(x)$. Therefore $\mathbb{C} \setminus \Omega \subseteq \rho_T(x)$, as desired.

(v) It is immediate.

(vi) From $\sigma(T \mid Y) \subseteq \Omega$ we obtain $\mathbb{C} \setminus \Omega \subseteq \rho(T \mid Y)$, so that for any $y \in Y$ we have $(\lambda I - T)(\lambda I - T \mid Y)^{-1}y = y$ for all $\lambda \in \mathbb{C} \setminus \Omega$. Obviously $f(\lambda) := (\lambda I - T \mid Y)^{-1}y$ is analytic for all $\lambda \in \mathbb{C} \setminus \Omega$, so $\mathbb{C} \setminus \Omega \subseteq \rho_T(y)$, consequently $\sigma_T(y) \subseteq \Omega$. ∎

Remark 2.7. It is easily seen that the absorbency result established in part (iv) of Theorem 2.6 implies $\ker (\lambda I - T)^n \subseteq X_T(\{\lambda\}$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. From this it follows that $\mathcal{N}^\infty(\lambda I - T) \subseteq X_T(\{\lambda\}$ for all $\lambda \in \mathbb{C}$. Later we shall see that $H_0(\lambda I - T) \subseteq X_T(\{\lambda\}$ for all $\lambda \in \mathbb{C}$.

We have already observed that 0 has an empty local spectrum. The next result shows that if T has the SVEP then 0 is the *unique* element of X having empty local spectrum. In fact, this property characterizes the SVEP.

Theorem 2.8. *Let $T \in L(X)$, X a Banach space. Then the following statements are equivalent :*

(i) *T has the SVEP;*

(ii) *$X_T(\varnothing) = \{0\}$;*

(iii) *$X_T(\varnothing)$ is closed.*

Proof (i) $\Leftrightarrow$ (ii) Suppose that T has the SVEP and $\sigma_T(x) = \varnothing$. Then $\rho_T(x) = \mathbb{C}$, so there exists an analytic function $f : \mathbb{C} \to X$ such that $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathbb{C}$. If $\lambda \in \rho(T)$ we have $f(\lambda) = (\lambda I - T)^{-1}x$, and hence, since $\|(\lambda I - T)^{-1}\| \to 0$ as $|\lambda| \to +\infty$, $f(\lambda)$ is a bounded function on $\mathbb{C}$. By Liouville's theorem $f(\lambda)$ is then constant, and therefore, since $(\lambda I - T)^{-1}x \to 0$ as $|\lambda| \to +\infty$, f is identically 0 on $\mathbb{C}$. This proves that $x = 0$. Since $0 \in X_T(\varnothing)$ we then conclude that $X_T(\varnothing) = \{0\}$.

Conversely, let $\lambda_0 \in \mathbb{C}$ be arbitrary and suppose that for every $0 \neq x \in X$ we have $\sigma_T(x) \neq \varnothing$. Consider any analytic function $f : \mathcal{U} \to X$ defined on an neighbourhhod $\mathcal{U}$ of λ_0 such that the equation $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathcal{U}$. From the equality

$$\sigma_T(f(\lambda)) = \sigma_T(0) = \varnothing,$$

see Theorem 2.2, we deduce that $f \equiv 0$ on $\mathcal{U}$ and therefore T has the SVEP at λ_0. Since λ_0 is arbitrary then T has the SVEP.

(ii) $\Rightarrow$(iii) Trivial.

(iii)⇒ (ii) Suppose that $X_T(\varnothing)$ is closed. From part (iii) of Theorem 2.6 we deduce that

$$(\lambda I - T)(X_T(\varnothing)) = X_T(\varnothing) \quad \text{for every } \lambda \in \mathbb{C}.$$

Now, let $\widetilde{T}$ denote the restriction $T\,|X_T(\varnothing)$. The operator $\lambda\widetilde{I}-\widetilde{T}$ is surjective and therefore semi-regular for every $\lambda \in \mathbb{C}$. This means that $\rho_{\rm se}(\widetilde{T})$ coincides with the whole complex field $\mathbb{C}$ and, by Theorem 1.75, that is true if and only if $X_T(\varnothing) = \{0\}$. ∎

If $T \in L(X)$ is a spectral operator on a Banach space X with spectral measure E, the local spectral subspaces $X_T(\Omega)$ associated with the closed subset $\Omega \subseteq \mathbb{C}$ may be precisely described. In fact, $X_T(\Omega)$ is the range of the projection $E(\Omega)$, see Corollary 1.2.25 of Laursen and Neumann [**214**], so that in this case $X_T(\Omega)$ is closed for all closed $\Omega \subseteq \mathbb{C}$.

Note that this last property in general is not true. Indeed, Theorem 2.8 shows, in particular, that the local spectral subspaces need not be closed, since $X_T(\varnothing)$ is not closed if T does not have SVEP.

Later the Example 2.32 will show that for a closed subset $\Omega \subseteq \mathbb{C}$ the local spectral subspaces $X_T(\Omega)$ need not be closed, also in the case that T has the SVEP.

Theorem 2.9. *Suppose that $T_i \in L(X_i)$, $i = 1, 2$, where X_i are Banach spaces. Then $T_1 \oplus T_2$ has the SVEP at λ_0 if and only if both T_1, T_2 have the SVEP at λ_0. If T_1, T_2 have the SVEP then*

$$\sigma_{T_1\oplus T_2}(x_1 \oplus x_2) = \sigma_{T_1}(x_1) \cup \sigma_{T_2}(x_2). \tag{34}$$

Proof First, suppose that T_1 and T_2 have the SVEP at λ_0 and let us consider an analytic function $f = f_1 \oplus f_2 : \mathcal{U} \to X_1 \oplus X_2$ on a neighbourhood $\mathcal{U}$ of λ_0, where $f_i : \mathcal{U} \to X_i$, $i = 1, 2$, are also analytic on $\mathcal{U}$. Obviously, for every $\lambda \in \mathcal{U}$ the condition $(\lambda I - T_1 \oplus T_2)f(\lambda) = 0$ implies that $(\lambda I - T_i)f_i(\lambda) = 0$, $i = 1, 2$. The SVEP of T_1 and T_2 then entails that $f_1 \equiv 0$ and $f_2 \equiv 0$ on $\mathcal{U}$. Thus $f \equiv 0$ on $\mathcal{U}$.

Conversely, assume that $T_1 \oplus T_2$ has the SVEP at λ_0 and let $f_i : \mathcal{U} \to X_i$, $\mathcal{U}$ a neighbourhood of λ_0, be two analytic functions which verify, for $i = 1, 2$, the equations

$$(\lambda I - T_i)f_i(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{U}.$$

For all $\lambda \in \mathcal{U}$ we have

$$0 = (\lambda I - T_1)f_1(\lambda) \oplus (\lambda I - T_2)f_2(\lambda) = (\lambda - T_1 \oplus T_2)[f_1(\lambda) \oplus (f_2(\lambda)],$$

so that the SVEP at λ_0 of $T_1 \oplus T_2$ implies $f_1(\lambda) \oplus (f_2(\lambda) \equiv 0$ on $\mathcal{U}$, and therefore $f_i \equiv 0$ on $\mathcal{U}$ for $i = 1, 2$.

To show the equality (34) , suppose that $T_1 \oplus T_2$ has the SVEP. Assume that $\lambda \in \rho_{T_1\oplus T_2}(x_1 \oplus x_2)$. Then there exists an open neighbourhood $\mathcal{U}$ of λ and an analytic function $f := f_1 \oplus f_2 : \mathcal{U} \to X_1 \oplus X_2$, with f_1 and f_2 analytic, such that

$$(\lambda I - T_1)f_1(\lambda) \oplus (\lambda I - T_2)f_2(\lambda) = (\lambda I - T_1 \oplus T_2)f(\lambda) = x_1 \oplus x_2.$$

Then $(\lambda I - T_i)f_i(\lambda) = x_i$, $i = 1, 2$, so $\lambda \in \rho_{T_1}(x_1) \cap \rho_{T_2}(x_2)$. This shows that $\sigma_{T_1}(x_1) \cup \sigma_{T_2}(x_2) \subseteq \sigma_{T_1 \oplus T_2}(x_1 \oplus x_2)$. The opposite inclusion has a similar proof. ∎

Corollary 2.10. *Suppose that $T \in L(X)$ has the SVEP and $X = M \oplus N$, where M and N are two closed and T-invariant subspaces. If $T_1 := T \mid M$ and $T_2 := T \mid N$, then for all closed subsets Ω of $\mathbb{C}$ we have*

$$X_T(\Omega) = M_{T_1}(\Omega) \oplus N_{T_2}(\Omega).$$

∎

The next result shows that the SVEP is stable under uniform convergence.

Theorem 2.11. *Suppose that the sequence $(T_n) \subset L(X)$, where X is a Banach space, converges to T in the uniform operator topology. If each T_n commutes with T and each T_n has the SVEP then T has the SVEP.*

Proof Let $f : \mathcal{U} \to X$ be an analytic function on the open set $\mathcal{U}$ such that

$$(\mu I - T)f(\lambda) = 0 \quad \text{for all } \mu \in \mathcal{U}. \tag{35}$$

Let $\lambda \in \mathcal{U}$ be arbitrary and for every $i = 1, 2$ let $\mathbf{D}(\lambda, r_i)$ denote a closed disc in $\mathbb{C}$ centered at λ with radius r_i such that $\mathbf{D}(\lambda, r_i) \subset \mathcal{U}$. Furthermore, assume that $r_2 < r_1$ and let $Q_n := T_n - T$ for every $n \in \mathbb{N}$. Clearly Q_n commutes with T_n for all $n \in \mathbb{N}$. By the uniform convergence we know that for $\varepsilon := \min\{r_2, r_1 - r_2\}$ there exists $n \in \mathbb{N}$ such that $\|Q_n\| < \varepsilon$. Set

$$K_\varepsilon := \{\nu \in \mathbb{C} : |\nu - \lambda| \le \varepsilon\}.$$

Since the spectral radius $r(Q_n)$ is less than ε, then for every $\mu \in \mathbb{C} \setminus K_\varepsilon$ we have $\mu - \lambda \in \rho(Q_n)$. Write

$$\mu I - T_n = (\mu - \lambda)I + (\lambda I - T_n) = (\mu - \lambda)I - Q_n + (\lambda I - T).$$

Since $(\lambda I - T)f(\lambda) = 0$, by (35), then

$$(\mu I - T_n)f(\lambda) = [(\mu - \lambda)I - Q_n]f(\lambda), \tag{36}$$

and hence, if $R(\mu - \lambda, Q_n) = [(\mu - \lambda)I - Q_n]^{-1}$,

$$(\mu I - T_n)R(\mu - \lambda), Q_n)f(\lambda) = R(\mu - \lambda), Q_n)(\mu I - T_n)f(\lambda) = f(\lambda). \tag{37}$$

Since the mapping $\mu \to R(\mu - \lambda, Q_n)f(\lambda)$ is analytic on $\mathbb{C} \setminus K_\varepsilon$ we then obtain that $\mu \in \rho_{T_n}(f(\lambda)$, so that $\sigma_{T_n}(f(\lambda)) \subseteq K_\varepsilon$.

In view of (36) and (37), by integration along the boundary of K_ε we obtain from the elementary functional calculus

$$(\mu I - T_n)\frac{1}{2\pi i}\int_{\partial K_\varepsilon} \frac{R(\mu - \nu, Q_n)f(\nu)}{\nu - \lambda}\, \mathrm{d}\nu = \frac{1}{2\pi i}\int_{\partial K_\varepsilon} \frac{f(\nu)}{\nu - \lambda}\, \mathrm{d}\nu = f(\lambda)$$

The function

$$\mu \to \frac{1}{2\pi i}\int_{\partial K_\varepsilon} \frac{R(\mu - \nu, Q_n)f(\nu)}{\nu - \lambda}\, \mathrm{d}\nu$$

is analytic in the open disc $\mathbb{D}(\lambda, r_2)$, so $\mathbb{D}(\lambda, r_2) \subseteq \rho_{T_n}(f(\lambda))$ and hence

$$\sigma_{T_n}(f(\lambda)) \subseteq \mathbb{C} \setminus \mathbb{D}(\lambda, r_2) \subseteq \mathbb{C} \setminus K_\varepsilon.$$

Since $\sigma_{T_n}(f(\lambda)) \subseteq K_\varepsilon$ the last inclusions imply that $\sigma_{T_n}(f(\lambda)) = \varnothing$. Since by assumption T_n has the SVEP, by Theorem 2.8 we then conclude that $f(\lambda) = 0$ for every $\lambda \in \mathcal{U}$. Hence T has the SVEP. ∎

Corollary 2.12. *Suppose that $T \in L(X)$ has the SVEP and Q is quasi-nilpotent commuting with T. Then $T + Q$ has the SVEP.*

Proof Suppose that $f : \mathbb{D} \to X$ is an analytic function on the set $\mathcal{U}$ such that $(\lambda I - T - Q)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. For $\mu \neq \lambda$, write $(\mu I - T)f(\lambda) = (\mu - \lambda + Q)f(\lambda)$. The assertion follows from the proof of Theorem 2.11 taking $Q_n := -Q$, $T_n := T$ and $K_\varepsilon = \{\lambda\}$. ∎

Remark 2.13. The result of Corollary 2.12 may be generalized as follows. If $T, S \in L(X)$, for every $n \in \mathbb{N}$ let us define

$$(T - S)^{[n]} := \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} T^k S^{n-k}.$$

Note that $(T - S)^{[n]}$ is not a function of $T - S$. However, if $TS = ST$ then $(T - S)^{[n]} = (T - S)^n$ for all $n \in \mathbb{N}$. The operators T, S are said to be *quasi-nilpotent equivalent* if

$$\lim_{n\to\infty} \|(T - S)^{[n]}\|^{1/n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(S - T)^{[n]}\|^{1/n} = 0.$$

Of course, two commuting operators T and S are quasi-nilpotent equivalent precisely when $T - S$ is quasi-nilpotent. Note that the relation defined above is actually an equivalence in $L(X)$, see Colojoară and Foiaş [**83**, p. 11]. Furthermore, if T, S are quasi-nilpotent equivalent then $\sigma(T) = \sigma(S)$, see [**83**, Theorem 2.2], and if T has the SVEP then also S has the SVEP and $\sigma_T(x) = \sigma_S(x)$ for every $x \in X$, see Theorem 2.3 and Theorem 2.4 of [**83**].

We next show that the SVEP is preserved by some trasforms.

Definition 2.14. *An operator $U \in L(X, Y)$ between the Banach spaces X and Y is said to be a* quasi-affinity *if U is injective and has dense range. The operator $S \in L(Y)$ is said to be a* quasi-affine transform *of $T \in L(X)$ if there is a quasi-affinity $U \in L(Y, X)$ such that $TU = US$.*

Theorem 2.15. *If $T \in L(X)$ has the SVEP at $\lambda_0 \in \mathbb{C}$ and $S \in L(Y)$ is a quasi-affine transform of T then S has the SVEP at λ_0.*

Proof Let $f : \mathcal{U} \to Y$ be an analytic function defined on an open neighbourhood $\mathcal{U}$ of λ_0 such that $(\mu I - S)f(\mu) = 0$ for all $\mu \in \mathcal{U}$. Then $U(\lambda I - S)f(\mu) = (\mu I - T)Uf(\mu) = 0$ and the SVEP of T at λ_0 entails that $Uf(\mu) = 0$ for all $\mu \in \mathcal{U}$. Since U is injective then $f(\mu) = 0$ for all $\mu \in \mathcal{U}$, hence S has the SVEP at λ_0. ∎

Next we want establish a local decomposition property that will be needed later. To establish this property we need first a preliminary result.

Lemma 2.16. *Let $K \subset \mathbb{C}$ be a compact set and suppose that Γ is a contour in the complement $\mathbb{C} \setminus K$ that surrounds K. If T is a bounded operator on a Banach space X and $f : \mathbb{C} \setminus K \to X$ is an analytic function for which $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus K$, then*

$$x = \frac{1}{2\pi i} \int_\Gamma f(\lambda)\ d\lambda.$$

Proof Let $\mathcal{U} := \mathbb{C} \setminus K$. We may suppose that Γ is contained in the unbounded connected component of the open set $\mathcal{U}$. In fact, f is analytic on $\mathcal{U}$ and therefore by Cauchy's theorem only the part of Γ which lies in the unbounded component of $\mathcal{U}$ contributes to the integral $\int_\Gamma f(\lambda)\ \mathrm{d}\lambda$.

Let Λ be the boundary, positively oriented, of a disc centered at 0 and having radius large enough to include in its interior both Γ and $\sigma(T)$. From Cauchy's theorem we have

$$\int_\Gamma f(\lambda)\ \mathrm{d}\lambda = \int_\Lambda f(\lambda)\ \mathrm{d}\lambda,$$

and from an elementary property of the Riesz functional calculus it follows that the last integral is $2\pi i I$. ∎

Theorem 2.17. *Suppose that $T \in L(X)$, X a Banach space, has the SVEP. If Ω_1 and Ω_2 are two closed and disjoint subsets of $\mathbb{C}$ then*

$$X_T(\Omega_1 \cup \Omega_2) = X_T(\Omega_1) \oplus X_T(\Omega_2),$$

where the direct sum is in algebraic sense.

Proof The inclusion $X_T(\Omega_1) \oplus X_T(\Omega_2) \subseteq X_T(\Omega_1 \cup \Omega_2)$ is obvious.

To show the reverse inclusion observe first that we may assume that Ω_1 and Ω_2 are both compact, since by part (ii) of Theorem 2.6 we have $X_T(\Omega) = X_T(\Omega \cap \sigma(T))$ for all subsets Ω of $\mathbb{C}$. Now, if $x \in X_T(\Omega_1 \cup \Omega_2)$ the SVEP ensures that there exists an analytic function $f : \mathbb{C} \setminus (\Omega_1 \cup \Omega_2) \to X$ such that

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus (\Omega_1 \cup \Omega_2).$$

Let Λ_1, Λ_2 be two compact disjoint sets such that Λ_i for $i = 1, 2$ is a neighbourhood of Ω_i and the boundary Γ_i of Λ_i is a contour surrounding Ω_i. From Lemma 2.16 it follows that $x = x_1 + x_2$, where

$$x_i := \frac{1}{2\pi i} \int_{\Gamma_i} f(\lambda)\ \ \mathrm{d}\lambda \quad \text{for } i = 1, 2.$$

We claim that $x_i \in X_T(\Lambda_i)$. In fact, set

$$g_i(\mu) := \frac{1}{2\pi i} \int_{\Gamma_i} \frac{f(\lambda)}{\mu - \lambda}\ \ \mathrm{d}\lambda \quad \text{for all } \mu \in \mathbb{C} \setminus \Lambda_i.$$

The functions $g_i(\lambda) : \mathbb{C} \setminus \Lambda_i \to X$ are analytic. Furthermore, for every $\mu \in \mathbb{C} \setminus \Lambda_i$ we have

$$\begin{aligned}(\mu I - T)g_i(\mu) &= \frac{1}{2\pi i}\int_{\Gamma_i} (\mu - \lambda + \lambda - T)\frac{f(\lambda)}{\mu - \lambda}\ \mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_i} (\lambda - T)\frac{f(\lambda)}{\mu - \lambda}\ \mathrm{d}\lambda + \frac{1}{2\pi i}\int_{\Gamma_i} f(\lambda)\ \mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_i} \frac{x}{\mu - \lambda}\ \mathrm{d}\lambda + x_i.\end{aligned}$$

But from the Cauchy's theorem we have that

$$\frac{1}{2\pi i}\int_{\Gamma_i} \frac{x}{\mu - \lambda}\ \mathrm{d}\lambda = 0 \qquad \text{for all } \mu \in \mathbb{C} \setminus \Lambda_i,$$

thus $(\mu I - T)g_i(\mu) = x_i$ for $i = 1, 2$ and this implies $x_i \in X_T(\Lambda_i)$, as claimed.

To conclude the proof observe that, again by Cauchy's theorem, the definition of x_1 and x_2 does not depend on the particular choice of Λ_1 and Λ_2, with the properties required above. This implies that $x_i \in X_T(\Lambda_i)$ for every compact neighbourhood Λ_i of Ω_i, so that $x_i \in X_T(\Omega_i)$ for $i = 1, 2$. Hence $x = x_1 + x_2$, where $x_i \in X_T(\Omega_i)$, as was desired. To see that the sum is direct, observe that since $\Omega_1 \cap \Omega_2 = \varnothing$, from Theorem 2.6, part (v), we have

$$X_T(\Omega_1) \cap X_T(\Omega_2) = X_T(\varnothing) = \{0\},$$

since, by assumption, T has the SVEP. ∎

The next result exhibits a simple characterization of the elements of the analytical core $K(T)$ by means of the local resolvent $\rho_T(x)$.

Theorem 2.18. *Let $T \in L(X)$, X a Banach space. Then*

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \in \rho_T(x)\}.$$

Proof Let $x \in K(T)$. We can suppose that $x \neq 0$. According to the definition of $K(T)$, let $\delta > 0$ and $(u_n) \subset X$ be a sequence for which

$$x = u_0,\ Tu_{n+1} = u_n, \quad \|u_n\| \le \delta^n \|x\| \quad \text{for every } n \in \mathbb{Z}_+.$$

Then the function $f : \mathbb{D}(0, 1/\delta) \to X$, where $\mathbb{D}(0, 1/\delta)$ is the open disc centered at 0 and radius $1/\delta$, defined by

$$f(\lambda) := -\sum_{n=1}^{\infty} \lambda^{n-1} u_n \qquad \text{for all } \lambda \in \mathbb{D}(0, 1/\delta),$$

is analytic and verifies the equation $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathbb{D}(0, 1/\delta)$. Consequently $0 \in \rho_T(x)$.

Conversely, if $0 \in \rho_T(x)$ then there exists an open disc $\mathbb{D}(0, \varepsilon)$ and an analytic function $f : \mathbb{D}(0, \varepsilon) \to X$ such that

$$(\lambda I - T)f(\lambda) = x \quad \text{for every } \lambda \in \mathbb{D}(0, \varepsilon). \tag{38}$$

Since f is analytic on $\mathbb{D}(0,\varepsilon)$ there exists a sequence $(u_n) \subset X$ such that

$$f(\lambda) = -\sum_{n=1}^{\infty} \lambda^{n-1} u_n \quad \text{for every } \lambda \in \mathbb{D}(0,\varepsilon). \tag{39}$$

Clearly $f(0) = -u_1$. and taking $\lambda = 0$ in (38) we obtain

$$Tu_1 = -T(f(0)) = x.$$

On the other hand

$$x = (\lambda I - T)f(\lambda) = Tu_1 + \lambda(Tu_2 - u_1) + \lambda^2(Tu_3 - u_2) + \cdots$$

for all $\lambda \in \mathbb{D}(0,\varepsilon)$. Since $x = Tu_1$ we conclude that

$$Tu_{n+1} = u_n \qquad \text{for all } n = 1, 2, \cdots .$$

Hence letting $u_0 = x$ the sequence (u_n) satisfies for all $n \in \mathbb{Z}_+$ the first of the conditions which define $K(T)$.

It remains to prove the condition $\|u_n\| \leq \delta^n \|x\|$ for a suitable $\delta > 0$ and for all $n \in \mathbb{Z}_+$. Take $\mu > 1/\varepsilon$. Since the series (39) converges then $|\lambda|^{n-1}\|u_n\| \to 0$ as $n \to \infty$ for all $\|\lambda\| < \varepsilon$ and, in particular, $1/\mu^{n-1}\|u_n\| \to 0$, so that there exists a $c > 0$ such that

$$\|u_n\| \leq c\ \mu^{n-1} \quad \text{for every } n \in \mathbb{N}. \tag{40}$$

From the estimates (40) we easily obtain

$$\|u_n\| \leq \left(\mu + \frac{c}{\|x\|}\right)^n \|x\|$$

and therefore $x \in K(T)$. ∎

For a bounded operator $T \in L(X)$ on a Banach space X and a closed set $\Omega \subseteq \mathbb{C}$, let $\mathcal{X}_T(\Omega)$ denote the set of all $x \in X$ such that there is an analytic function $f : \mathbb{C} \setminus \Omega \to X$ such that

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus \Omega.$$

It is easy to verify that $\mathcal{X}_T(\Omega)$ is a linear subspace of X. Clearly

$$\mathcal{X}_T(\Omega) \subseteq X_T(\Omega) \qquad \text{for every closed subset } \Omega \subseteq \mathbb{C}\ . \tag{41}$$

The set $\mathcal{X}_T(\Omega)$ is a linear subspace of X, called the *glocal spectral subspace* of T associated with Ω. This subspace is more appropriate for certain general questions of local spectral theory than the classical analytic subspace $X_T(\Omega)$.

In the following theorem we show few basic properties of the glocal subspaces. Some of these properties are rather similar to those of local spectral subspaces. The interested reader may be found further results on glocal spectral subspaces in Laursen and Neumann [**214**].

Theorem 2.19. *For an operator $T \in L(X)$, X a Banach space, the following statements hold:*

(i) $\mathcal{X}_T(\varnothing) = \{0\}$ *and* $\mathcal{X}_T(\sigma(T)) = X$*;*

(ii) $\mathcal{X}_T(\Omega) = \mathcal{X}_T(\Omega \cap \sigma(T))$ *and* $(\lambda I - T)\mathcal{X}_T(\Omega) = \mathcal{X}_T(\Omega)$ *for every closed set* $\Omega \subseteq \mathbb{C}$ *and all* $\lambda \in \mathbb{C} \setminus \Omega$*;*

(iii) $\mathcal{X}_T(\Omega_1 \cup \Omega_2) = \mathcal{X}_T(\Omega_1) + \mathcal{X}_T(\Omega_2)$ *for all disjoint closed subsets* Ω_1 *and* Ω_2 *of* $\mathbb{C}$*;*

(iv) T *has the SVEP if and only if* $\mathcal{X}_T(\Omega) = X_T(\Omega)$*, for every closed subset* $\Omega \subseteq \mathbb{C}$*.*

Proof (i) Suppose that $x \in \mathcal{X}_T(\varnothing)$ and let $f : \mathbb{C} \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C}$. Then $f(\lambda)$ coalesces with the resolvent function $R(\lambda, T) := (\lambda I - T)^{-1}$ on $\rho(T)$, so $f(\lambda) \to 0$ as $|\lambda| \to \infty$. By the vector-valued version of Liouville's theorem $f \equiv 0$, and therefore $x = 0$. The second equality of part (i) is straightforward.

The proof of (ii) easily follows from Theorem 2.2, whilst the proof of the decomposition (iii) is similar to the proof of that given for spectral local subspaces.

(iv) Clearly, if T has the SVEP and $\Omega \subseteq \mathbb{C}$ is closed then $\mathcal{X}_T(\Omega) = X_T(\Omega)$. Conversely, if $X_T(\Omega) = \mathcal{X}_T(\Omega)$ for all closed sets $\Omega \subseteq \mathbb{C}$ then

$$X_T(\varnothing) = \mathcal{X}_T(\varnothing) = \{0\},$$

so, by Theorem 2.8, T has the SVEP. ∎

Let $\mathbf{D}_\varepsilon$ denote the closed unit disc of $\mathbb{C}$ centered at 0 with radius $\varepsilon \geq 0$. The space $\mathcal{X}_T(\mathbf{D}_\varepsilon)$ may be characterized in the following way.

Theorem 2.20. *For every bounded operator* $T \in L(X)$*,* X *a Banach space, we have*

$$\mathcal{X}_T(\mathbf{D}_\varepsilon) = \left\{ x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} \leq \varepsilon \right\}. \tag{42}$$

In particular, $H_0(T) = \mathcal{X}_T(\{0\})$ *and if* T *has the SVEP then*

$$H_0(T) = X_T(\{0\}) = \{x \in X : \sigma_T(x) \subseteq \{0\}\}. \tag{43}$$

Proof Let $x \in X$ such that $\rho_T(x) := \limsup_{n\to\infty} \|T^n x\|^{1/n} \leq \varepsilon$. The series

$$f(\lambda) := \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} x, \quad \lambda \in \mathbb{C} \setminus \mathbf{D}_\varepsilon,$$

converges locally uniformly, so it defines an X-valued function on the set $\mathbb{C} \setminus \mathbf{D}_\varepsilon$. Evidently

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbf{D}_\varepsilon,$$

so $x \in \mathcal{X}_T(\mathbf{D}_\varepsilon)$. Conversely, assume that $x \in \mathcal{X}_T(\mathbf{D}_\varepsilon)$ and consider an analytic function $f : \mathbb{C} \setminus \mathbf{D}_\varepsilon \to X$ such that $(\lambda I - T)f(\lambda) = x$ holds for all $\lambda \in \mathbb{C} \setminus \mathbf{D}_\varepsilon$. For every $|\lambda| >$ max $\{\varepsilon, \|T\|\}$ we then obtain

$$f(\lambda) = (\lambda I - T)^{-1} x = \sum_{n}^{\infty} \lambda^{-n} T^{n-1} x,$$

and therefore $f(\lambda) \to 0$ as $|\lambda| \to \infty$. Consider the open disc $\mathbb{D}(0, 1/\varepsilon)$ of $\mathbb{C}$ centered at 0 with radius ε. The analytic function $g : \mathbb{D}(0, 1/\varepsilon) \to X$ defined by

$$g(\mu) := \begin{cases} f(\dfrac{1}{\lambda}) & \text{if } 0 \neq \mu \in \mathbb{D}(0, 1/\varepsilon), \\ 0 & \text{if } \mu = 0, \end{cases}$$

verifies the equality

$$g(\mu) = \sum_{n=1}^{\infty} \mu^n T^{n-1} x \quad \text{for all } |\mu| < \frac{1}{\max\{\varepsilon, \|T\|\}}. \tag{44}$$

Since g is analytic on $\mathbb{D}(0, 1/\varepsilon)$ it follows, exactly as in the scalar setting, from Cauchy's integral formula that the equality (44) holds even for all $\mu \in \mathbf{D}(0, 1/\varepsilon)$. This shows that the radius of convergence of the power series representing $g(\mu)$ is greater then $1/\varepsilon$. The standard formula for the radius of convergence of a vector valued power series then implies that $\rho_T(x) < \varepsilon$. Therefore the equality (42) holds.

The latter assertions are clear, by part (ii) of Theorem 2.19 and taking $\varepsilon = 0$ in the equality (42). ∎

Later we shall give an example of operator T such that $H_0(T)$ is not closed. From the equality $H_0(T) = \mathcal{X}_T(\{0\}$ we also obtain an example of operator for which the glocal spectral subspaces need not be closed.

Given an element $x \in X$ and $T \in L(X)$, the quantity

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{1/n}$$

is called the *local spectral radius* of T at x and the choice of this denomination is justified by the following fact:

Theorem 2.21. *Suppose that $T \in L(X)$ has the SVEP. Then*

$$r_T(x) = \ max\{|\lambda| : \lambda \in \sigma_T(x)\}. \tag{45}$$

Proof Let $x \in X$ be a non zero element. From Theorem 2.8 we know that $\sigma_T(x)$ is a non-empty compact subset of $\mathbb{C}$. Put $r := \max\{|\lambda| : \lambda \in \sigma_T(x)\}$ and let $\mathbf{D}(0, r)$ denote the closed disc of $\mathbb{C}$ centered at 0 and radius r. Then since T has the SVEP $x \in X_T(\mathbf{D}(0, r)) = \mathcal{X}_T(\mathbf{D}(0, r))$, and hence $r_T(x) \leq r$ by Theorem 2.20. Again, from Theorem 2.20 we obtain that x belongs to the glocal spectral space associated with the closed disc $\mathbf{D}(0, r_T(x))$ centered at 0 and radius $r_T(x)$, and therefore $x \in \mathcal{X}_T(\mathbf{D}(0, r_T(x))$. Therefore $\sigma_T(x) \subseteq \mathbf{D}(0, r_T(x))$. From this we easily conclude that $r \leq r_T(x)$, so the proof is complete. ∎

2. The SVEP at a point

We have seen in Theorem 2.8 that T has the SVEP precisely when for every element $0 \neq x \in X$ we have $\sigma_T(x) = \varnothing$. The next fundamental result establishes a localized version of this result.

Theorem 2.22. *Suppose that $T \in L(X)$, X a Banach space. Then the following conditions are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) $\ker\,(\lambda_0 I - T) \cap X_T(\varnothing) = \{0\}$*;*

iii) $\ker\,(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$*;*

(iv) *For every $0 \neq x \in \ker\,(\lambda_0 I - T)$ we have $\sigma_T(x) = \{\lambda_0\}$.*

Proof By replacing T with $\lambda_0 I - T$ we may assume without loss of generality $\lambda_0 = 0$.

(i) $\Leftrightarrow$ (ii) Assume that for $x \in \ker T$ we have $\sigma_T(x) = \varnothing$. Then $0 \in \rho_T(x)$, so there is an open disc $\mathbb{D}(0, \varepsilon)$ and an analytic function $f : \mathbb{D}(0, \varepsilon) \to X$ such that $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathbb{D}(0, \varepsilon)$. Then

$$T((\lambda I - T)f(\lambda)) = (\lambda I - T)T(f(\lambda)) = Tx = 0$$

for every $\lambda \in \mathbb{D}(0, \varepsilon)$. Since T has the SVEP at 0 then $Tf(\lambda) = 0$, and therefore $T(f(0)) = x = 0$.

Conversely, suppose that for every $0 \neq x \in \ker T$ we have $\sigma_T(x) \neq \varnothing$. Let $f : \mathbb{D}\ (0, \varepsilon) \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$ for every $\lambda \in \mathbb{D}\ (0, \varepsilon)$. Then $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$ for a suitable sequence $(u_n) \subset X$. Clearly $Tu_0 = T(f(0)) = 0$, so $u_0 \in \ker T$. Moreover, from the equalities $\sigma_T(f(\lambda)) = \sigma_T(0) = \varnothing$ for every $\lambda \in \mathbb{D}\ (0, \varepsilon)$ we obtain that

$$\sigma_T(f(0)) = \sigma_T(u_o) = \varnothing,$$

and therefore by the assumption we conclude that $u_0 = 0$. For all $0 \neq \lambda \in \mathbb{D}\ (0, \varepsilon)$ we have

$$0 = (\lambda I - T)f(\lambda) = (\lambda I - T)\sum_{n=1}^{\infty} \lambda^n u_n = \lambda(\lambda I - T)\sum_{n=1}^{\infty} \lambda^n u_{n+1},$$

and therefore

$$0 = (\lambda I - T)(\sum_{n=0}^{\infty} \lambda^n u_{n+1}) \quad \text{for every } 0 \neq \lambda \in \mathbb{D}\ (0, \varepsilon).$$

By continuity this is still true for every $\lambda \in \mathbb{D}\ (0, \varepsilon)$. At this point, by using the same argument as in the first part of the proof, it is possible to show that $u_1 = 0$, and by iterating this procedure we conclude that $u_2 = u_3 = \cdots = 0$. This shows that $f \equiv 0$ on $\mathbb{D}\ (0, \varepsilon)$, and therefore T has the SVEP at 0.

(ii) $\Leftrightarrow$ (iii) It suffices to prove the equality

$$\ker T \cap K(T) = \ker T \cap X_T(\varnothing).$$

To see this observe first that by Theorem 2.20 we have $\ker T \subseteq H_0(T) \subseteq X_T(\{0\})$. From Theorem 2.18 it follows that

$$\ker T \cap K(T) = \ker T \cap X_T(\mathbb{C} \setminus \{0\}) \subseteq X_T(\{0\}) \cap X_T(\mathbb{C} \setminus \{0\}) = X_T(\varnothing).$$

Since $X_T(\varnothing) \subseteq X_T(\mathbb{C} \setminus \{0\}) = K(T)$ we then conclude that

$$\ker T \cap K(T) = \ker T \cap K(T) \cap X_T(\varnothing) = \ker T \cap X_T(\varnothing),$$

as required.

(ii) $\Rightarrow$ (iv) Since $\ker T \subseteq H_0(T)$, from Theorem 2.20 it follows that $\sigma_T(x) \subseteq \{0\}$ for every $0 \neq x \in \ker T$. By assumption $\sigma_T(x) \neq \varnothing$, so $\sigma_T(x) = \{0\}$.

(iv) $\Rightarrow$ (ii) Obvious. ∎

The following corollary is a more detailed version of the result established in Theorem 2.8.

Corollary 2.23. *Let $T \in L(X)$, X a Banach space. Then T does not have SVEP if and only if there exists $\lambda \in \sigma_{\mathrm{p}}(T)$ and a corresponding eigenvector $x_0(\neq 0)$ such that $\sigma_T(x_0) = \varnothing$. In such a case T does not have SVEP at λ_0.* ∎

Clearly, if $\lambda_0 I - T$ is injective then T has the SVEP at λ_0. The next result shows that if $\lambda_0 I - T$ is surjective then T has the SVEP at λ_0 if and only if λ_0 belongs to the resolvent $\rho(T)$.

Corollary 2.24. *Let $T \in L(X)$, X a Banach space, be such that $\lambda_0 I - T$ is surjective. Then T has the SVEP at λ_0 if and only if $\lambda_0 I - T$ is injective.*

Proof We can assume $\lambda_0 = 0$. Assume that T is onto and has the SVEP at 0. Then $K(T) = X$ and by Theorem 2.22 $\ker T \cap X = \ker T = \{0\}$, so T is injective. The converse is clear. ∎

An immediate consequence of Corollary 2.24 is that every unilateral left shift on the Hilbert space $\ell_2(\mathbb{N})$ fails to have SVEP at 0. In the next chapter we shall see that other examples of operators which do not have SVEP are semi-Fredholm operators on a Banach space having index strictly greater than 0. Another example of operator which does not have SVEP at 0 is the adjoint $T^\star$ of an isometric non-unitary operator $T \in L(H)$ on a Hilbert space H, see Colojoară and Foiaş [**83**, Example 1.7].

Remark 2.25. Evidently if Y is a closed subspace of the Banach space X such that $(\lambda_0 I - T)(Y) = Y$ and the restriction $(\lambda_0 I - T) \,|Y$ does not have SVEP at λ_0 then also T does not have the same property at λ_0.

This property, together with Corollary 2.24, suggests how to obtain operators without the SVEP: if for an operator $T \in L(X)$ there exists a closed subspace Y such that

$$(\lambda_0 I - T)(Y) = Y \quad \text{and} \quad \ker\,(\lambda_0 I - T) \cap Y \neq \{0\}$$

then T does not have SVEP at λ_0.

In the remaining part of this section we want show that the relative positions of all the subspaces introduced in the previous chapter are intimately related to the SVEP at a point.

To see that let us consider, for an arbitrary $\lambda_0 \in \mathbb{C}$ and an operator $T \in L(X)$ the following increasing chain of kernel type of spaces:

$$\ker\,(\lambda_0 I - T) \subseteq \mathcal{N}^\infty(\lambda_0 I - T) \subseteq H_0(\lambda_0 I - T) \subseteq X_T(\{\lambda_0\}),$$

and the decreasing chain of the range type of spaces:

$$X_T(\varnothing) \subseteq X_T(\mathbb{C} \setminus \{\lambda_0\}) = K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)(X).$$

The next corollary is an immediate consequence of Theorem 2.22 and the inclusions considered above.

Corollary 2.26. *Suppose that $T \in L(X)$, X a Banach space, verifies one of the following conditions:*

(i) $\mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\}$;

(ii) $\mathcal{N}^\infty(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;

(iii) $\mathcal{N}^\infty(\lambda_0 I - T) \cap X_T(\varnothing) = \{0\}$;

(iv) $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;

(v) $\ker\,(\lambda_0 I - T) \cap (\lambda_0 I - T)(X) = \{0\}$.

Then T has the SVEP at λ_0. ∎

The SVEP may be characterized as follows.

Theorem 2.27. *Let $T \in L(X)$, X a Banach space. Then T has the SVEP if and only if $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$.*

Proof Suppose first that T has the SVEP. From Theorem 2.18 we know that

$$K(\lambda I - T) = X_{\lambda I - T}(\mathbb{C} \setminus \{0\}) = X_T(\mathbb{C} \setminus \{\lambda\}) \quad \text{for every } \lambda \in \mathbb{C},$$

and, by Theorem 2.20,

$$H_0(\lambda I - T) = X_{\lambda I - T}(\{0\}) = X_T(\{\lambda\}) \quad \text{for every } \lambda \in \mathbb{C}\ .$$

Consequently by Theorem 2.8

$$H_0(\lambda I - T) \cap K(\lambda I - T) = X_T(\{\lambda\}) \cap X_T(\mathbb{C} \setminus \{\lambda\}) = X_T(\varnothing) = \{0\}\ .$$

The converse implication is clear by Corollary 2.26. ∎

Corollary 2.28. *Suppose that $T \in L(X)$, X a Banach space. If T is quasi-nilpotent then $K(T) = \{0\}$.*

Proof If T is quasi-nilpotent then $H_0(T) = X$ by Theorem 1.68. On the other hand, since T has the SVEP, from Theorem 2.27 we conclude that $\{0\} = K(T) \cap H_0(T) = K(T)$. ∎

Theorem 2.29. *Let $T \in L(X)$, X a Banach space, be essentially semi-regular and quasi-nilpotent. Then X is finite-dimensional and T is nilpotent. In particular, this holds for semi-Fredholm operators.*

Proof Suppose that (M, N) is a GKD for T such that $T|N$ is nilpotent and N is finite-dimensional. Since T is quasi-nilpotent, from Corollary 1.69 we have $X = H_0(T) = H_0(T|M) \oplus N$. Moreover, $T|M$ is semi-regular, hence by Theorem 1.70 and Theorem 1.41 $H_0(T|M) \subseteq K(T|M) = K(T)$. But $K(T) = \{0\}$ by Corollary 2.28, so $H_0(T|M) = \{0\}$ and this implies that $X = \{0\} \oplus N = N$. Therefore X is finite-dimensional and T is nilpotent. ∎

Example 2.30. The next example, based on theory of weighted shifts, shows that the SVEP at a point does not necessarily implies that $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$.
Let $\beta := (\beta_n)_{n\in\mathbb{Z}}$ be the sequence of real numbers defined as follows:

$$\beta_n := \begin{cases} 1 + |n| & \text{if } n < 0, \\ e^{-n^2} & \text{if } n \geq 0. \end{cases}$$

Let $X := L_2(\beta)$ denote the Hilbert space of all formal Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n \quad \text{for which} \quad \sum_{n=-\infty}^{\infty} |\alpha_n|^2 {\beta_n}^2 < \infty \ ,$$

Let us consider the *bilateral weighted right shift* defined by

$$T(\sum_{n=-\infty}^{\infty} a_n z^n) := \sum_{n=-\infty}^{\infty} a_n z^{n+1} \ ,$$

or equivalently, $Tz^n := z^{n+1}$ for every $n \in \mathbb{Z}$. The operator T is bounded on $L_2(\beta)$ and

$$\|T\| = \sup \left\{ \frac{\beta_{n+1}}{\beta_n} : n \in \mathbb{Z} \right\} = 1 \ .$$

Clearly T is injective, so it has the SVEP at 0. The following argument shows that $H_0(T) \cap K(T) \neq \{0\}$. From $\|z^n\|_\beta = \beta_n$ for all $n \in \mathbb{Z}$ we obtain that

$$\lim_{n\to\infty} \|z^{n-1}\|_\beta{}^{1/n} = 0$$

and

$$\lim_{n\to\infty} \|z^{-n-1}\|_\beta{}^{1/n} = 1 \ .$$

By the formula for the radius of convergence of a power series we then conclude that the two series

$$f(\lambda) := \sum_{n=1}^{\infty} \lambda^{-n} z^{n-1} \quad \text{and} \quad g(\lambda) := -\sum_{n=1}^{\infty} \lambda^{n} z^{-n-1}$$

converge in $L_2(\beta)$ for all $|\lambda| > 0$ and $|\lambda| < 1$, respectively. Clearly the function f is analytic on $\mathbb{C} \setminus \{0\}$, and

$$(\lambda I - T) f(\lambda) = -\sum_{n=1}^{\infty} \lambda^{-n} z^{n} - \sum_{n=1}^{\infty} \lambda^{1-n} z^{n-1} = 1 \quad \text{for all } \lambda \neq 0 \ ,$$

whilst the function g is analytic on the open unit disc $\mathbb{D}$ and verifies

$$(\lambda I - T)g(\lambda) = \sum_{n=0}^{\infty} \lambda^n z^{-n} - \sum_{n=0}^{\infty} \lambda^{1+n} z^{-n-1} = 1 \quad \text{for all } \lambda \in \mathbb{D} .$$

This means that $1 \in \mathcal{X}_T(\{0\}) \cap \mathcal{X}_T(\mathbb{C} \setminus \mathbb{D}) = H_0(T) \cap K(T)$, where the last equality follows from Theorem 2.18 and Theorem 2.20.

Theorem 2.31. *Suppose that $T \in L(X)$, where X is a Banach space, has a closed quasi-nilpotent part $H(\lambda_0 I - T)$ or that $H_0(\lambda_0 - T) \cap K(\lambda_0 I - T)$ is closed. Then $H_0(\lambda_0 - T) \cap K(\lambda_0 I - T) = \{0\}$ and hence T has the SVEP at λ_0.*

Proof Without loss of generality we may consider $\lambda_0 = 0$.

Assume first that $H_0(T)$ is closed. Let $\widetilde{T}$ denote the restriction of T to the Banach space $H_0(T)$. Clearly, $H_0(T) = H_0(\widetilde{T})$, so $\widetilde{T}$ is quasi-nilpotent and hence $K(\widetilde{T}) = \{0\}$, by Corollary 2.28. On the other hand it is easily seen that $H_0(T) \cap K(T) = K(\widetilde{T})$.

Assume now that $Y := H_0(T) \cap K(T)$ is closed. Clearly Y is invariant under T, so we can consider the restriction $S := T \mid Y$. If $y \in Y$ then $\|S^n y\|^{1/n} = \|T^n y\|^{1/n} \to 0$ as $n \to \infty$, so $y \in H_0(S)$ and hence $H_0(S) = Y$. From Theorem 1.68 we infer that S is quasi-nilpotent and hence, by Corollary 2.28 $K(S) = \{0\}$. We show that $Y = K(S)$. To prove this equality choose $y \in Y = H_0(T) \cap K(T)$. By the definition of $K(T)$ there is then a sequence $(y_n) \subset X$ and a $\delta > 0$ such

$$y_0 = y, \; Ty_n = y_{n-1} \quad \text{and} \quad \|y_n\| \le \delta^n \|y\|$$

for all $n \in \mathbb{Z}_+$. Since $y \in Y \subseteq H_0(T)$, from Lemma 1.67 we obtain that $y_n \in H_0(T)$ for all $n \in \mathbb{N}$. Moreover, since $y \in K(T) = X_T(\mathbb{C} \setminus \{0\}$, from part (iv) of Theorem 2.6 we also obtain that $y_n \in K(T) =$ for all $n \in \mathbb{Z}_+$, so that $y_n \in Y$ and therefore $y \in K(S)$. This shows that $Y \subseteq K(S)$.

The opposite inclusion is clear since $K(S) = K(T) \cap Y \subseteq Y$. Thus $Y = K(T) \cap H_0(T) = K(S) = \{0\}$, and hence by Corollary 2.26 T has the SVEP at 0.

The last assertion is clear from Corollary 2.26. ∎

The next example shows that an operator $T \in L(X)$ may have the SVEP at the point λ_0 but fails the property of having a closed quasi-nilpotent part $H_0(\lambda_0 I - T)$.

Example 2.32. Let $X := \ell_2 \oplus \ell_2 \cdots$ provided with the norm

$$\|x\| := \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2} \qquad \text{for all } x := (x_n) \in X,$$

and define

$$T_n e_i := \begin{cases} e_{i+1} & \text{if } i = 1, \cdots, n , \\ \dfrac{e_{i+1}}{i-n} & \text{if } i > n. \end{cases}$$

It is easy to verify that

$$\|T_n^{n+k}\| = 1/k! \quad \text{and} \quad (1/k!)^{1/n+k} \text{ as } k \to \infty.$$

From this it follows that $\sigma(T_n) = \{0\}$. Moreover, T_n is injective and the point spectrum $\sigma_{\rm p}(T_n)$ is empty, so T_n has the SVEP.

Now let us define $T := T_1 \oplus \cdots \oplus T_n \oplus \cdots$. From the estimate $\|T_n\| = 1$ for every $n \in \mathbb{N}$, we easily obtain $\|T\| = 1$. Moreover, since $\sigma_{\rm p}(T_n) = \varnothing$ for every $n \in \mathbb{N}$, it also follows that $\sigma_{\rm p}(T) = \varnothing$.

Let us consider the sequence $x = (x_n) \subset X$ defined by $x_n := e_1/n$ for every n. We have

$$\|x\| = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty \ ,$$

which implies that $x \in X$. Moreover,

$$\|T^n x\|^{1/n} \geq \|T_n^n \ \frac{e_1}{n}\|^{1/n} = (1/n)^{1/n}$$

and the last term does not converge to 0. From this it follows that $\sigma_T(x)$ contains properly $\{0\}$ and therefore, by Theorem 2.20, $x \notin H_0(T)$. Finally,

$$\ell_2 \oplus \ell_2 \cdots \oplus \ell_2 \oplus \{0\} \cdots \subset H_0(T),$$

where the non-zero terms are n. This holds for every $n \in \mathbb{N}$, so $H_0(T)$ is dense in X. Since $H_0(T) \neq X$ it follows that $H_0(T)$ is not closed.

Theorem 2.33. *Suppose that for a bounded operator $T \in L(X)$, the sum $H_0(\lambda_0 I - T) + (\lambda_0 I - T)(X)$ is norm dense in X. Then $T^\star$ has the SVEP at λ_0.*

Proof Also here we assume that $\lambda_0 = 0$. From Theorem 1.70 we know that $K(T^\star) \subseteq H_0(T)^\perp$. From a standard duality argument we now obtain

$$\ker\ (T^\star) \cap K(T^\star) \subseteq T(X)^\perp \cap H_0(T)^\perp = (T(X) \cap H_0(T))^\perp.$$

If the subspace $H(T) + T(X)$ is norm-dense in X, then the last annihilator is zero, so that $\ker T^\star \cap K(T^\star) = \{0\}$, and consequently by Theorem 2.22 $T^\star$ has the SVEP at 0. ∎

Corollary 2.34. *Suppose either that $H_0(\lambda_0 I - T) + K(\lambda_0 I - T)$ or $\mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$ is norm dense in X. Then $T^\star$ has the SVEP at λ_0.* ∎

It is easy to find an example of an operator for which $T^\star$ has the SVEP at a point λ_0 and such that $\mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$ is not norm dense in X.

Example 2.35. Let T denote the Volterra operator on the Banach space $X := C[0,1]$ defined by

$$(Tf)(t) := \int_0^t f(s) \mathrm{d}s \quad \text{for all } f \in C[0,1] \quad \text{and } t \in [0,1].$$

T is injective and quasi-nilpotent. Consequently $\mathcal{N}^\infty(T) = \{0\}$ and $K(T) = \{0\}$ by Corollary 2.28. It is easy to check that

$$T^\infty(X) = \{f \in C^\infty[0,1] : f^{(n)}(0) = 0,\ n \in \mathbb{Z}_+\},$$

thus $T^\infty(X)$ is not closed and hence is strictly larger than $K(T) = \{0\}$. Clearly the sum $\mathcal{N}^\infty(T) + T^\infty(X)$ is not norm dense in X, whilst $T^\star$ has the SVEP because it is quasi-nilpotent.

The next theorem is, in a certain sense, dual to Theorem 2.33.

Theorem 2.36. *Suppose that for a bounded operator $T \in L(X)$ the sum $H_0(\lambda_0 I^\star - T^\star) + (\lambda_0 I^\star - T^\star)(X^\star)$ is weak * dense in $X^\star$. Then T has the SVEP at λ_0.*

Proof From Theorem 1.70 we know that $K(T) \subseteq^\perp H_0(T^\star)$. Therefore

$$\ker T \cap K(T) \subseteq^\perp T^\star(X^\star) \cap^\perp H_0(T^\star) =^\perp (T^\star(X^\star) + H_0(T^\star)).$$

But the sum $H_0(T^\star) + T^\star)(X^\star)$ is weak* dense in $X^\star$, so by the Hahn–Banach theorem the last annihilator is zero and therefore T has the SVEP at 0, again by Theorem 2.22. ∎

Corollary 2.37. *Suppose that for a bounded operator $T \in L(X)$, either $H_0(\lambda_0 I - T^\star) + K(\lambda_0 I - T^\star)$ or $\mathcal{N}^\infty(\lambda_0 - T^\star) + (\lambda_0 - T^\star)^\infty(X^\star)$ is weak* dense in $X^\star$. Then T has the SVEP at λ_0.* ∎

3. A local spectral mapping theorem

Given an operator $T \in L(X)$, X a Banach space, and an analytic function f defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$, and let $f(T)$ denote the corresponding operator defined by the functional calculus. The classical spectral theorem states that $f(\sigma(T)) = \sigma(f(T))$. We also know that an analogous formula holds for the semi-regular spectrum $\sigma_{\rm se}(T)$ and the essential semi-regular spectrum $\sigma_{\rm es}(T)$. One may be tempted to conjecture an analogous result for the local spectrum, $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ for all $x \in X$, but it can be easily seen that in general that is not true. Indeed, if we consider the constant function $f \equiv c$ on the neighborhood $\mathcal{U}$ and an operator T without the SVEP, then there exists by Theorem 2.18 a vector $0 \neq x \in X$ such that $\sigma_T(x) = \varnothing$. Clearly $f(\sigma_T(x) = \varnothing$, whilst

$$\sigma_{f(T)}(x) = \sigma(f(T)) = \{c\} \neq \varnothing.$$

However, in the next remark, where we collect some information about the local spectra $\sigma_{f(T)}(x)$, we see that the spectral theorem for the local spectrum holds under certain additional assumptions on the function f or on T .

Remark 2.38. Let $T \in L(X)$, X a Banach space, and let f be an analytic function f on the open neighborhood $\mathcal{U}$ of $\sigma(T)$. We have

(i) $f(\sigma_T(x)) \subseteq \sigma_{f(T)}(x)$ for all $x \in X$ [**214**, Theorem 3.3.8].

(ii) If T has the SVEP or if the function f is non-constant on each of the connected components of $\mathcal{U}$ then

$$f(\sigma_T(x)) = \sigma_{f(T)}(x) \quad \text{for all } x \in X,$$

see Laursen and Neumann [**214**, Theorem 3.3.8]. The proof of this equality depend upon the glocal spectral subspaces canonical behaviour with respect to Riesz functional calculus, i.e., if f is analytic on an open neighbourhood of $\sigma(T)$ then $\mathcal{X}_{f(T)}(\Omega) = \mathcal{X}_T(f^{-1}(\Omega))$ for all closed sets $\Omega \subseteq \mathbb{C}$. The proof of this deep result may be found in Laursen and Neumann [**214**, Theorem 3.3.6].

For an arbitrary operator $T \in L(X)$ on a Banach space X let

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \ \text{ does not have the SVEP at } \lambda\} .$$

From the identity theorem for analytic functions it readily follows that $\Xi(T)$ is open and consequently is contained in the interior of the spectrum $\sigma(T)$. Clearly $\Xi(T)$ is empty precisely when T has the SVEP.

Theorem 2.39. *Let $T \in L(X)$, X a Banach space. Let $f : \mathcal{U} \to \mathbb{C}$ be an analytic function on the open neighborhood $\mathcal{U}$ of $\sigma(T)$. Suppose that f is non-constant on each of the connected components of $\mathcal{U}$. Then $f(T)$ has the SVEP at $\lambda \in \mathbb{C}$ if and only if T has the SVEP at every point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. Moreover, $f(\Xi(T)) = \Xi((f(T))$.*

Proof Suppose first that $f(T)$ has the SVEP at $\lambda_0 \in \mathbb{C}$. By Theorem 2.22 then

$$\ker\,(\lambda_0 I - f(T)) \cap X_{f(T)}(\varnothing) = \{0\}.$$

Suppose now that for some $\mu_0 \in \sigma(T)$ we have $f(\mu_0) = \lambda_0$. To show the SVEP of T at μ_0 it suffices, again by Theorem 2.22, to show that $\ker\,(\mu_0 I - T) \cap X_{\mu_0 I - T}(\varnothing) = \{0\}$.

Let $x \in \ker\,(\mu_0 I - T) \cap X_T(\varnothing)$ be arbitrarily given and define by $h(\mu) := \lambda_0 - f(\mu)$ for all $\mu \in \mathcal{U}$. Then $h(T) = \lambda_0 I - f(T)$ and, since $h(\mu_0) = 0$ we can write $h(\mu) = (\mu_0 - \mu)g(\mu)$, where g is analytic on $\mathcal{U}$. Clearly

$$h(T) = (\mu_0 I - T)g(T) = g(T)(\mu_0 I - T),$$

so that $x \in \ker\, h(T) = \ker\,(\lambda_0 I - f(T)$. On the other hand, from $x \in X_T(\varnothing)$ we obtain $\sigma_T(x) = \varnothing$, and hence by part (ii) of Remark 2.38

$$\sigma_{f(T)}(x) = f(\sigma_T(x) = f(\varnothing) = \varnothing,$$

so $x \in X_{f(T)}(\varnothing)$. Therefore

$$\ker\,(\mu_0 I - T) \cap X_T(\varnothing) \subseteq \ker\,(\lambda_0 I - f(T)) \cap X_{f(T)}(\varnothing) = \{0\},$$

which shows that T has the SVEP at λ_0.

Conversely, let $\lambda_0 \in \mathbb{C}$ and assume that T has the SVEP at every $\mu_0 \in \sigma(T)$ for which $f(\mu_0) = \lambda_0$. Write $h(\mu) := \lambda_0 - f(\mu)$, where $\mu \in \mathcal{U}$. By assumption f is non-constant on each connected component of $\mathcal{U}$, so, by the identity theorem for analytic functions, the function h has only finitely many zeros in $\sigma(T)$ and these zeros are of finite multiplicity. Hence there exists an analytic function g defined on $\mathcal{U}$ without zeros in $\sigma(T)$ and a polynomial p of the form

$$p(\mu) = (\mu_1 - \mu) \cdots (\mu_n - \mu),$$

with not necessarily distinct elements $\mu_1, \cdots, \mu_n \in \sigma(T)$ such that

$$h(\mu) = \lambda_0 - f(\mu) = p(\mu)g(\mu) \quad \text{for all } \mu \in \mathcal{U}.$$

Assume that $x \in \ker(\lambda_0 I - f(T)) \cap X_{f(T)}(\varnothing)$. In order to prove that $f(T)$ has the SVEP at λ_0 it suffices to show, again by Theorem 2.22, that $x = 0$. From the classical spectral mapping theorem we know that $g(T)$ is invertible, so the equality

$$\lambda_0 I - f(T) = p(T)g(T) = g(T)p(T)$$

implies that $p(T)x \in \ker g(T) = \{0\}$. If we put $q(\mu) := (\mu_2 - \mu) \cdots (\mu_n - \mu)$ and $y = q(T)x$ then $(\mu_1 I - T)y = 0$.

On the other hand, $x \in X_{f(T)}(\varnothing)$ and f is non-constant on each of the connected components of $\mathcal{U}$. Part (ii) of Remark 2.38 then ensures that

$$f(\sigma_T(x)) = \sigma_{f(T)}(x) = \varnothing$$

and therefore since T and $q(T)$ commute

$$\sigma_T(y) = \sigma_T(q(T)x) \subseteq \sigma_T(x) = \varnothing.$$

But T has the SVEP at μ_1, by assumption, so, again by Theorem 2.22, $y = 0$. A repetition of this argument for $\mu_2, \cdots, \mu_n$ then leads to the equality $x = 0$, thus $f(T)$ has the SVEP at λ_0.

The last claim is obvious, being nothing else than a reformulation of the equivalence proved above. ∎

Theorem 2.40. *Let $T \in L(X)$, X a Banach space, and $f : \mathcal{U} \to \mathbb{C}$ an analytic function on the open neighborhood $\mathcal{U}$ of $\sigma(T)$. If T has the SVEP then $f(T)$ has the SVEP. If f is non-constant on each of the connected components of $\mathcal{U}$, then T has the SVEP if and only if $f(T)$ has the SVEP.*

Proof The second assertion is immediate from Theorem 2.39, so we have only to show the first assertion.

Assume that T has the SVEP and let f be an analytic defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$. We may assume that f is not identically 0 on each component of $\mathcal{U}$. Since T has the SVEP the inclusion $f(\sigma_T(x)) \subseteq \sigma_{f(T)}(x)$ holds for any analytic function. Proceeding exactly as in the second part of the proof of Theorem 2.39 we easily obtain that $f(T)$ has the SVEP. ∎

An immediate consequence of Theorem 2.39 is that, in the characterization of the SVEP at a point $\lambda_0 \in \mathbb{C}$ given in Theorem 2.22, the kernel $\ker(\lambda_0 I - T)$ may be replaced by the hyper-kernel $\mathcal{N}^\infty(\lambda_0 I - T)$.

Corollary 2.41. *For every bounded operator on a Banach space X the following properties are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) *T^n has the SVEP at λ_0 for every $n \in \mathbb{N}$.*

(iii) *$\mathcal{N}^\infty(\lambda_0 I - T) \cap X_T(\varnothing) = \{0\}$;*

(iv) *$\mathcal{N}^\infty(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;*

Proof The equivalence (i) $\Leftrightarrow$ (ii) is obvious from Theorem 2.39. Combining this equivalence with Theorem 2.22 we then obtain that T has the SVEP at λ_0 if and only if $\ker\,(\lambda_0 I - T)^n \cap X_T(\varnothing) = \{0\}$, for every $n \in \mathbb{N}$. Therefore the equivalence (i) $\Leftrightarrow$ (iii) is proved. The equivalence (i) $\Leftrightarrow$ (iv) follows from Theorem 2.22 in a similar way. ∎

Note that in the condition (ii) of Corollary 2.41 the power T^n may be replaced by $f(T)$, where f is any analytic function on some neighborhood $\mathcal{U}$ of $\sigma(T)$ such that f is non-constant on each of the connected components of $\mathcal{U}$ and such that 0 is the only zero of f in $\sigma(T)$.

The spectrum of a bounded linear operator can be divided into subsets in many different ways. In this section we shall consider some other parts of the spectrum which play a relevant role in local spectral theory. In the sequel we shall denote by

$$\sigma_{\rm su}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\},$$

the *surjectivity spectrum* of T. It should be noted that the surjectivity spectrum has been called something else by several authors, e.g., *approximate defect spectrum*. Since the terminology does not appear standard, we prefer to point out its purely algebraic nature by using the term of surjectivity spectrum .

The *approximate point spectrum* of $T \in L(X)$, is defined to be the set

$$\sigma_{\rm ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

We have already observed that $\lambda I - T$ is bounded below if and only if there exists $K > 0$ such that $\|\lambda I - Tx\| \geq K\|x\|$ for all $x \in X$. From this it easily follows that $\lambda \in \sigma_{\rm ap}(T)$ if and only if there exists a sequence $(x_n) \subset X$ such that $\|x_n\| = 1$ and $(\lambda I - T)x_n \to 0$ as $n \to \infty$.

The next result gives further informations on $\sigma_{\rm ap}(T)$.

Theorem 2.42. *If $T \in L(X)$, X a Banach space, then $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T^\star)$ and $\sigma_{\rm ap}(T) = \sigma_{\rm su}(T^\star)$. Moreover, $\sigma_{\rm ap}(T)$, as well as $\sigma_{\rm su}(T)$, is a non-empty compact subset of $\mathbb{C}$ containing the topological boundary of $\sigma(T)$.*

Proof The equalities $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T^\star)$ and $\sigma_{\rm ap}(T) = \sigma_{\rm su}(T^\star)$ are obvious from Lemma 1.30, part (i). From Lemma 1.30, part (ii), we also obtain that both $\sigma_{\rm ap}(T)$ and $\sigma_{\rm su}(T)$ are closed because have open complements. Obvious, the two spectra are compact, since both are subsets of $\sigma(T)$.

That $\sigma_{\rm ap}(T)$ contains the topological boundary $\partial\sigma(T)$ of the spectrum is an obvious consequence of part (ii) of Theorem 1.75, once observed that $\sigma_{\rm ap}(T) \supseteq \sigma_{\rm se}(T)$. Furthermore, from the equality $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T^\star)$ we conclude that $\sigma_{\rm su}(T)$ contains the topological boundary of $\sigma(T^\star) = \sigma(T)$. ∎

The following theorem shows that the surjectivity spectrum of an operator is closely related to the local spectra.

Theorem 2.43. *For every operator $T \in L(X)$ on a Banach space X we have*

$$\sigma_{\rm su}(T) = \bigcup_{x\in X} \sigma_T(x).$$

Proof If $\lambda \notin \bigcup_{x\in X} \sigma_T(x)$ then $\lambda \in \rho_T(x)$ for every $x \in X$ and hence, directly from the definition of $\rho_T(x)$, we conclude that $(\lambda I - T)y = x$ always admits a solution for every $x \in X$, $\lambda I - T$ is surjective. Thus $\lambda \notin \sigma_{\rm su}(T)$.

Conversely, suppose $\lambda \notin \sigma_{\rm su}(T)$. Then $\lambda I - T$ is surjective and therefore $X = K(\lambda I - T)$. From Theorem 2.18 it follows that $0 \notin \sigma_{\lambda I-T}(x)$ for every $x \in X$, and consequently $\lambda \notin \sigma_T(x)$ for every $x \in X$. ∎

Corollary 2.44. *If X is a Banach space and $T \in L(X)$ then $\sigma(T) = \Xi(T) \cup \sigma_{\rm su}(T)$. In particular, $\sigma_{\rm su}(T)$ contains $\partial\Xi(T)$, the topological boundary of $\Xi(T)$.*

Proof The inclusion $\Xi(T) \cup \sigma_{\rm su}(T) \subseteq \sigma(T)$ is obvious. Conversely, if $\lambda \notin \Xi(T) \cup \sigma_{\rm su}(T)$ then $\lambda I - T$ is surjective and T has the SVEP at λ, so by Corollary 2.24 $\lambda I - T$ is also injective. Hence $\lambda \notin \sigma(T)$.

The last claim is immediate, $\partial\Xi(T) \subseteq \sigma(T)$ and since $\Xi(T)$ is open it follows that $\partial\Xi(T) \cap \Xi(T) = \varnothing$. This obviously implies that $\partial\Xi(T) \subseteq \sigma_{\rm su}(T)$. ∎

Corollary 2.45. *Let X be a Banach space and $T \in L(X)$ has the SVEP. We have*

(i) *If T has the SVEP then $\sigma_{\rm su}(T) = \sigma(T)$ and $\sigma_{\rm se}(T) = \sigma_{\rm ap}(T)$.*

(ii) *If $T^\star$ has the SVEP then $\sigma_{\rm ap}(T) = \sigma(T)$ and $\sigma_{\rm se}(T) = \sigma_{\rm su}(T)$.*

(iii) *If both T and $T^\star$ have the SVEP then*

$$\sigma(T) = \sigma_{\rm su}(T) = \sigma_{\rm ap}(T) = \sigma_{\rm se}(T).$$

Proof The first equality (i) is an obvious consequence of Corollary 2.44, since $\Xi(T)$ is empty. To prove the second equality of (i) observe first that the inclusion $\sigma_{\rm se}(T) \subseteq \sigma_{\rm ap}(T)$ is trivial, since every bounded below operator is semi-regular . Conversely, let $\lambda \notin \sigma_{\rm se}(T)$. From the definition of semi-regularity and Theorem 1.10 we have

$$\ker\,(\lambda I - T) \subseteq (\lambda I - T)^\infty(X) = K(\lambda I - T)$$

and therefore, since $\ker\,(\lambda I - T) \subseteq H_0(\lambda I - T)$ for all $\lambda \in \mathbb{C}$,

$$\ker\,(\lambda I - T) \subseteq K(\lambda I - T) \cap H_0(\lambda I - T)$$

From Corollary 2.26 we then conclude that $\ker(\lambda I - T) = \{0\}$, i.e. $\lambda I - T$ is injective. This implies, since $\lambda I - T$ has closed range by assumption, that $\lambda \notin \sigma_{\rm ap}(T)$.

The two equalities of part (ii) are obvious consequence of Theorem 2.42, whilst part (iii) follows from part (i) and part (ii). ∎

Note that if T fails the SVEP then the local spectral formula (45) is not valid. This is obvious, once it is observed that in this case $\sigma_T(x)$ is empty for some non-zero $x \in X$ by Theorem 2.8. Other informations about the local spectral radius $r_T(x)$ may be found in Chapter 3 of the book by Laursen and Neumann [**214**].

It is easily seen that if $X_T(\Omega) = \{0\}$ then $\Omega \cap \sigma_{\rm p}(T) = \varnothing$. In fact, suppose that $X_T(\Omega) = \{0\}$ and assume that there is $\lambda_0 \in \Omega \cap \sigma_{\rm p}(T)$. Then there is $0 \neq x \in \ker(\lambda_0 I - T)$. Clearly $\sigma_T(x) \subseteq \{\lambda_0\}$, and since $\lambda_0 \in \Omega$ this implies that $x \in X_T(\Omega) = \{0\}$, a contradiction.

We also have that $X_T(\Omega) = X$ precisely when $\sigma_{\rm su}(T) \subseteq \Omega$. In fact, if $X_T(\Omega) = X$ and $\lambda \notin \Omega$ then

$$K(\lambda I - T) = X_T(\mathbb{C} \setminus \{\lambda\}) \supseteq X_T(\Omega \setminus \{\lambda\}) = X_T(\Omega) = X,$$

so that $X = K(\lambda I - T)$ and hence $\lambda I - T$ is surjective, namely $\lambda \notin \sigma_{\rm su}(T)$. Conversely, suppose that $\sigma_{\rm su}(T) \subseteq \Omega$. By Theorem 2.43 we obtain that $\sigma_T(x) \subseteq \Omega$ for all $x \in X$ so that $X = X_T(\Omega)$.

One of the deep results of local spectral theory shows that analogous results hold for the glocal subspaces in the case where Ω is closed subset of $\mathbb{C}$. In the next theorem we only state this result and refer for a proof of it to Laursen and Neumann [**214**, Theorem 3.3.12].

Theorem 2.46. *Suppose that $T \in L(X)$, where X is a Banach space, and Ω is a closed subset of $\mathbb{C}$. Then the following assertions hold:*

(i) $\mathcal{X}_T(\Omega) = X$ *if and only if* $\sigma_{\rm su}(T) \subseteq \Omega$;

(ii) $\mathcal{X}_T(\Omega) = \{0\}$ *implies that* $\Omega \cap \sigma_{\rm p}(T) = \varnothing$;

(iii) *If* $\Omega \cap \sigma_{\rm ap}(T) = \varnothing$ *then* $\mathcal{X}_T(\Omega) = \{0\}$. ∎

An interesting consequence of Theorem 2.43 is that the approximate point spectrum $\sigma_{\rm ap}(T)$ and the surjectivity spectrum $\sigma_{\rm su}(T)$ behave canonically under the Riesz functional calculus.

To see this we need first a preliminary remark. Suppose that a Banach space X is the direct sum $X = M \oplus N$, where the closed subspaces M and N are T-invariant, and let P_M denote the projection of X onto M. Clearly P_M commutes with T. It is easily seen that

$$\ker T = \ker T|M \oplus \ker T|N \quad \text{and} \quad T(X) = T(M) \oplus T(N),$$

so that T is injective if and only if both the restrictions $T|M$ and $T|N$ are injective. Moreover, $T(X)$ is closed if and only if $T(M)$ is closed in M and $T(N)$ is closed in N. In fact, if $T(X)$ is closed then

$$T(M) = TP_M(X) = P_M(T(X)) = T(X) \cap M$$

so $T(M)$ is closed, and analogously $T(N)$ is closed.

Conversely, assume that $T(M)$ is closed in M and $T(N)$ is closed in N. Since the mapping $\Psi : M \times N \to M \oplus N$, defined by $\Psi(x, y) := x + y$ is a topological isomorphism, then the image

$$\Psi(T(M) \times T(N)) = T(M) \oplus T(N) = T(X)$$

is closed in X. Combining all these resuls we obtain

$$T \text{ is bounded below} \Leftrightarrow T|M,\ T|N \text{ are bounded below,}$$

and hence

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T|M) \cup \sigma_{\mathrm{ap}}(T|N).$$

Analogously, from the equality $T(X) = T(M) \oplus T(N)$ we easily deduce that T is onto if and only if both $T|M$ and $T|N$ are onto, so that

$$\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(T|M) \cup \sigma_{\mathrm{su}}(T|N).$$

Clearly

$$\sigma(T) = \sigma(T|M) \cup \sigma(T|N).$$

Next we want to establish the spectral mapping theorem for the approximate point spectrum and the surjectivity spectrum. To prepare this result we first consider the very special case that the analytic function f is constant on the connected components of its domain of definition.

Lemma 2.47. *Let $T \in L(X)$, X a Banach space, and suppose that the function f is constant on each connected component of an open set $\mathcal{U}$ containing $\sigma(T)$. Then*

$$f(\sigma_{\mathrm{su}}(T)) = \sigma_{\mathrm{su}}(f(T)) = f(\sigma(T)) = \sigma(f(T)).$$

Analogously

$$\sigma_{\mathrm{ap}}(f(T)) = \sigma_{\mathrm{ap}}(f(T)) = f(\sigma(T)) = \sigma(f(T)).$$

Proof We shall prove first the equality for $\sigma_{\mathrm{su}}(T)$.

Since $\sigma(T)$ is compact we may assume that $\mathcal{U}$ has only a finite number of components, say $\Omega_1, \cdots, \Omega_n$. Assume also that

$$\sigma_i := \Omega_i \cap \sigma(T) \neq \varnothing, \qquad i = 1, \dots, n.$$

Let $f(\lambda) = c_i$ for every $\lambda \in \Omega_i$ and denote by χ_i the characteristic function of Ω_i. From the elementary functional calculus we know that $P_i := \chi_i(T)$ is the spectral projection associated with the spectral set σ_i and that the decomposition $X = M_1 \oplus \cdots \oplus M_n$ holds, where $M_i := P_i(X)$. The subspaces M_i are invariant under T and under $f(T)$. Moreover,

$$f(\lambda) = \sum_{i=1}^{n} c_i \chi_i(\lambda) \quad \text{and} \quad f(T) = \sum_{i=1}^{n} c_i P_i,$$

so that $f(T)|M_i = c_i I_i$, I_i the identity on M_i. Obviuously,,

$$\sigma_{\mathrm{su}}(f(T)|M_i) = \sigma(f(T)|M_i) = \{c_i\},$$

from which we obtain

$$\sigma(f(T)) = \sigma_{\rm su}(f(T)) = \{c_1, \dots, c_n\}.$$

On the other hand, $\sigma_{\rm su}(T) \subseteq \sigma(T)$ implies

$$f(\sigma_{\rm su}(T)) \subseteq f(\sigma(T)) = \{c_1, \dots, c_n\}. \tag{46}$$

The reverse inclusion of (46) is also true, since every Ω_i contains some points λ of $\sigma_{\rm su}(T|M_i)$, since the latter set is non-empty, and therefore

$$c_i = f(\lambda) \in f(\sigma_{\rm su}(T|M_i) \subseteq f(\sigma_{\rm su}(T)) \quad \text{for all } i = 1, \dots, n.$$

Hence $f(\sigma_{\rm su}(T)) = \sigma_{\rm su}(f(T)) = f(\sigma(T))$. The equality $\sigma_{\rm ap}(f(T)) = f(\sigma_{\rm ap}(T))$ easily follows by duality, since $\sigma_{\rm ap}(f(T)) = \sigma_{\rm su}(f(T^*))$ and $f(\sigma_{\rm ap}(T)) = f(\sigma_{\rm su}(T^*))$. ∎

Theorem 2.48. *Let $T \in L(X)$, X a Banach space, and suppose that the function $f : \mathcal{U} \to \mathbb{C}$ defined on an open set $\mathcal{U}$ containing $\sigma(T)$. Then*

$$\sigma_{\rm su}(f(T)) = f(\sigma_{\rm su}(T)) \quad \textit{and} \quad \sigma_{\rm ap}(f(T)) = f(\sigma_{\rm ap}(T)).$$

Proof Assume first that f is non-constant on each component of $\mathcal{U}$. As noted in part (ii) of Remark 2.38 we have $\sigma_{f(T))}(x) = f(\sigma_T(x))$ for all $x \in X$. Taking the union over all $x \in X$, we obtain from Theorem 2.43 that $\sigma_{\rm su}(f(T)) = f(\sigma_{\rm su}(T))$.

To show the general case consider the other possibility, i.e., suppose that f is constant on some components of $\mathcal{U}$. Denote by $\Omega_1, \cdots, \Omega_n$ all the components where f is constant, say $f(\lambda) = c_n$ for $\lambda \in \Omega_n$. Clearly we may assume that $\sigma_i := \Omega_i \cap \sigma(T) \neq \varnothing$ for every $i = 1, \dots, n$. Define

$$\Omega := \bigcup_{i=1}^n \Omega_i \quad \text{and} \quad \sigma := \bigcup_{i=1}^n \sigma_i.$$

If P, Q denote the spectral projection associated with σ and $\sigma(T) \setminus \sigma$, respectively, then $X = M \oplus N$, where $M := P(X)$ and $N := Q(X)$. Moreover,

$$\sigma(T|M) = \sigma \subseteq \Omega, \quad \sigma(T|N) = \sigma(T) \setminus \sigma \subseteq \mathcal{U} \setminus \Omega.$$

Let g be the restriction of f on Ω and h the restriction of f onto $\mathcal{U} \setminus \Omega$. The functions g and h are analytic on an open set containing $\sigma(T|M)$ and $\sigma(T|N)$, respectively. Furthermore,

$$g(T|M) = f(T|M) = f(T)|M$$

and

$$h(T|N) = f(T|N) = f(T)|N.$$

Clearly g is constant on every connected component of Ω, whilst h is non-constant on every connected component of $\mathcal{U} \setminus \Omega$. From Lemma 3.68 and from the first part of the proof, we then obtain

$$\begin{aligned}\sigma_{\mathrm{su}}(f(T)) &= \sigma_{\mathrm{su}}(f(T)|M) \cup \sigma_{\mathrm{su}}(f(T)|N) = \sigma_{\mathrm{su}}(g(T|M)) \cup \sigma_{\mathrm{su}}(h(T|N)) \\ &= g(\sigma_{\mathrm{su}}(T|M)) \cup h(\sigma_{\mathrm{su}}(T|N)) = f(\sigma_{\mathrm{su}}(T|M)) \cup f(\sigma_{\mathrm{su}}(T|N)) \\ &= f(\sigma_{\mathrm{su}}(T|M) \cup \sigma_{\mathrm{su}}(T|N)) = f(\sigma_{\mathrm{su}}(T)),\end{aligned}$$

where the equality $f(\sigma_{\mathrm{su}}(T|M)) \cup f(\sigma_{\mathrm{su}}(T|N)) = f(\sigma_{\mathrm{su}}(T|M) \cup \sigma_{\mathrm{su}}(T|N))$ follows since $\sigma_{\mathrm{su}}(T|M)$ and $\sigma_{\mathrm{su}}(T|N)$ are disjoint. Therefore the proof of the spectral mapping theorem for $\sigma_{\mathrm{su}}(T)$ is complete.

By duality it easily follows that $\sigma_{\mathrm{ap}}(f(T)) = f(\sigma_{\mathrm{ap}}(T))$, so also the second equality is proved. ∎

The following result shows that the SVEP at a point λ_0 may be characterized in a very simple way in the special case that T is semi-regular.

Theorem 2.49. *Suppose that $\lambda_0 I - T$ is a semi-regular operator on the Banach space X. Then the following equivalences hold:*

(i) *T has the SVEP at λ_0 precisely when $\lambda_0 I - T$ is injective or, equivalently, when $\lambda_0 I - T$ is bounded below;*

(ii) *$T^\star$ has the SVEP at λ_0 precisely when $\lambda_0 I - T$ is surjective.*

Proof (i) Assume that $\lambda_0 = 0$. Evidently we have only to prove that if T has the SVEP at 0 then T is injective. Suppose that T is not injective. The semi-regularity of T entails $T^\infty(X) = K(T)$ by Theorem 1.24, and $\{0\} \neq \ker T \subseteq T^\infty(X) = K(T)$, thus T does not have the SVEP at 0 by Theorem 2.22.

(ii) We know that if $\lambda_0 I - T$ is semi-regular then also $\lambda_0 I^\star - T^\star$ is semi-regular and by Theorem 2.42 $\lambda_0 I - T$ is surjective if and only if $\lambda_0 I^\star - T^\star$ is bounded below . ∎

Clearly Theorem 2.49 generalizes Corollary 2.24 because every surjective operator is semi-regular.

Corollary 2.50. *Let X be a Banach space and $T \in L(X)$. The following assertions hold:*

(i) *If $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{ap}}(T)$ then T has the SVEP at λ_0, but $T^\star$ fails to have the SVEP at λ_0.*

(ii) *If $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{su}}(T)$ then $T^\star$ has the SVEP at λ_0, but T fails to have the SVEP at λ_0.*

Proof The condition $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{ap}}(T)$ implies that $\lambda_0 I - T$ has closed range, is injective but not surjective, so we can apply Theorem 2.49. Analogously, if $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{su}}(T)$ then $\lambda_0 I - T$ is surjective but not injective, so we can apply again Theorem 2.49. ∎

From Theorem 1.31 we know that the semi-regular resolvent $\rho_{\mathrm{se}}(T)$ is an open subset of $\mathbb{C}$, so it may be decomposed in connected disjoint open non-empty components.

Theorem 2.51. *Let $T \in L(X)$, X a Banach space, and Ω a component of $\rho_{\rm se}(T)$. Then we have the following alternative:*

(i) *T has the SVEP at every point of Ω. In this case $\sigma_{\rm p}(T) \cap \Omega = \varnothing$;*

(ii) *For every $\lambda \in \Omega$, T does not have the SVEP. In this case $\sigma_{\rm p}(T) \supseteq \Omega$.*

Proof Suppose that T has the SVEP at a point $\lambda_0 \in \Omega$ and consider an arbitrary point λ of Ω. In order to show that T has the SVEP at λ it suffices to show, by Theorem 2.49, that $\lambda I - T$ is injective. By Theorem 2.49 $\lambda_0 I - T$ is injective, so $\mathcal{N}^\infty(\lambda_0 I - T) = \{0\}$ and therefore $H_0(\lambda_0 I - T) = \{0\}$, by part (ii) of Theorem 1.70. From Theorem 1.72 we know that the subspaces $\overline{H_0(\lambda I - T)}$ are constant for λ ranging through Ω, so that $H_0(\lambda I - T) = \{0\}$ for every $\lambda \in \Omega$. This shows that T has the SVEP at every $\lambda \in \Omega$.

The assertions on the point spectrum are clear from Theorem 2.49. ■

A very special situation is given when $\sigma_{\rm ap}(T)$ and $\sigma_{\rm su}(T)$ are contained in the boundary $\partial\sigma(T)$ of the spectrum, or, equivalently, are equal since both contain $\partial\sigma(T)$. Later we shall see that this situation is fulfilled by several classes of operators. Note first that both T and $T^\star$ have the SVEP at every point $\lambda \in \overline{\rho(T)}$.

Theorem 2.52. *Suppose that for a bounded operator $T \in L(X)$, X a Banach space, we have $\sigma_{\rm ap}(T) = \partial\sigma(T)$. Then T has the SVEP whilst $\Xi(T^\star)$ coincides with the interior of $\sigma(T)$. Similarly, if $\sigma_{\rm su}(T) = \partial\sigma(T)$ then $T^\star$ has the SVEP whilst $\Xi(T)$ coincides with the interior of $\sigma(T)$.*

Proof Suppose that $\sigma_{\rm ap}(T) = \partial\sigma(T)$. If λ belongs to the interior of $\sigma(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{\rm ap}(T)$, hence T has the SVEP at λ whilst $T^\star$ does not have the SVEP at λ, by part (i) of Corollary 2.50. Similarly the last claim is a consequence of part (ii) of Corollary 2.50. ■

Theorem 2.52 has another nice application to the so called *Césaro operator* C_p defined on the classical Hardy space $H_p(\mathbb{D})$, $\mathbb{D}$ the open unit disc and $1 < p < \infty$. The operator C_p is defined by

$$(C_p f)(\lambda) := \frac{1}{\lambda} \int_0^\lambda \frac{f(\mu)}{1-\mu}\, {\rm d}\mu \quad \text{for all } f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D}.$$

As noted by T.L. Miller, V.G. Miller and Smith [**237**], the spectrum of the operator C_p is the entire closed disc Γ_p, centered at $p/2$ with radius $p/2$, and $\sigma_{\rm ap}(C_p)$ is the boundary $\partial\Gamma_p$. Hence, the Césaro operator has the SVEP, whilst its adjoint does not have the SVEP at any point of the interior of Γ_p.

In order to find applications to Theorem 2.52, let us consider, for an arbitrary operator $T \in L(X)$ on a Banach space X, the so called *lower bound* of T defined by

$$k(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

It is obvious that if T is invertible then $k(T) = \|T^{-1}\|$. Clearly

$$k(T^n)k(T^m) \leq k(T^{n+m}) \quad \text{for all } n, m \in \mathbb{N} \tag{47}$$

and consequently $k(T) = 0$ whenever $k(T^n) = 0$ for some $n \in \mathbb{N}$. The converse is also true: if $k(T) = 0$ then $0 \in \sigma_{\rm ap}(T)$ and therefore $k(T^n) = 0$ for all $n \in \mathbb{N}$.

Theorem 2.53. *If $T \in L(X)$ then*

$$\lim_{n\to\infty} k(T^n)^{1/n} = \sup_{n\in\mathbb{N}} k(T^n)^{1/n}. \tag{48}$$

Proof Fix $m \in \mathbb{N}$ and write for all $n \in \mathbb{N}$, $n = mq + r$, $0 \leq r \leq m$, where $q := q(n)$ and $r := r(n)$ are functions of n. Note that

$$\lim_{n\to\infty} \frac{q(n)}{n} = \frac{1}{m} \quad \text{and} \quad \lim_{n\to\infty} \frac{r(n)}{n} = 0.$$

From (47) we obtain that $k(T^n) \geq k(T^m)^q k(T)^r$ and hence

$$\lim_{n\to\infty} \inf (k(T^n))^{1/n} \geq k(T^m)^{1/m} \quad \text{for all } m \in \mathbb{N}.$$

Therefore

$$\lim_{n\to\infty} \inf (k(T^n))^{1/n} \geq \sup_{n\in\mathbb{N}} k(T^n)^{1/n} \geq \lim_{n\to\infty} \sup (k(T^n))^{1/n},$$

from which the equality (48) follows. ∎

Put

$$i(T) := \lim_{n\to\infty} k(T^n)^{1/n}.$$

If $r(T)$ denotes the spectral radius of T it is obvious that $i(T) \leq r(T)$. For every bounded operator $T \in L(X)$, X a Banach space, let us consider the (possible degenerated) closed annulus

$$\Lambda(T) := \{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\}.$$

The next result shows that the approximate point spectrum is located in $\Lambda(T)$.

Theorem 2.54. *For every bounded operator $T \in L(X)$, X a Banach space, we have $\sigma_{\rm ap}(T) \subseteq \Lambda(T)$.*

Proof Clearly, if $\lambda \in \sigma_{\rm ap}(T)$ then $|\lambda| \leq r(T)$. Assume $|\lambda| < i(T)$ and let $c > 0$ be such that $|\lambda| < c < i(T)$. Take $n \in \mathbb{N}$ such that $c^n \leq k(T^n)$. For every $x \in X$ we have $c^n\|x\| \leq \|T^n x\|$ and hence

$$\|(\lambda^n I - T^n)x\| \geq \|T^n x\| - |\lambda^n| \, \|x\| \geq (c^n - |\lambda^n|)\|x\|,$$

thus $\lambda^n I - T^n$ is bounded below, $\lambda^n \notin \sigma_{\rm ap}(T)$. Writing

$$\lambda^n I - T^n = (\lambda I - T)(T^{n-1} + \lambda T^{n-2} + \cdots + \lambda^n I),$$

we then conclude that $\lambda \notin \sigma_{\rm ap}(T)$. ∎

As usual by $\mathbb{D}(0, \varepsilon)$ and $\mathbf{D}(0, \varepsilon)$ we shall denote the open disc and the closed disc centered at 0 and radius ε, respectively.

Theorem 2.55. *For a bounded operator $T \in L(X)$, X a Banach space, the following properties hold:*

(i) *If T is invertible then $\mathbb{D}(0, i(T)) \subseteq \rho(T)$, and consequently $\sigma(T) \subseteq \Lambda(T)$. If T is non-invertible then $\mathbf{D}(0, i(T)) \subseteq \sigma(T)$;*

(ii) *Suppose that $i(T) = r(T)$. If T is invertible then $\sigma(T) \subseteq \partial\mathbb{D}(0, r(T))$, whilst if T is non-invertible then*

$$\sigma(T) = \mathbf{D}(0, r(T)) \quad \text{and} \quad \sigma_{\rm ap}(T) = \partial\sigma(T).$$

Proof (i) Let T be invertible and assume that there is some $\lambda \in \sigma(T)$ such that $|\lambda| < i(T)$. Since $0 \in \rho(T)$ then there is some μ in the boundary of $\sigma(T)$ such that $|\mu| \leq |\lambda| < i(T)$. But this is impossible since, again by Theorem 2.42, $\mu \in \sigma_{\rm ap}(T)$ and hence by Theorem 2.54 $|\mu| \geq i(T)$. This shows the first assertion of (i).

Suppose now that T is non-invertible and that there is an element $\lambda \in \rho(T)$ for which $|\lambda| \leq i(T)$. By assumption $0 \in \sigma(T)$ and $\rho(T)$ is open, so there exists $0 \leq c < 1$ such that $c\lambda$ belongs to the boundary of $\sigma(T)$. From Theorem 2.42 we then conclude that $c\lambda \in \sigma_{\rm ap}(T)$. On the other hand, $|c\lambda| < i(T)$, so by Theorem 2.54 $c\lambda \notin \sigma_{\rm ap}(T)$, a contradiction. This shows the second assertion of part (i).

(ii) The inclusion $\sigma(T) \subseteq \partial\mathbb{D}(0, r(T))$, if T is invertible, and the equality $\sigma(T) = \mathbf{D}(0, r(T))$, if T is non-invertible, are simple consequences of part (i).

Suppose now that if T is non-invertible there exists some $\lambda \in \sigma(T)$ such that $|\lambda| = 1$ and $\lambda \notin \sigma_{\rm ap}(T)$. By Corollary 2.50 T^* then fails the SVEP at λ and this contradicts the fact that λ belongs to the boundary of the spectrum. Therefore $\partial\sigma(T) \subseteq \sigma_{\rm ap}(T)$ and from Theorem 2.54 it then follows that $\partial\sigma(T) = \sigma_{\rm ap}(T)$. ∎

Theorem 2.56. *Let $T \in L(X)$, X a Banach space, and suppose that $\lambda \in \mathbb{C}$ is a point for which $|\lambda| < i(T)$. Then T has the SVEP at λ, whilst $T^\star$ has the SVEP at λ if and only if T is invertible.*

Proof From Theorem 2.54 we know that if $|\lambda| < i(T)$ then $\lambda \notin \sigma_{\rm ap}(T)$. Hence the assertions easily follow from Corollary 2.50. ∎

The following corollary describes the SVEP in the special case $i(T) = r(T)$.

Corollary 2.57. *Let $T \in L(X)$, X a Banach space, and suppose that $i(T) = r(T)$. Then the following dichotomy holds:*

(i) *If T is invertible then both T and $T^\star$ have the SVEP;*

(ii) *If T is non-invertible then T has the SVEP, whist $T^\star$ has the SVEP at a point λ precisely when $|\lambda| \geq r(T)$.*

Corollary 2.57 applies, in particular to an arbitrary isometry $T \in L(X)$. Hence every isometry has the SVEP, whilst the adjoint of a non-invertible isometry has the SVEP at a point $\lambda \in \mathbb{C}$ if and only if $|\lambda| \geq 1$.

Trivially, an operator $T \in L(X)$ has the SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it easily follows that $T \in L(X)$ has the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. Hence, we have the implications:

$$\sigma_{\mathrm{ap}}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T \text{ has the SVEP at } \lambda_0, \tag{49}$$

and

$$\sigma_{\mathrm{su}}(T) \text{ does not cluster at } \lambda_0 \Rightarrow T^\star \text{ has the SVEP at } \lambda_0. \tag{50}$$

The first implication may be proved by using the same argument of the proof of part (c) of Remark 2.4. The second implication is an immediate consequence of the equality $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T^\star)$.

It is easily seen that none of the implications (49) and (50) may be reversed in general. In fact, as observed in Theorem 2.42, the boundary $\partial\sigma(T)$ of $\sigma(T)$ is contained in $\sigma_{\mathrm{ap}}(T)$ and in $\sigma_{\mathrm{su}}(T)$. Consequently, if $\sigma(T)$ contains a non-isolated boundary point λ_0 of $\sigma(T)$ then $\sigma_{\mathrm{ap}}(T)$, as well as $\sigma_{\mathrm{su}}(T)$, clusters at λ_0 but, as observed before, both T and $T^\star$ have the SVEP at λ_0. In the next chapter we shall see that for operators of Kato type the implications (49) and (50) may be reversed.

From the implications (49) and (50) we know that T and $T^\star$ have the SVEP outside $\sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{su}}(T)$, respectively. The question of the extent to which an operator T or its adjoint $T^\star$ may have SVEP, or not, at the points inside the approximate point or surjectivity spectrum is a more delicate issue.

The next result on non-invertible isometries will be useful to settle this question in the case of certain operators.

Theorem 2.58. *Let $T \in L(X)$ be a non-invertible isometry and suppose that $f : \mathcal{U} \to \mathbb{C}$ is a non-constant analytic function on some connected open neighborhood of the closed unit disc. Then the following assertions hold:*

(i) *$\sigma(f(T)) = f(\overline{\mathbb{D}})$ and $\sigma_{\mathrm{ap}}(f(T)) = f(\partial\mathbb{D})$, where $\mathbb{D}$ denotes the open unit disc of $\mathbb{C}$.*

(ii) *$f(T)$ has the SVEP.*

(iii) *$f(T)^\star$ has the SVEP at a point λ if and only if $\lambda \notin f(\mathbb{D})$.*

(iv) *$f(\partial\mathbb{D}) \cap f(\mathbb{D}) = \{\lambda \in \mathbb{C} : f(T)^\star$ does not have the SVEP at $\lambda\}$.*

Proof Since $\sigma(f(T)) = \overline{\mathbb{D}}$ and, by Theorem 2.54, $\sigma_{\mathrm{ap}}(T) = \partial\mathbb{D}$, the equalities (i) follow from the canonical spectral mapping theorem and Theorem 2.48, respectively. The assertion (ii) is a consequence of Corollary 2.57 and the spectral mapping Theorem 2.39.

(iii) Since $f(T)^\star = f(T^\star)$, from Theorem 2.39 it follows that $f(T)^\star$ has the SVEP at the point $\lambda \in \mathbb{C}$ if and only if $T^\star$ has the SVEP at each point $\mu \in \mathcal{U}$ for which $f(\mu) = \lambda$. Corollary 2.57 then ensures that the latter condition holds precisely when $\lambda \notin f(\mathbb{D})$.

The assertion (iv) easily follows from part (i) and part (iii). ∎

Part (iv) of Theorem 2.58 leads to many examples in which the SVEP for the adjoint fails to hold at points which belong to the approximate point spectrum of T. In fact, if f is a non-constant analytic function on some connected open neighborhood $\mathcal{U}$ of the closed unit disc and $\Lambda := f(\partial\mathbb{D}) \cap f(\mathbb{D})$ is non empty then for every $\lambda \in \Lambda$ the adjoint of $f(T)$ does not have SVEP at λ. This situation is, for instance, fulfilled for every function of the form

$$f(\lambda) := (\lambda - \gamma)(\lambda - \omega)g(\lambda) \quad \text{for } \lambda \in \mathcal{U},$$

where g is an arbitrary analytic function on $\mathcal{U}$, $|\gamma| = 1$ and $|\omega| < 1$.

In the next example we give an application of Theorem 2.58 to operators which act on the Hardy space $H_2(\mathbb{D})$.

Example 2.59. Let $\mathbb{D}$ denote the open unit disc of $\mathbb{C}$ and $1 \leq p < infty$. If $g \in H^\infty(\mathbb{D})$. The operator T_f on $H^p(\mathbb{D})$ defined by the assignment

$$T_f g := fg \quad \text{for every } g \in H^p(\mathbb{D})$$

is called the *analytic Toeplitz operator with symbol* f.

Let f be a non-constant analytic function on some connected open neighborhood of the closed unit disc, and consider the multiplication Toeplitz operator T_f on $H^2(\mathbb{D})$. It is clear that $T_f = f(T)$, where T denotes the operator of multiplication by the independent variable,

$$(Tg)(\lambda) := \lambda g(\lambda) \quad \text{for all } g \in H^2(\mathbb{D}),\ \lambda \in \mathbb{D}.$$

The operator T is a non-invertible isometry (actually, is unitary equivalent to the unilateral right shift on $\ell^2(\mathbb{N})$), thus by part (i) of Theorem 2.58 we have

$$\sigma(T_f) = f(\sigma(T)) = f(\overline{\mathbb{D}}) \quad \text{and} \quad \sigma_{\mathrm{ap}}(T_f) = f(\partial\mathbb{D}).$$

Moreover, from part (iv) of Theorem 2.58, we conclude that the adjoint $T_f{}^\star$ has the SVEP at $\lambda \in \mathbb{C}$ if and only if $\lambda \notin f(\mathbb{D})$.

Note that similar results holds for Toeplitz operators with arbitrary bounded analytic symbols. In fact, if $f \in H^\infty(\mathbb{D})$ the approximate point spectrum $\sigma_{\mathrm{ap}}(T_f)$ coincides with the essential range of the boundary function that is obtained by taking non-tangential limits of f almost everywhere on the unit circle, and operator the T_f does not have the SVEP at any $\lambda \notin f(\mathbb{D})$. These results may be established using standard tools from theory of Hardy spaces, see [**265**].

Example 2.60. We conclude this section by mentioning an example from theory of composition operators on Hardy spaces. It is well known that every analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ on the open unit disc $\mathbb{D}$ induces a bounded linear operator on $H^p(\mathbb{D})$ defined by

$$T_\varphi(f) := f \circ \varphi \quad \text{for all } f \in H^p(\mathbb{D}).$$

The operator T_φ is an isometry which is invertible if and only if φ is an automorphism of $\mathbb{D}$, a mapping of the form

$$\varphi(\lambda) = \frac{a\lambda + \overline{b}}{b\lambda + \overline{a}} \quad \text{for all } \lambda \in \mathbb{D} \ ,$$

where a and b are complex numbers for which $|a|^2 - |b|^2 = 1$. The automorphisms φ are classified as follows:

- φ is *elliptic* if $|\operatorname{Im} a| > |b|$;
- φ is *parabolic* if $|\operatorname{Im} \ a| = |b|$;
- φ is *hyperbolic* if $|\operatorname{Im} a| < |b|$.

If φ is either elliptic or parabolic then a result of Smith [**300**] shows that the composition operator T_φ on $H^p(\mathbb{D})$ and its adjoint has the SVEP (actually we have much more, T_φ is generalized scalar and therefore decomposable, see § 1.5 of Laursen and Neumann [**214**]).

On the other hand, from an inspection of the proof of Theorem 6 of Nordgreen [**249**] and Theorems 1.4 and 2.3 of Smith [**300**] it easily follows that if φ is hyperbolic the spectrum of the corresponding composition operator T_φ on $H^2(\mathbb{D})$ is the annulus $\Gamma_r := \{\lambda \in \mathbb{C} : 1/r \leq |\lambda| \leq r\}$ for some $r > 1$. Moreover, T_φ does not have the SVEP at λ if and only if λ belongs to the interior of Γ_r. We mention that the adjoint ${T_\varphi}^\star$ has the SVEP since it is subnormal, see [**214**].

In contrast, if an automorphism of $\mathbb{D}$ is either elliptic or parabolic, then, as shown by Smith [**300**], the corresponding composition operator is generalized scalar on $H^p(\mathbb{D})$ for arbitrary $1 \leq p < \infty$, thus both T and $T^\star$ have SVEP.

4. Algebraic spectral subspaces

In this section we shall introduce an important class of subspaces related to an operator $T \in L(X)$, where X is a Banach space. These spaces are defined in purely algebraic terms.

Definition 2.61. *Let X be a vector space and $T : X \to X$ be a linear mapping. Given an arbitrary subset $\Omega \subseteq \mathbb{C}$ the* algebraic spectral subspace *$E_T(\Omega)$ is defined as the algebraic sum of all subspaces M of X with the property that $(\lambda I - T)(M) = M$ for every $\lambda \in \mathbb{C} \setminus \Omega$.*

Evidently $E_T(\Omega)$ is the largest subspace of X on which all the restrictions of $\lambda I - T$, $\lambda \in \mathbb{C} \setminus \Omega$, are surjective, in particular

$$(\lambda I - T)(E_T(\Omega)) = E_T(\Omega) \quad \text{for every } \lambda \in \mathbb{C} \setminus \Omega.$$

Note that

$$E_T(\mathbb{C} \setminus \{\lambda\}) = C(\lambda I - T)$$

where, as usual, $C(\lambda I - T)$ denotes the algebraic core of $\lambda I - T$. If $\lambda I - T$ is semi-regular then by Theorem 1.24

$$E_T(\mathbb{C} \setminus \{\lambda\}) = K(\lambda I - T) = (\lambda I - T)^\infty(X).$$

Clearly, if $\Omega_1 \subseteq \Omega_2$ are two subsets of $\mathbb{C}$ then $E_T(\Omega_1) \subseteq E_T(\Omega_2)$.

The subspace $E_T(\varnothing)$ is of particular interest. By definition $E_T(\varnothing)$ is the largest linear subspace M of X for which $(\lambda I - T)(M) = M$ holds for every $\lambda \in \mathbb{C}$. It is easily seen that $E_T(\varnothing) = E_T(\Omega)$ whenever $\sigma(T) \cap \Omega = \varnothing$. The space $E_T(\varnothing)$, which is usually called the *largest divisible subspace* for T, plays a remarkable role in theory of automatic continuity where it is often necessary to exclude the existence of non-trivial divisible subspaces. For an extensive treatment of this theory we refer to the monograph by Laursen and Neumann [**214**].

Our next results concern some other basic properties of the subspaces $E_T(\Omega)$.

Theorem 2.62. *Let X be a Banach space and $T \in L(X)$. For each proper subset $\Omega \subseteq \mathbb{C}$ the subspace $E_T(\Omega)$ is hyper-invariant under T, i.e., $E_T(\Omega)$ is invariant under any linear map S commuting with T.*

Proof In fact, for $\lambda \notin \Omega$ we have

$$(\lambda I - T)S(E_T(\Omega)) = S(\lambda I - T)(E_T(\Omega)) = E_T(\Omega),$$

thus $S(E_T(\Omega)) \subseteq E_T(\Omega)$. ∎

Corollary 2.63. *For each proper subset $\Omega \subseteq \mathbb{C}$ we have $E_T(\Omega) = E_T(\Omega \cap \sigma(T))$.*

Proof Clearly, for each $\lambda \in \Omega \setminus \sigma(T)$ the operator $(\lambda I - T)^{-1}$ commutes with T. Hence $(\lambda I - T)^{-1}(E_T(\Omega)) \subseteq E_T(\Omega)$, and so $E_T(\Omega) = (\lambda I - T)E_T(\Omega)$ for all $\lambda \in \Omega \setminus \sigma(T)$ as well as for all $\lambda \notin \Omega$. It follows that $E_T(\Omega) \subseteq E_T(\Omega \cap \sigma(T))$. The other inclusion is obvious. ∎

Theorem 2.64. *Let T be a linear operator on a vector space X , $\Omega \subseteq \mathbb{C}$ and $\lambda_0 \in \Omega$. Then $x \in E_T(\Omega)$ if and only if $(\lambda_0 I - T)x \in E_T(\Omega)$.*

Proof Let $\lambda_0 \in \Omega$ and suppose that $x \in E_T(\Omega)$. Since $(\lambda I - T)E_T(\Omega) = E_T(\Omega)$ for every $\lambda \notin \Omega$, we see that

$$(\lambda_0 I - T)x = (\lambda I - T)x + (\lambda_0 - \lambda)x \in E_T(\Omega).$$

Conversely, let $y_0 \in X$ such that $(\lambda_0 I - T)y_0 \in E_T(\Omega)$. Let $Y := E_T(\Omega) + \mathbb{C}y_0$ and consider $z := y + \alpha y_0 \in Y$, with $y \in E_T(\Omega)$ and $\alpha \in \mathbb{C}$.

Clearly, if $\lambda \notin \Omega$ we have

$$y - \frac{\alpha}{\lambda - \lambda_0}(\lambda_0 I - T)y_0 \in E_T(\Omega).$$

From the equality $(\lambda I - T)(E_T(\Omega)) = E_T(\Omega)$ we obtain an element $y_1 \in E_T(\Omega)$ such that

$$(\lambda I - T)y_1 = y - \frac{\alpha}{\lambda - \lambda_0}(\lambda_0 I - T)y_0.$$

For the element

$$z_0 := y_1 + \frac{\alpha}{\lambda - \lambda_0} y_0$$

we have

$$\begin{aligned}(\lambda I - T)z_0 &= (\lambda I - T)y_1 + \frac{\alpha}{\lambda - \lambda_0}(\lambda I - T)y_0 \\ &= y - \frac{\alpha}{\lambda - \lambda_0}(\lambda_0 I - T)y_0 + \frac{\alpha}{\lambda - \lambda_0}(\lambda_0 I - T)y_0 + \alpha y_0 \\ &= y + \alpha y_0 = z.\end{aligned}$$

Hence $z \in (\lambda I - T)(Y)$ and therefore the inclusion $Y \subseteq (\lambda I - T)(Y)$ is verified for every $\lambda \notin \Omega$. It is easy to see that also the inverse inclusion holds. Indeed, for every element $z = y + \alpha y_0 \in Y$ and $\lambda \notin \Omega$ we have

$$(\lambda I - T)z = (\lambda I - T)y + (\lambda I - T)\alpha y_0 = (\lambda I - T)y +$$
$$\alpha(\lambda_0 I - T)y_0 + \alpha(\lambda - \lambda_0)y_0 \in E_T(\Omega) \ + \mathbb{C}y_0 = Y$$

This implies that $Y \subseteq E_T(\Omega)$ and therefore $y_0 \in E_T(\Omega)$. ■

Corollary 2.65. ker $(\lambda_0 I - T) \subseteq E_T(\Omega)$ *for every* $\lambda_0 \in \Omega$. ■

Theorem 2.66. *Let X be a Banach space, $T \in L(X)$, $\Omega \subseteq \mathbb{C}$. Then*

$$\sigma(T\ |E_T(\Omega)) \subseteq \sigma(T\ |\overline{E_T(\Omega)}) \subseteq \sigma(T).$$

Proof Let $Y := \overline{E_T(\Omega)}$ and suppose $\lambda \in \rho(T|Y)$. Then $\lambda I - T$ is injective on $E_T(\Omega)$. Let $y \in E_T(\Omega)$ and let $z \in Y$ be chosen so that $(\lambda I - T)z = y$. Observe that if $\lambda \notin \Omega$ then by Theorem 2.64 $z \in E_T(\Omega)$ (indeed there is a pre-image of y in $E_T(\Omega)$ and since $\lambda \in \rho(T|Y)$ this pre-image is unique in Y, so it must be z). Hence $\lambda I - T$ is injective and onto $E_T(\Omega)$. Therefore

$$\rho(T|Y) \subseteq \mathbb{C} \setminus \ \sigma(T|E_T(\Omega))$$

and hence $\sigma(T|E_T(\Omega)) \subseteq \sigma(T|Y)$.

The second inclusion is almost obvious: if $\lambda \notin \sigma(T)$ then $\lambda \notin \sigma(T) \cap \Omega$, and consequently,

$$(\lambda I - T)(E_T(\Omega)) = (\lambda I - T)(E_T(\sigma(T) \cap \Omega)) = E_T(\sigma(T) \cap \Omega) = E_T(\Omega).$$

Therefore $(\lambda I - T)(Y) = Y$. Since $\lambda I - T$ is injective, this shows that $\lambda \notin \sigma(T|Y)$. ■

Next we want show that the algebraic spectral subspace of the intersection of any family of proper subsets (Ω_j) of $\mathbb{C}$ coincides with the intersection of all $E_T(\Omega_j)$. First we need some preliminary results.

Let $\pi : X \to X/E_T(\Omega)$ denote the canonical quotient map and let $\widetilde{T} : X/E_T(\Omega) \to X/E_T(\Omega)$ be the map

$$\widetilde{T}\tilde{x} := \widetilde{Tx}, \text{ with } x \in \tilde{x}.$$

It is evident that $\widetilde{T}\pi = \pi T$.

Lemma 2.67. *Let $\Omega \subseteq \Omega$ be proper subsets of $\mathbb{C}$. Then $E_{\widetilde{T}}(\Omega) = \pi(E_T(\Omega))$.*

Proof Observe first that if $\lambda \notin \Omega$ then

$$(\lambda\widetilde{I} - \widetilde{T})(\pi(E_T(\Omega))) = \pi(\lambda I - T)(E_T(\Omega)) = \pi(E_T(\Omega)),$$

thus $\pi(E_T(\Omega)) \subseteq E_{\widetilde{T}}(\Omega)$.

To prove the inverse inclusion, let us consider $M := \pi^{-1}E_{\widetilde{T}}(\Omega)$. We shall prove that $(\lambda I - T)(M) = M$ for each $\lambda \notin \Omega$. Clearly,

$$\pi(\lambda I - T)(M) = (\lambda\widetilde{I} - \widetilde{T})(\pi(M)) = (\lambda\widetilde{I} - \widetilde{T})(E_{\widetilde{T}}(\Omega)) = E_{\widetilde{T}}(\Omega),$$

and therefore

$$(\lambda I - T)(M) = \pi^{-1}(E_{\widetilde{T}}(\Omega)) = M.$$

On the other hand, if $x \in M$ then $\tilde{x} = \pi x \in E_{\widetilde{T}}(\Omega)$, thus there exists an element $\tilde{y} = \pi y \in E_{\widetilde{T}}(\Omega)$ such that $(\lambda\widetilde{I} - \widetilde{T})\tilde{y} = \tilde{x}$ This implies that $x - (\lambda I - T)y \in E_T(\Omega)$. Since $\lambda \notin \Omega$ the equality $(\lambda I - T)(E_T(\Omega) = E_T(\Omega)$ also holds, so there exists $z \in E_T(\Omega)$ for which

$$x = (\lambda I - T)y + (\lambda I - T)z = (\lambda I - T)(y + z).$$

From the equality $\pi(y + z) = \pi y \in E_{\widetilde{T}}(\Omega)$ follows that $y + z \in M$ so $M \subseteq (\lambda I - T)(M)$. Hence $M = (\lambda I - T)(M)$ for every $\lambda \notin M$ and this implies $M = \pi^{-1}(E_{\widetilde{T}}(\Omega)) \subseteq E_T(\Omega)$.

Hence also the inclusion $E_{\widetilde{T}}(\Omega) \subseteq \pi(E_T(\Omega))$ holds, so the proof is complete. ■

Lemma 2.68. *Let $\{\Omega_1, \dots, \Omega_n\}$ be a finite family of proper subsets of $\mathbb{C}$. Then*

$$E_T\left(\bigcap_{k=1}^{n} \Omega_k\right) = \bigcap_{k=1}^{n} E_T(\Omega_k).$$

Proof Obviously it suffices to consider only the case of two sets Ω_1 and Ω_2. Moreover, by Corollary 2.63 we may assume that Ω_1 and Ω_2 are subsets of $\sigma(T)$.

Suppose first that $\Omega_1 \cup \Omega_2 = \sigma(T)$. Let $Y := E_T(\Omega_1) \cup E_T(\Omega_2)$ and let $\lambda \notin \Omega_1 \cap \Omega_2$. As we have seen in the proof of Corollary 2.63, $E_T(\Omega) = (\lambda I - T)E_T(\Omega)$ for any $\Omega \subset \mathbb{C}$ and $\lambda \notin \sigma(T)$. So we may assume $\lambda \in \sigma(T)$ and hence $\lambda \in \Omega_1 \setminus \Omega_2$ or $\lambda \in \Omega_2 \setminus \Omega_1$.

Consider the case $\lambda \in \Omega_1 \setminus \Omega_2$. If $x_0 \in Y$ then there exists $y \in E_T(\Omega_2)$ so that $x_0 = (\lambda I - T)y$. But $\lambda \in \Omega_1$ and $x_0 \in E_T(\Omega_1)$, so by Theorem 2.64 we have $y \in E_T(\Omega_1)$. Hence $(\lambda I - T)(Y) = Y$ for every $\lambda \notin \Omega_1 \cap \Omega_2$ and, by maximality, $Y \subseteq E_T(\Omega_1 \cap \Omega_2)$. As the other inclusion is obvious we conclude that $Y = E_T(\Omega_1 \cap \Omega_2)$.

Now let us consider the general case Ω_1 and Ω_2 are arbitrary subsets of $\sigma(T)$. Let $\Omega_3 := \Omega_2 \cup (\sigma(T) \setminus \Omega_1)$. Then

$$\begin{aligned} E_T(\Omega_1) \cap E_T(\Omega_2) &\subseteq E_T(\Omega_1) \cap E_T(\Omega_3) = E_T(\Omega_1 \cap \Omega_3) \\ &= E_T(\Omega_1 \cap \Omega_2) \subseteq E_T(\Omega_1) \cap E_T(\Omega_2), \end{aligned}$$

so the proof is complete. ■

Theorem 2.69. *Suppose $(\Omega_j)_{j\in J})$ is a family of proper subsets of $\mathbb{C}$. Then*

$$E_T\left(\bigcap_{j\in J}\Omega_j\right) = \bigcap_{j\in J} E_T(\Omega_j).$$

Proof Let $Y := \bigcap_{j\in J} E_T(\Omega_j)$. Clearly $\bigcap_{j\in J} E_T(\Omega_j) \subseteq Y$. Since the statement has already been proved whenever J is a finite set, there is no harm in replacing the index set J by the set of finite subsets of J, ordered by inclusion. Thus we may assume without loss of generality that J is a directed set and that $j_1 \geq j_2$ implies $\Omega_{j_1} \subseteq \Omega_{j_2}$.

Let $\lambda \notin \bigcap_{j\in J}\Omega_j$ and choose j_o such that $\lambda \notin \Omega_{j_o}$. Let

$$Z := \bigcap_{j\geq j_o} E_T(\Omega_j).$$

Clearly $Y \subseteq Z$. On the other hand, if $z \in Z$ and if $E_T(\Omega_j)$ is given then there exists some $j' \geq j_o$ for which $j' \geq j$ also holds. From this we then obtain that $z \in E_T(\Omega_{j'}) \subseteq E_T(\Omega_j)$, so $z \in Y$. To show that $(\lambda I - T)(Y) = Y$ holds for each $\lambda \notin \bigcap_{j\in J}\Omega_j$ it therefore suffices to show that $(\lambda I - T)(Z) = Z$ for the λ chosen above.

Consider first the case $E_T(\varnothing) = \{0\}$. Since $\lambda \notin \bigcap_{j\in J}\Omega_j$ there corresponds to our given $z \in Z$ an element $z_j \in E_T(\Omega_j)$ such that

$$(\lambda I - T)z_j = z \quad \text{for every} \quad j \geq j_o.$$

Let us consider $j', j'' \geq j_o$ and take $j''' \geq j'$, $j''' \geq j''$. If $z_{j'} = z_{j''}$ then $z_{j'''} = z_{j'}$, say. Hence $z_{j'''} - z_{j'} \in \ker\,(\lambda I - T) \subseteq E_T(\{\lambda\})$ and therefore

$$z_{j'''} - z_{j'} \in E_T(\Omega_{j'}) \cap E_T(\{\lambda\}) = E_T(\varnothing).$$

From this it follows that if $(\lambda I - T)z_j = z$ then $z_j \in E_T(\Omega_j)$ is independent of $j \geq j_o$. Thus $z_j \in Z$ and the surjectivity of $\lambda I - T$ has been established for every $\lambda \notin \bigcap_{j\in J}\Omega_j$. This completes the proof in the case $E_T(\varnothing) = \{0\}$.

To remove this additional assumption we can invoke Lemma 2.67. In $X \setminus E_T(\varnothing)$ we have $E_{\widetilde{T}}(\varnothing) = \pi(E_T(\varnothing)) = \{0\}$. Hence, by our previous results,

$$E_{\widetilde{T}}\left(\bigcap_{j\in J}\Omega_j\right) = \bigcap_{j\in J} E_{\widetilde{T}}(\Omega_j)$$

and so

$$E_T\left(\bigcap_{j\in J}\Omega_j\right) = \pi^{-1}\left(E_{\widetilde{T}}(\bigcap_{j\in J}\Omega_j)\right) = \bigcap_{j\in J}\pi^{-1}(E_{\widetilde{T}}(\Omega_j)) = \bigcap_{j\in J} E_T(\Omega_j).$$

■

As an immediate consequence of Theorem 2.69, we can express an algebraic spectral subspace as intersection of cores.

Corollary 2.70. *For every proper subset* Ω *of* $\mathbb{C}$ *we have*

$$E_T(\mathbb{C}\setminus\Omega)=\bigcap_{\lambda\in\Omega}E_T(\mathbb{C}\setminus\{\lambda\})=\bigcap_{\lambda\in\Omega}C(\lambda I-T)$$

Next, we shall introduce a property, introduced by Dunford, which is strictly stronger than the SVEP. First we give some additional informations on the relationships between the SVEP and local spectral subspaces $X_T(\Omega)$.

Theorem 2.71. *Let* $T\in L(X)$, X *a Banach space, has the SVEP and let* $\Omega\subseteq\mathbb{C}$ *be a closed subset such that* $X_T(\Omega)$ *is closed. Then*

$$\sigma(T|X_T(\Omega))\subseteq\Omega\cap\sigma(T).$$

Proof Let $\widetilde{T}$ denote the restriction of T to the invariant closed subspace $X_T(\Omega)$. Clearly, $\lambda\widetilde{I}-\widetilde{T}$ has the SVEP and by Theorem 2.6, part (iii), $\lambda\widetilde{I}-\widetilde{T}$ is surjective for every $\lambda\notin\Omega$. By Corollary 2.24, then $\lambda I-\widetilde{T}$ is invertible for every $\lambda\notin\Omega$, hence $\sigma(\widetilde{T})\subseteq\Omega$.

It remains to prove that $\sigma(\widetilde{T})\subseteq\sigma(T)$. Suppose that $\lambda\notin\sigma(T)$. Clearly, if $\lambda\notin\Omega$ from the inclusion $\sigma(\widetilde{T})\subseteq\Omega$, it follows that $\lambda\notin\sigma(\widetilde{T})$. Consider the other case $\lambda\in\Omega$. Since $\lambda I-T$ is bijective, for every $y\in X_T(\Omega)$ there exists $x\in X$ such that $y=(\lambda I-T)x$. From Theorem 2.6, part (iv), it then follows that $x\in X_T(\Omega)$, thus $X_T(\Omega)\subseteq(\lambda I-T)(X_T(\Omega))$. Since the reverse inclusion is always true, see Theorem 2.6, part (i), it follows that $X_T(\Omega)=(\lambda I-T)(X_T(\Omega))$, so $\lambda I-\widetilde{T}$ is bijective. Hence also in this case $\lambda\notin\sigma(\widetilde{T})$. ■

Theorem 2.72. *For every bounded operator* $T\in L(X)$, X *a Banach space, we have*

$$X_T(\Omega)\subseteq E_T(\Omega)\quad\text{for every }\Omega\subseteq\mathbb{C}.$$

Moreover, if $\Omega\subseteq\mathbb{C}$ *is closed and* $E_T(\Omega)$ *is closed then* $E_T(\Omega)=X_T(\Omega)$.

Proof Consider any subset $\Omega\subseteq\mathbb{C}$. If $x\in X_T(\Omega)$ then $(\lambda I-T)x\in X_T(\Omega)$ for every $\lambda\in\rho_T(x)$. The inclusion $\rho_T(x)\supseteq\mathbb{C}\setminus\Omega$ then ensures that

$$X_T(\Omega)\subseteq(\lambda I-T)X_T(\Omega)\quad\text{ for all }\lambda\in\mathbb{C}\setminus\Omega.$$

The reverse inclusion is an obvious consequence of part(i) of Theorem 2.6. Hence

$$(\lambda I-T)X_T(\Omega)=X_T(\Omega)\quad\text{for every }\lambda\in\mathbb{C}\setminus\Omega$$

and this implies $X_T(\Omega)\subseteq E_T(\Omega)$.

To prove the second assertion, suppose that $\Omega\subseteq\mathbb{C}$ is closed and that $E_T(\Omega)$ is closed. Let $Y:=E_T(\Omega)$. By Theorem 2.43 we know that

$$\sigma_{\rm su}(T|Y)=\bigcup_{y\in Y}\sigma_{T|Y}(x).$$

Obviously, if $x \in Y$ then $\rho_{T|Y}(x) \subseteq \rho_T(y)$ and therefore

$$\sigma_T(x) \subseteq \sigma_{T|Y}(x) \subseteq \sigma_{\mathrm{su}}(T|Y).$$

Since $Y = E_T(\Omega)$, by definition the restriction $\lambda I - T$ on Y is surjective for every $\lambda \notin \Omega$, so $\sigma_{\mathrm{su}}(T|Y) \subseteq \Omega$. This proves that inclusion $E_T(\Omega)$ is contained in $X_T(\Omega)$, and since the inverse inclusion has already been proved, the proof is complete. ∎

Later we shall exhibit an example for which $X_T(\Omega)$ is a proper subset of $E_T(\Omega)$. In Chapter 7 we shall prove that for the super-decomposable operators $T \in L(X)$, for which $E_T(\varnothing) = \{0\}$, then the equality $E_T(\Omega) = X_T(\Omega)$ holds for every closed set $\Omega \subseteq \mathbb{C}$.

Theorem 2.73. *Let X be a Banach space, $T \in L(X)$ and $\Omega \subseteq \mathbb{C}$ a connected component of the semi-regular resolvent $\rho_{\mathrm{se}}(T)$. For any $\lambda \in \Omega$ we have*

$$\begin{aligned}(\lambda I - T)^\infty(X) &= K(\lambda I - T) = E_T(\mathbb{C} \setminus \Omega) \\ &= X_T(\mathbb{C} \setminus \Omega) = \bigcap_{|\mu|<\varepsilon} (\mu I - T)(X),\end{aligned}$$

where $\varepsilon > 0$ is such that $\mu \in \Omega$ for all $|\lambda| < \varepsilon$.

Proof By Theorem 1.36 the spaces $C(\lambda I - T) = K(\lambda I - T)$ are constant as λ ranges through Ω, and hence by Corollary 2.70 $E_T(\mathbb{C} \setminus \Omega) = K(\lambda I - T)$. By Theorem 1.24 we know that $K(\lambda I - T)$ is closed, and therefore by Theorem 2.72

$$K(\lambda I - T) = E_T(\mathbb{C} \setminus \Omega) = X_T(\mathbb{C} \setminus \Omega).$$

Let $\varepsilon > 0$ be such that $\mu \in \Omega$ for all $|\mu| < \varepsilon$. From Theorem 1.39 it easily follows that

$$K(\lambda I - T) \supseteq \bigcap_{|\mu|<\varepsilon} (\mu I - T)(X). \tag{51}$$

On the other hand, again by Theorem 1.24 we have

$$K(\lambda I - T) = K(\mu I - T) = (\mu I - T)^\infty(X) \subseteq (\mu I - T)(X)$$

for arbitrary λ, $\mu \in \Omega$, so the reverse inclusion of (51) holds. ∎

The following result may be viewed as the local version of the inclusion $\partial\sigma(T) \subseteq \sigma_{\mathrm{se}}(T)$ established in Theorem 1.75.

Corollary 2.74. *Let $T \in L(X)$ be a bounded operator on a Banach space X. Then $\partial\sigma_T(x) \subseteq \sigma_{\mathrm{se}}(T)$ for every $x \in X$.*

Proof Let $x \in X$ be arbitrarily given and suppose that there is a point $\lambda \in \partial\sigma_T(x)$ for which $\lambda I - T$ is semi-regular. Let Ω be the component which contains λ. The set Ω is open and $\lambda \in \partial\sigma_T(x) \cap \Omega$, thus Ω cannot be contained in $\sigma_T(x)$. Consider a point $\mu \in \Omega \setminus \sigma_T(x)$. By Theorem 2.73 it follows that $x \in X_T(\mathbb{C} \setminus \{\mu\}) = X_T(\mathbb{C} \setminus \Omega)$ and hence $\sigma_T(x) \subseteq \mathbb{C} \setminus \Omega$, but this is impossible since $\lambda \in \partial\sigma_T(x) \cap \Omega$. ∎

Corollary 2.75. *Suppose that $T \in L(X)$, X a Banach space, has a spectrum $\sigma(T)$ with connected interior Γ such that $\sigma(T) = \overline{\Gamma}$. If T has the SVEP and $\sigma_{\rm ap}(T) = \partial\sigma(T)$ then for every $x \in X$ either $\sigma_T(x) = \sigma(T)$ or $\sigma_T(x) = \partial\sigma(T)$.*

Proof From Corollary 2.45 we know that $\sigma_{\rm ap}(T) = \sigma_{\rm se}(T)$. Suppose that $\sigma_T(x) \neq \sigma(T)$. If $\sigma_T(x)$ is not included in the boundary $\partial\sigma(T)$ then there exists two points $\lambda_1 \in \Gamma \setminus \sigma_T(x)$ and $\lambda_2 \in \sigma_T(x) \setminus \Gamma$. Since Γ is connected then there is a third point $\mu \in \partial\sigma_T(x) \cap \Gamma$. From Theorem 2.74 we then conclude that $\mu \in \sigma_T(x) \cap \Gamma$, which contradicts our assumption that $\Gamma \cap \sigma_{\rm ap}(T) = \varnothing$.
∎

As has been observed before, the local spectral subspaces $X_T(\Omega)$ need not be closed, also in the case that T has the SVEP. In fact, the operator T of Example 2.32 has the SVEP, its quasi-nilpotent part $H_0(T)$ is not closed and by Theorem 2.20 $H_0(T) = X_T(\{0\}$.

Definition 2.76. *A bounded operator $T \in L(X)$, X a Banach space, is said to have the Dunford property (C), shortly the property (C), if the analytic subspace $X_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbb{C}$.*

Clearly, every spectral operator T with spectral measure $E(\dot{})$ has the property (C), since $X_T(\Omega)$ for every closed Ω is the range of the projection $E(\Omega)$.

The property (C) dates back to the earliest days of local spectral theory, and was introduced first by Dunford. In the book by Dunford and Schwartz [**97**] this property plays a fundamental role since, as noted above, every spectral operator has the property (C).

Trivially, by Theorem 2.8, we have the following relevant fact:

Theorem 2.77. *If $T \in L(X)$, X a Banach space, has the property (C) then T has the SVEP.*

Note that if an operator T has the property (C), and hence the SVEP, then the quasi-nilpotent part $H_0(T)$ is closed since $H_0(T) = X_T(\{0\}$, see Theorem 2.20. The operator T considered in Example 2.32 shows that the implication of Theorem 2.77 cannot be reversed in general. Further examples of operators with the SVEP but without the property (C) will be given amongst the class of all multipliers of semi-simple commutative Banach algebras. These operators, as we shall show later, have the SVEP, whilst the property (C) plays a distinctive role in this context.

A first example of operators which have the property (C) is given by quasi-nilpotent operators.

Theorem 2.78. *Let $T \in L(X)$ be a quasi-nilpotent operator on a Banach space X. Then T has the property (C).*

Proof Consider any closed subset of $\Omega \subseteq \mathbb{C}$. Consider first the case $0 \notin \Omega$. Then since T has the SVEP

$$X_T(\Omega) = X_T(\Omega \cap \sigma(T)) = X_T(\varnothing) = \{0\}$$

is trivially closed. On the other hand, if $0 \in \Omega$ then by Theorem 1.68 and Theorem 2.20

$$X_T(\Omega) = X_T(\Omega \cap \sigma(T)) = X_T(\{0\}) = H_0(T) = X.$$

Hence, also in this case $X_T(\Omega)$ is closed. ∎

The next result shows that the property (C) is inherited by restrictions to closed invariant subspaces.

Theorem 2.79. *Suppose that $T \in L(X)$, where X is a Banach space, has the property (C). If Y is a T-invariant closed subspace of X then the restriction $T|Y$ has the property (C).*

Proof Set $S := T \mid Y$ and let Ω be a closed subset of $\mathbb{C}$. Suppose that the sequence $(x_n) \subset Y_S(\Omega)$ converges at $x \in X$. We have to show that $x \in Y_S(\Omega)$. Evidently $Y_S(\Omega) \subseteq X_T(\Omega) \cap Y$, so that $x \in Y_S(\Omega) \subseteq X_T(\Omega)$. By Theorem 2.77 we know that T has the SVEP, so there exists an analytic function $f : \mathbb{C} \setminus \Omega \to X$ such that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus \Omega$.

To show that $x \in Y_S(\Omega)$ it suffices to prove that $f(\lambda)$ belongs to Y for all $\lambda \in \mathbb{C} \setminus \Omega$. Since T has the SVEP, for every $n \in \mathbb{N}$ there exists an analytic function $f_n : \mathbb{C} \setminus \Omega \to Y$ such that $(\lambda I - T)f_n(\lambda) = x_n$ for all $\lambda \in \mathbb{C} \setminus \Omega$. The elements x and x_n belong to $X_T(\Omega)$, so Theorem 2.2 implies that $f(\lambda)$ and $f_n(\lambda)$ belong to $X_T(\Omega)$ for all $\lambda \in \mathbb{C} \setminus \Omega$ and $n \in \mathbb{N}$. From Theorem 2.71 we know that $\sigma(T \mid X_T(\Omega)) \subseteq \Omega$, so the bounded operator $\lambda I - T \mid X_T(\Omega)$ on $X_T(\Omega)$ has an inverse $(\lambda I - T \mid X_T(\Omega))^{-1}$ for every $\lambda \in \mathbb{C} \setminus \Omega$.

From this we then obtain that $f_n(\lambda) = (\lambda I - T \mid X_T(\Omega))^{-1} x_n$ converges to the element $(\lambda I - T \mid X_T(\Omega))^{-1} x$, as $n \to \infty$. Therefore $f(\lambda) \in Y$, so the proof is complete. ∎

Note that if T has the property (C)then so does $f(T)$ for every function f analytic on an open neighbourhood $\mathcal{U}$ of $\sigma(T)$, see Theorem 3.3.6 of Laursen and Neumann [**214**]. It could be reasonable to expect that the converse is true if we assume that f is non-constant on each connected component of $\mathcal{U}$, as it happens, by Theorem 2.40, for the SVEP; but this is not known.

In chapter 6 we shall see that every decomposable operator has the property (C), and, in particular, that every bounded operator having a totally disconnected spectrum on a Banach space, enjoys the property of being super-decomposable, a property which is stronger than the property (C). Furthermore, the property (C) will be studied in the framework of multipliers of commutative Banach algebras.

The following example shows that the inclusion $X_T(\Omega) \subseteq E_T(\Omega)$ may be strict.

Example 2.80. Let $X := C[0,1]$ and T be the quasi-nilpotent Volterra operator defined in Example 2.35. By Theorem 2.78 T has the property (C) and hence $X_T(\varnothing) = \{0\}$. On the other hand, the non-trivial subspace

$$T^\infty(X) := \{f \in C^\infty[0,1] : f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{Z}_+\}$$

is T-divisible for X, $Y \subseteq E_T(\varnothing)$. Hence $X_T(\varnothing) \neq E_T(\varnothing)$.

We conclude this section by giving a result which shows the connection between algebraic local spectral subspaces and local spectral subspaces associated with closed subsets of $\mathbb{C}$.

Theorem 2.81. *For an arbitrary operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) $E_T(\Omega)$ *is closed for every closed subset* $\Omega \subseteq \mathbb{C}$*;*

(ii) T *has the property* (C) *and* $E_T(\Omega) = X_T(\Omega)$ *for every closed subset* $\Omega \subseteq \mathbb{C}$*.*

Proof (i) $\Rightarrow$ (ii) Suppose that $E_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbb{C}$. From the inclusion $X_T(\varnothing) \subseteq E_T(\varnothing) = \{0\}$ and Theorem 2.8 we then infer that T has the SVEP . By Theorem 2.45 we now have $\sigma(T\ |E_T(\Omega) = \sigma_{\mathrm{su}}(T\ |E_T(\Omega) \subseteq \Omega$ and therefore from Theorem 2.6, part (vi), $E_T(\Omega) \subseteq X_T(\Omega)$. Hence the equality $E_T(\Omega) = X_T(\Omega)$ holds for every closed subset $\Omega \subseteq \mathbb{C}$ and, since $E_T(\Omega)$ is closed by assumption, T has the property (C).

The implication (ii) $\Rightarrow$ (i) is trivial. ∎

5. Weighted shift operators and SVEP

In this section we want explore the question of the SVEP for both unilateral and bilateral weighted shifts. For a thorough discussion of the basic theory of weighted shifts in the Hilbert setting we refer to Shields [**297**], or Laursen and Neumann [**214**] for weighted shifts in $\ell^p(\mathbb{N})$.

In the first part we shall establish some results in the more general situation of operators $T \in L(X)$, X a Banach space, for which the condition $T^\infty(X) = \{0\}$ holds. This condition may be viewed, in a certain sense, as an abstract shift condition, since it is satisfied by every weighted right shift operator T on $\ell^p(\mathbb{N})$. Clearly the condition $T^\infty(X) = \{0\}$ entails that T is non-surjective and hence non-invertible. This condition also implies that $K(T) = \{0\}$, since $K(T)$ is a subset of $T^\infty(X)$, but the quasi-nilpotent Volterra operator defined in Example 2.35 shows that in general the converse is not true. In fact, for this operator we have by Corollary 2.28 $K(T) = \{0\}$, whilst $T^\infty(X) \neq \{0\}$, see Example 2.80.

As usual, in the sequel we shall denote by $\mathbf{D}(0, i(T))$ the closed disc centered at 0 with radius $i(T)$.

Theorem 2.82. *Suppose that for $T \in L(X)$ we have $T^\infty(X) = \{0\}$. Then:*

(i) $\ker\,(\lambda I - T) = \{0\}$ *for all* $0 \neq \lambda \in \mathbb{C}$*;*

(ii) *T has the SVEP ;*

(iii) *$\sigma_T(x)$ and $\sigma(T)$ are connected, and the closed disc $\mathbf{D}(0, i(T))$ is contained in $\sigma_T(x)$ for all $x \neq 0$.*

(iv) *$H_0(\lambda I - T) = \{0\}$ for all $0 \neq \lambda \in \mathbb{C}$.*

Proof (i) For every $\lambda \neq 0$ we have ker $(\lambda I - T) \subseteq T^\infty(X)$.

(ii) This may be seen in several ways, for instance from Theorem 2.22, since ker $(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$.

(iii) It is easy to see that $0 \in \sigma_T(x)$ for every non-zero $x \in X$. Indeed, from Theorem 2.18 we have

$$\{0\} = K(T) = \{x \in X : 0 \in \rho_T(x)\},$$

and hence $0 \in \sigma_T(x)$ for every $x \neq 0$. Now, suppose that $\sigma_T(x)$ is non-connected for some element $x \neq 0$. Then there exists two non-empty closed subsets Ω_1, Ω_2 of $\mathbb{C}$ such that:

$$\sigma_T(x) = \Omega_1 \cup \Omega_2, \quad \text{and} \quad \Omega_1 \cap \Omega_2 = \varnothing.$$

From the local decomposition property established in Theorem 2.17, there exist two elements $x_1, x_2 \in X$ such that

$$x = x_1 + x_2 \qquad \text{with } \sigma_T(x_i) \subseteq \Omega_i \ (i = 1, 2),$$

Now, from Theorem 2.17 we have $x_1 \neq 0$ and $x_2 \neq 0$, and hence

$$0 \in \sigma_T(x_1) \cap \sigma_T(x_2) \subseteq \Omega_1 \cap \Omega_2 = \varnothing,$$

a contradiction. Hence $\sigma_T(x)$ is connected.

To prove that $\sigma(T)$ is connected observe that since T has the SVEP we have by Theorem 2.43 and Corollary 2.45

$$\sigma(T) = \sigma_{\mathrm{su}}(T) = \bigcup_{x \in X} \sigma_T(x).$$

By Theorem 2.82 the local spectra $\sigma_T(x)$ are connected for every non-zero $x \in X$, and $\sigma(0) = \varnothing$, thus $\sigma(T)$ is connected.

It remains to prove the inclusion $\mathbf{D}(0, i(T)) \subseteq \sigma_T(x)$ for all $x \neq 0$. By Theorem 2.74 we have $\partial \sigma_T(x) \subseteq \sigma_{\mathrm{se}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ for all $x \in X$. Since $i(T) \leq |\lambda|$ for all $\lambda \in \sigma_{\mathrm{ap}}(T)$, it follows easily that $\mathbf{D}(0, i(T)) \subseteq \sigma_T(x)$, as desired.

(iv) Since T has the SVEP then $H_0(\lambda I - T) = \{x \in X : \sigma_T(x) \subseteq \{\lambda\}\}$ for every $\lambda \in \mathbb{C}$, see Theorem 2.20. Now, if $x \neq 0$ and $x \in H_0(\lambda I - T)$ the SVEP ensures that $\sigma_T(x) \neq \varnothing$, so $\sigma_T(x) = \{\lambda\}$. On the other hand, from part (iii) we have $0 \in \{\lambda\}$, a contradiction. ∎

It is evident that the proof of theorem 2.82 works also if we assume $K(T) = \{0\}$, a condition which is less restrictive with respect to the condition $T^\infty(X) = \{0\}$. However, the next result shows that these two conditions are equivalent if $i(T) > 0$.

Corollary 2.83. *Suppose that for a bounded operator $T \in L(X)$, X a Banach space, we have $i(T) > 0$. Then the following statements are equivalent:*

(i) $T^\infty(X) = \{0\}$;

(ii) $\mathbf{D}(0, i(T)) \subseteq \sigma_T(x)$ *for all* $x \neq 0$;

(iii) $K(T) = \{0\}$.

Proof the implication (i) ⇒ (ii) has been proved in Theorem 2.82, whilst (ii) ⇒ (iii) is obvious. It remains only to prove the implication (iii) ⇒ (i). From Theorem 2.54 the condition $i(T) > 0$ implies that $0 \notin \sigma_{\rm ap}(T)$, T is bounded below and therefore semi-regular. By Theorem 1.24 it follows that $T^\infty(X) = K(T) = \{0\}$. ∎

Corollary 2.84. *Suppose that for a bounded operator $T \in L(X)$, X a Banach space, we have $T^\infty(X) = \{0\}$ and $i(T) = r(T)$. Then we have*

$$\sigma_T(x) = \sigma(T) = \mathbf{D}(0, r(T)), \tag{52}$$

for every $x \neq 0$, where $\mathbf{D}(0, r(T))$ denotes the closed disc centered at 0 and radius $r(T)$. Furthermore, if $i(T) = r(T) > 0$ then the equalities (52) hold for every $x \neq 0$ if and only if $T^\infty(X) = \{0\}$.

Proof If $T^\infty(X) = \{0\}$ then T is non-invertible, so by Theorem 2.55 the condition $i(T) = r(T)$ entails that $\sigma(T) = \mathbf{D}(0, r(T))$, and therefore $\sigma_T(x) \subseteq \sigma(T) = \mathbf{D}(0, r(T))$. The opposite inclusion is true by part (iii) of Theorem 2.82, so (52) is satisfied. The equivalence in the last assertion is clear from Corollary 2.83. ∎

It should be noted that if $T \in L(X)$ satisfies the conditions of the preceding corollary then T has the property (C). In fact, for every closed subset Ω of $\mathbb{C}$ we have:

$$X_T(\Omega) = \begin{cases} X & \text{if } \Omega \supseteq \mathbf{D}(0, r(T)), \\ \{0\} & \text{otherwise}, \end{cases}$$

and hence all $X_T(\Omega)$ are closed.

Let $1 \leq p < \infty$ and denote by $\omega := \{\omega_n\}_{n \in \mathbb{N}}$ any bounded sequence of strictly positive real numbers. The corresponding *unilateral weighted right shift* operator on the Banach space $\ell^p(\mathbb{N})$ is the operator defined by:

$$Tx := \sum_{n=1}^{\infty} \omega_n x_n e_{n+1} \quad \text{for all } x := (x_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N}).$$

It is easily seen that T does not admit eigenvalues, thus T has the SVEP. Furthermore, the lower bound and the norms of the iterates T^n may be easily computed as follows:

$$k(T^n) = \inf_{k \in \mathbb{N}} \omega_k \cdots \omega_{k+n-1} \quad \text{for all } n \in \mathbb{N},$$

and

$$\|T^n\| = \sup_{k\in\mathbb{N}} \omega_k \cdots \omega_{k+n-1} \qquad \text{for all } n \in \mathbb{N}.$$

Moreover, a routine calculation shows that the numbers $i(T)$ and $r(T)$ may be computed as follows:

$$i(T) = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (\omega_k \cdots \omega_{k+n-1})^{1/n}$$

and

$$r(T) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} (\omega_k \cdots \omega_{k+n-1})^{1/n}.$$

To determine further properties of the spectrum of an unilateral weighted right shift we recall two simple facts which will be used in the sequel.

Remark 2.85. (i) Let $\alpha \in \mathbb{C}$, with $|\alpha| = 1$ and define, on $\ell^p(\mathbb{N})$, the linear operator $U_\alpha x := (\alpha^n x_n)_{n\in\mathbb{N}}$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Evidently, $\lambda T U_\alpha = U_\alpha T$ and

$$U_\alpha U_{\overline{\alpha}} = U_{\overline{\alpha}} U_\alpha = I.$$

From this it follows that the operators αT and T are similar, and consequently have the same spectrum. This also shows that $\sigma(T)$ is circular symmetry about the origin.

(ii) Let K be a non-empty compact subset of $\mathbb{C}$. If K is connected and invariant under circular symmetry about the origin, then there are two real numbers a and b, with $0 \le a \le b$, such that $K = \{\lambda \in \mathbb{C} : a \le |\lambda| \le b\}$.

Theorem 2.86. *For an arbitrary unilateral weighted right shift T on $\ell_p(\mathbb{N})$ we have $\sigma(T) = \mathbf{D}(0, r(T))$.*

Proof We know by Theorem 2.82 that $\sigma(T)$ is connected and contains the closed disc $\mathbf{D}(0, i(T))$. Since, by part (i) of Remark 2.85 $\sigma(T)$ is circularly symmetric about the origin, from part (ii) of the same Remark we deduce that $\sigma(T)$ is the whole closed disc $\mathbf{D}(0, r(T))$. ∎

Remark 2.87. Theorem 2.54 shows that the approximate point spectrum is located in the annulus $\Lambda(T)$. For an arbitrary unilateral weighted right shift we can say more. In fact, in this case

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : i(T) \le |\lambda| \le r(T)\}.$$

For a proof of this result we refer to Proposition 1.6.15 of Laursen and Neumann [**214**].

It is easily seen that the adjoint of an unilateral weighted right shift T is the *unilateral weighted left shift* on $\ell^q(\mathbb{N})$ defined by:

$$T^\star x := \sum_{n=1}^{\infty} \omega_n x_{n+1} e_n \qquad \text{for all } x := (x_n)_{n\in\mathbb{N}} \in \ell^q(\mathbb{N}),$$

where, as usual, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $\ell^q(\mathbb{N})$ is canonically identified with the dual $(\ell^p(\mathbb{N}))^\star$ of $\ell^p(\mathbb{N})$. Finally, from Corollary 2.57 we deduce that $T^\star$ does not have SVEP whenever $i(T) > 0$.

To investigate more precisely the question of the SVEP for $T^\star$ we introduce the following quantity:

$$c(T) := \lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n}.$$

It is clear that $i(T) \le c(T) \le r(T)$.

The next result shows that every unilateral weighted right shift on $\ell^p(\mathbb{N})$ has the SVEP and gives a precise description of the SVEP for $T^\star$.

Theorem 2.88. *Let T be an unilateral weighted right shift on $\ell^p(\mathbb{N})$ for some $1 \le p < \infty$. Then $T^\star$ has the SVEP at a point $\lambda \in \mathbb{C}$ precisely when $|\lambda| \ge c(T)$. In particular, $T^\star$ has the SVEP if and only if $c(T) = 0$.*

Proof By the classical formula for the radius of convergence of a vector-valued power series we see that the series

$$f(\lambda) := \sum_{n=1}^{\infty} \frac{e_n\ \lambda^{n-1}}{\omega_1 \cdots \omega_{n-1}}$$

converges in $\ell^q(\mathbb{N})$ for every $|\lambda| < c(T)$. Moreover, this series defines an analytic function f on the open disc $\mathbb{D}(0, c(T))$.

Clearly

$$(\lambda I - T^\star) f(\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{D}(0, c(T)),$$

and hence $\Xi(T^\star) \subseteq \mathbb{D}(0, c(T))$.

On the other hand, it is not difficult to check that $T^\star$ has no eigenvalues outside the closed disc $\mathbf{D}(0, c(T))$. This implies that $T^\star$ has the SVEP at every point λ for which $|\lambda| \ge c(T)$, so the proof is complete. ∎

The preceding result has a certain interest, since for every triple of real numbers i, c, and r for which $0 \le i \le c \le r$ it is possible to find a weighted right shift T on $\ell^p(\mathbb{N})$ for which $i(T) = i$, $c(T) = c$ and $r(T) = r$. The details for the construction of the sequences $\{\omega_n\}_{n\in\mathbb{N}}$ for which the corresponding weighted right shift T has these properties are outlined in Shields [**297**].

It is clear that for every weighted right shift operator T on $\ell^p(\mathbb{N})$ we have $e_1 \in \ker T^\star \cap T^{\star\infty}(X)$, so $\mathcal{N}^\infty(T^\star) \cap T^{\star\infty}(X)$ is non-trivial. On the other hand, if we consider a weighted right shift T such that $c(T) = 0$ then by Theorem 2.88 $T^\star$ has the SVEP at 0 whilst $\mathcal{N}^\infty(T^\star) \cap T^{\star\infty}(X) \neq \{0\}$. This observation illustrates that the implication established in Corollary 2.26 cannot be reversed in general.

The next result shows that also the converse of the implications provided in Theorem 2.33 and Theorem 2.36 fails to be true in general.

Theorem 2.89. *Let $1 \le p < \infty$ arbitrarily given and let T be a weighted right shift operator on $\ell^p(\mathbb{N})$ with weight sequence $\omega := (\omega_n)_{n\in\mathbb{N}}$. Then:*

(i) *$H_0(T) + T(X)$ is norm dense in $\ell^p(\mathbb{N})$ if and only if*

$$\lim_{n\to\infty} \sup(\omega_1 \cdots \omega_n)^{1/n} = 0; \tag{53}$$

(ii) *$T^\star$ has the SVEP at* 0 *if and only if*

$$\lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n} = 0.$$

Proof By Theorem 2.88 we need only to prove the equivalence (i). Since

$$\|T^n e_1\| = \omega_1 \cdots \omega_n \quad \text{for all } n \in \mathbb{N},$$

the equality (53) holds precisely when $e_1 \in H_0(T)$. From this it follows that (53) implies that the sum $H_0(T) + T(X)$ is norm dense in $\ell^p(\mathbb{N})$ because $e_n \in T(X)$ for all $n \geq 2$.

Conversely, suppose that $H_0(T) + T(X)$ is norm dense in $\ell^p(\mathbb{N})$, and for every $k \in \mathbb{N}$ choose $u_k \in H_0(T)$ and $v_k \in T(X)$ such that $u_k + v_k \to e_1$ as $k \to \infty$. Let P denote the projection on $\ell^p(\mathbb{N})$ defined by

$$Px := x_1 e_1 \quad \text{for every } x := (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}).$$

It is clear that P vanishes on $T(X)$ and leaves $H_0(T)$ invariant. Moreover, the subspace $H_0(T) \cap T(X)$ is closed, since its dimension is at most 1. Finally,

$$P(u_k + v_k) \to Pe_1 = e_1 \quad \text{as } k \to \infty,$$

so that $e_1 \in H_0(T)$, which concludes the proof. ∎

Every weighted right shift operator T on $\ell^p(\mathbb{N})$ is injective, thus $\mathcal{N}^\infty(T) = \{0\}$. Moreover, $T^\infty(\ell^p(\mathbb{N})) = \{0\}$, and consequently $K(T) = \{0\}$. From this it follows that for these operators the implications provided in Corollary 2.34 and the implications provided in Corollary 2.37 are considerably weaker than those provided in Theorem 2.33 and Theorem 2.36.

We conclude this section with a brief discussion on the SVEP for bilateral weighted right shift operators. Given a two-sided bounded sequence $\omega := \{\omega_n\}_{n\in\mathbb{Z}}$ of strictly positive real numbers, the corresponding *bilateral weighted right shift* operator on $\ell^p(\mathbb{Z})$, for $1 \leq p \leq \infty$, is defined by

$$Tx := (\omega_{n-1} x_{n-1})_{n\in\mathbb{Z}} \quad \text{for all } x = (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}).$$

Contrary the unilateral case, a bilateral weighted right shift T may admit eigenvalues. In fact, we shall see that T need not have SVEP. To see that, first we define the following bilateral sequence:

$$\alpha_o := 1, \quad \alpha_n := \omega_o \cdots \omega_{n-1}, \quad \text{and} \quad \alpha_{-n} := \omega_{-n} \cdots \omega_{-1},$$

for all $n \in \mathbb{N}$. Define

$$c^\pm(T) := \lim_{n\to\infty} \inf {\alpha_{\pm n}}^{1/n} \quad \text{and} \quad d^\pm(T) := \lim_{n\to\infty} \sup {\alpha_{\pm n}}^{1/n}.$$

Theorem 2.90. *Let T be a bilateral weighted right shift on $\ell^p(\mathbb{Z})$ for some $1 \leq p \leq \infty$. Then*

$$\Xi(T) = \{\lambda \in \mathbb{C} : d^+(T) < |\lambda| < c^-(T)\}. \tag{54}$$

In particular, T has the SVEP precisely when $c^-(T) < d^+(T)$.

Proof Suppose that λ is an eigenvalue of T and consider a corresponding non-zero eigenvector $x \in \ell^p(\mathbb{Z})$. A simple computation shows that $\lambda \neq 0$ and that the equations

$$x_n = \frac{x_0\ \alpha_n}{\lambda^n} \qquad \text{and} \qquad x_{-n} = \frac{x_0\ \lambda^n}{\alpha_{-n}}$$

hold for every $n \in \mathbb{N}$. From $x \in \ell^p(\mathbb{Z})$ it then follows that $d^+(T) < |\lambda| < c^-(T)\}$. On the other hand, if $d^+(T) < c^-(T)\}$ we can define the analytic function

$$f(\lambda) := \sum_{n=0}^{\infty} \frac{e_n\ \alpha_n}{\lambda^n} + \sum_{n=1}^{\infty} \frac{e_{-n}\ \lambda_n}{\alpha_{-n}}$$

for all $\lambda \in \mathbb{C}$ with $d^+(T) < |\lambda| < c^-(T)$. Proceeding as in the proof of Theorem 2.89, the classical formula for the radius of convergence of a series guarantees that the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{holds for all} \ \ d^+(T) < |\lambda| < c^-(T)\},$$

which shows the identity (2.90).
The last assertion is obvious. ∎

The following result establishes the SVEP for the dual of the bilateral weighted right shift on $\ell^p(\mathbb{Z})$.

Theorem 2.91. *For every bilateral weighted right shift on $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, the following properties hold:*

(i) $\Xi(T^\star) = \{\lambda \in \mathbb{C} : d^-(T) < |\lambda| < c^+(T)\}$*;*

(ii) $T^\star$ *has the SVEP if and only if* $c^+(T) \leq d^-(T)$*;*

(iii) *At least one of the operators T or $T^\star$ has the SVEP.*

Proof It is clear that the adjoint of T is bilateral weighted left shift on $\ell^q(\mathbb{Z})$, defined by

$$T^\star x := (\omega_n x_{n+1})_{n\in\mathbb{Z}} \qquad \text{for all } x = (x_n)_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}),$$

where, as usual, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Moreover, if we choose

$$\widehat{\omega} := (\omega_{-n-1})_{n\in\mathbb{Z}} \qquad \text{and} \qquad Sx := (x_{-n})_{n\in\mathbb{Z}}$$

for all $x = (x_n)_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$, it follows that

$$(ST^\star S)x = (\omega_{n-1}x_{n-1})_{n\in\mathbb{Z}} \qquad \text{for all} \ \ x = (x_n)_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

This shows that $T^\star$ is similar to the bilateral weighted right shift on $\ell^q(\mathbb{Z})$ with weight sequence $\widehat{\omega}$. In the sense of the right shift representation of $T^\star$, we obtain the identities

$$c^\pm(T^\star) = c^\mp(T) \qquad \text{and} \quad d^\pm(T^\star) = d^\mp(T),$$

because $\hat{\alpha_n} = \alpha_{-n}$ for all $n \in \mathbb{Z}$. From Theorem 2.90 we then conclude that the assertion (i) is valid.

The assertion (ii) is clear from (i).

To prove (iii) assume that both T and $T^\star$ fail to have the SVEP. Then the preceding results entail that $d^+(T) < c^-(T)$ and $d^-(T) < c^+(T)$. But this is an obvious contradiction because $c^-(T) < d^-(T)$ and $c^+(T) < d^+(T)$. ∎

It should be noted that in part (iii) of the preceding theorem it is possible that both T and $T^\star$ have the SVEP . In fact, there are several examples of bilateral weighted shift which are decomposable and for these operators, as remarked before, both T and $T^\star$ have the SVEP.

5.1. Comments. A modern and extensive treatment of the role of the local spectral subspaces in theory of spectral decomposition may be found in the recent monograph of Laursen and Neumann [**214**]. This book also provides a large variety of examples and applications to several concrete cases. The local decomposition property given in Theorem 2.17 dates back to Radjabalipour [**269**], whilst the characterization of the analytical core of an operator given in Theorem 2.18 is owed to Vrbová [**313**] and Mbekhta [**227**].

The concept of glocal spectral subspace dates back to the early days of local spectral theory, see Bishop [**70**]. However, the precise relationship between local spectral subspaces and glocal spectral subspaces has been established, together with some other basic property, only recently by Laursen and Neumann [**212**] and [**213**]. Theorem 2.20 is owed to L. Miller and Neumann [**236**], see also Laursen [**201**]. The equality $H_0(T) = X_T(\{0\}$ for an operator having the SVEP may be also found in Mbekhta [**227**].

The counter example of the bilateral right shift on Hilbert space of all formal Laurent series is owed to Aiena, L. Miller, and Neumann [**30**]. Note that this counter example, which shows that the condition $K(T) \cap H_0(T) = \{0\}$ does not characterize the SVEP at 0, corrects a claim made in Theorem 1.4 of Mbekhta [**229**].

The localized SVEP at a point has been introduced by Finch [**115**]. The characterization of the SVEP at a single point λ_0 given in Theorem 2.22 is taken from Aiena and Monsalve [**31**], whilst the classical result of Corollary 2.24 is owed to Finch [**115**].

Except for Theorem 2.27 and Corollary 2.28, owed to Mbekhta [**229**], the source of the results of the second section is essentially that of Aiena, T.L. Miller and Neumann [**30**], and Aiena, Colasante, and González [**16**]. Also the spectral mapping result of Theorem 2.39, as well as all the material on isometries, Toeplitz operators, and shift operators is taken from Aiena, T.L. Miller, and Neumann [**30**], see also V.G. Miller, T.L. Miller and Neumann [**239**].

The relations between the local spectrum and the surjectivity spectrum established in Theorem 2.43 are taken from Laursen and Vrbová [**215**] and Vrbová [**313**]. The propertty of a decomposable operator the aproximate point spectrum being the entire spectrum dates back to Colojoară and Foiaş [**83**]. Theorem 2.49 and the alternative for the SVEP on the components of the semi-regular resolvent established in Theorem 2.51 are from Aiena and Monsalve [**31**].

The concept of algebraic spectral subspace was first introduced by Johnson and Sinclair [**178**]. This concept has been revisited by several authors, but the definition considered in this book was first given by Laursen and Neumann [**209**]. The material developed here, in the section of the algebraic spectral subspaces, is part of [**200**], whilst the results of the section on local spectral subspaces are mostly contained in Laursen [**200**] and Laursen and Vrbová [**215**]. Further informations on algebraic and divisible subspaces may be found in Bade, Curtis, and Laursen [**54**].

The property (C) was introduced by Dunford and plays a large role in the development of theory of spectral operators. In the book by Dunford and Schwartz [**97**] the property (C) was one of the three basic conditions used in the abstract characterization of spectral operators, and another was the SVEP. Note that it has been observed only recently, by Laursen and Neumann [**212**], that the SVEP is actually a consequence of the property (C).

Theorem 2.73 extends to semi-regular operators a result of Ó Searcóid and West [**253**], which showed that the hyper-range of a semi-regular semi-Fredholm operator is the intersection of neighbouring ranges. The more general case, here established in Theorem 2.73, is obtained by adapting to the local spectral language previous results of Goldman and Kračkovskiĭ [**143**], and Förster [**116**] .

The final results on algebraic spectral subspaces are owed to Laursen [**200**]. For more results on algebraic spectral subspaces see also Pták and Vrbová [**267**].

CHAPTER 3

The SVEP and Fredholm theory

An operator which does not have the single-valued extension property hides in its spectrum a pathology which does not permit the construction of a satisfactory spectral theory. For this reason it is useful to find conditions for an operator which ensure that this property is satisfied.

In the first part of this chapter we shall see that some of these conditions are related to the finiteness of some classical quantities associated with an operator T. These quantities, such as the ascent, and the descent of an operator are defined in the first section and are the basic bricks in the construction of one of the most important branches of spectral theory, the theory of Fredholm operators.

In the preceeding chapter we have exhibited a variety of conditions which imply the SVEP at a point $\lambda_0 \in \mathbb{C}$. In this chapter we shall see that for semi-Fredholm operators, or more generally for operators of Kato type, all these conditions are actually equivalent to the SVEP at λ_0. These equivalences also show how deeply Fredholm theory and local spectral theory interact.

In fact, many classical results of Fredholm theory may be explained in terms of the SVEP, for instance the classification of the connected open components Ω of the semi-Fredholm resolvent $\rho_{\rm sf}(T) := \mathbb{C} \setminus \sigma_{\rm sf}(T)$, or more generally of the Kato type of resolvent $\rho_{\rm k}(T)$, is a consequence of the SVEP at a point $\lambda_0 \in \Omega$ implying the SVEP at every point of Ω. Since this classification is established in the more general framework of the Kato type of operators, these results subsume some classical results contained in standard texts on Fredholm theory, such as Kato's book [**182**] or Heuser's book [**159**].

The SVEP at a point for operators of Kato type may be also characterized by means of the approximate point spectrum, to be precise, if $\lambda_0 I - T$ is of Kato type then T has the SVEP at a point λ_0 precisely when the approximate point spectrum $\sigma_{\rm ap}(T)$ does not cluster at λ_0. Dually, $T^\star$ has the SVEP at λ_0 precisely when the surjectivity spectrum $\sigma_{\rm su}(T)$ does not cluster at λ_0. These two properties lead to many results on the cluster points of some distinguished parts of the spectrum.

In the fifth section we shall study some spectra originating from the classical Fredholm theory, the semi-Browder spectra and the Browder spectrum, defined by means of the ascent and the descent, and the Weyl spectrum. We establish some characterizations of these spectra as the intersection of spectra of compact perturbations, as well as of compressions. The more remarkable result of this section shows that a classical result on the spectral theory of normal operators on Hilbert spaces may be extended to the more general case of operators T having the SVEP together with its adjoint $T^\star$. In fact, the Browder spectrum and the Weyl spectrum coincide when T or $T^\star$ has the SVEP, and coincide with the Fredholm spectrum $\sigma_{\rm f}(T)$ if T and $T^\star$ have both the SVEP.

The Riesz functional calculus yields for an analytic function f defined on an open subset containing the spectrum an operator $f(T) \in L(X)$. The classical spectral mapping theorem asserts that $f(\sigma(T)) = \sigma(f(T))$ and a natural question is whether a similar result holds for distinguished parts of the spectrum. We shall prove the spectral mapping theorem for many of the spectra mentioned above and, in particular, we shall show, *via* the local spectral theory, that the spectral mapping theorem holds for the semi-Browder spectra and the Browder spectrum.

An interesting situation is obtained when the two subspaces $K(\lambda_0 I - T)$ and $H_0(\lambda_0 I - T)$ are relative to an isolated point λ_0 of the spectrum $\sigma(T)$. In this case we establish a very illuminating description of these two subspaces. Indeed, they coincide with the kernel and the range of the spectral projection associated with λ_0, respectively. These properties lead to a very simple characterization of the poles of the resolvent in terms of the single-valued extension property at λ_0 . Moreover, these characterizations are useful tools for studyng some important classes of operators: the class of operators which satisfy Weyl's theorem and the class of all Riesz operators.

The last section of the chapter is devoted to operators $T \in L(X)$ having hyper-range $T^\infty(X) = \{0\}$. In particular, the results obtained apply to isometries and give useful information about the spectra of these operators.

1. Ascent, descent, and the SVEP

We have already seen that the kernels and the ranges of the iterates of a linear operator T, defined on a vector space X, form two increasing and decreasing chains, respectively. In this section we shall consider operators for which one, or both, of these chains becomes constant at some $n \in \mathbb{N}$.

Definition 3.1. *Given a linear operator T on a vector space X, T is said to have* finite ascent *if $\mathcal{N}^\infty(T) = \ker T^k$ for some positive integer k. Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that* $\ker T^p = \ker T^{p+1}$. *The positive integer p is called the* ascent *of T. If there is no such integer we set $p(T) := \infty$. Analogously, T is said to have* finite descent *if $T^\infty(X) = T^k(X)$ for some k. The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the* descent *of T. If there is no such integer we set $q(T) := \infty$.*

Clearly $p(T) = 0$ if and only if T is injective and $q(T) = 0$ if and only if T is surjective. The classical Riesz–Schauder theory asserts that $p(\lambda I - T) = q(\lambda I - T) < \infty$ for every compact operator T on a Banach space X, see Heuser [**159**, Chapter VI].

The following lemma establishes useful and simple charaterizations of finite ascent and finite descent.

Lemma 3.2. *Let T be a linear operator on a vector space X. For a naturalm $\in \mathbb{Z}_+$, the following assertions hold:*

(i) *$p(T) \leq m < \infty$ if and only if for every $n \in \mathbb{N}$ we have $T^m(X) \cap \ker T^n = \{0\}$;*

(ii) *$q(T) \leq m < \infty$ if and only if for every $n \in \mathbb{N}$ there exists a subspace $Y_n \subseteq \ker T^m$ such that $X = Y_n \oplus T^n(X)$.*

Proof (i) Suppose $p(T) \leq m < \infty$ and n any natural number. Consider an element $y \in T^m(X) \cap \ker T^n$. Then there exists $x \in X$ such that $y = T^m x$ and $T^n y = 0$. From that we obtain $T^{m+n}x = T^n y = 0$ and therefore $x \in \ker T^{m+n} = \ker T^m$. Hence $y = T^m x = 0$.

Conversely, suppose $T^m(X) \cap \ker T^n = \{0\}$ for some natural m and let $x \in \ker T^{m+1}$. Then $T^m x \in \ker T$ and therefore

$$T^m x \in T^m(X) \cap \ker T \subseteq T^m(X) \cap \ker T^n = \{0\}.$$

Hence $x \in \ker T^m$. We have shown that $\ker T^{m+1} \subseteq \ker T^m$. Since the opposite inclusion is verified for all operators we conclude that $\ker T^m = \ker T^{m+1}$.

(ii) Let $q := q(T) \leq m < \infty$ and Y be a complementary subspace to $T^n(X)$ in X. Let $\{x_j : j \in J\}$ be a basis of Y. For every element x_j of the basis there exists, since $T^q(Y) \subseteq T^q(X) = T^{q+n}(X)$, an element $y_j \in X$ such that $T^q x_j = T^{q+n} y_j$. Set $z_j := x_j - T^y_j$. Then

$$T^q z_j = T^q x_j - T^{q+n} y_j = 0.$$

From this it follows that the linear subspace Y_n generated by the elements z_j is contained in $\ker T^q$ and *a fortiori* in $\ker T^m$. From the decomposition $X = Y \oplus T^n(X)$ we obtain for every $x \in X$ a representation of the form

$$x = \sum_{j \in J} \lambda_j x_j + T^n y = \sum_{j \in J} \lambda_j (z_j + T^n y_j) + T^n y = \sum_{j \in J} \lambda_j z_j + T^n z,$$

so $X = Y_n + T^n(X)$. We show that this sum is direct. Indeed, suppose that $x \in Y_n \cap T^n(X)$. Then $x = \sum_{j \in J} \mu_j z_j = T^n v$ for some $v \in X$, and therefore

$$\sum_{j \in J} \mu_j x_j = \sum_{j \in J} \mu_j T^n y_j + T^n v \in T^n(X).$$

From the decomposition $X = Y \oplus T^n(X)$ we then obtain that $\mu_j = 0$ for all $j \in J$ and hence $x = 0$. Therefore Y_n is a complement of $T^n(X)$ contained in $\ker T^m$. Conversely, if for $n \in \mathbb{N}$ the subspace $T^n(X)$ has a complement $Y_n \subseteq \ker T^m$ then

$$T^m(X) = T^m(Y_n) + T^{m+n}(X) = T^{m+n}(X),$$

and therefore $q(T) \leq m$. ∎

Theorem 3.3. *If both $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$.*

Proof Set $p := p(T)$ and $q := q(T)$. Assume first that $p \le q$, so that the inclusion $T^q(X) \subseteq T^p(X)$ holds. Obviously we may assume $q > 0$. From part (ii) of Lemma 3.2 we have $X = \ker T^q + T^q(X)$, so every element $y := T^p(x) \in T^p(X)$ admits the decomposition $y = z + T^q w$, with $z \in \ker T^q$. From $z = T^p x - T^q w \in T^q(X)$ we then obtain that $z \in \ker T^q \cap T^q(X)$ and hence the last intersection is $\{0\}$ by part (i) of Lemma 3.2. Therefore $y = T^q w \in T^q(X)$ and this shows the equality $T^p(X) = T^q(X)$, from whence we obtain $p \ge q$, so that $p = q$.

Assume now that $q \le p$ and $p > 0$, so that $\ker T^q \subseteq \ker T^p$. From part (ii) of Lemma 3.2 we have $X = \ker T^q + T^p(X)$, so that an arbitrary element x of $\ker T^p$ admits the representation $x = u + T^p v$, with $u \in \ker T^q$. From $T^p x = T^p u = 0$ it then follows that $T^{2p} v = 0$, so that $v \in \ker T^{2p} = \ker T^p$. Hence $T^p v = 0$ and consequently $x = u \in \ker T^q$. This shows that $\ker T^q = \ker T^p$, hence $q \ge p$. Therefore $p = q$. ∎

Let $\Delta(X)$ denote the set of all linear operators on vector space X for which the nullity $\alpha(T)$ and $\beta(T)$ are both finite. We recall that for every $T \in \Delta(X)$, the index of T, defined by

$$\text{ind } T := \alpha(T) - \beta(T),$$

satisfies the basic *index theorem*:

$$\text{ind } (TS) = \text{ind } T + \text{ind } S \quad \text{for all } T, S \in \Delta(X),$$

see Theorem 23.1 of Heuser [**159**].

In the next theorem we establish the basic relationships between the quantities $\alpha(T)$, $\beta(T)$, $p(T)$ and $q(T)$.

Theorem 3.4. *If T is a linear operator on a vector space X then the following properties hold:*

(i) *If $p(T) < \infty$ then $\alpha(T) \le \beta(T)$;*

(ii) *If $q(T) < \infty$ then $\beta(T) \le \alpha(T)$;*

(iii) *If $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinite);*

(iv) *If $\alpha(T) = \beta(T) < \infty$ and if either $p(T)$ or $q(T)$ is finite then $p(T) = q(T)$.*

Proof (i) Let $p := p(T) < \infty$. Obviously if $\beta(T) = \infty$ there is nothing to prove. Assume that $\beta(T) < \infty$. It is easy to check that also $\beta(T^n)$ is finite. By Lemma 3.2, part (i), we have $\ker T \cap T^p(X) = \{0\}$ and this implies that $\alpha(T) < \infty$. From the index theorem we obtain for all $n \ge p$ the following equality:

$$n \cdot \text{ind } T = \text{ind } T^n = \alpha(T^p) - \beta(T^n).$$

Now suppose that $q := q(T) < \infty$. For all integers $n \ge \max\{p, q\}$ the quantity $n \cdot \text{ind } T = \alpha(T^p) - \beta(T^p)$ is then constant, so that $\text{ind } T = 0$, $\alpha(T) = \beta(T)$. Consider the other case $q = \infty$. Then $\beta(T^n) \to 0$ as $n \to \infty$, so $n \cdot \text{ind}\, T$ eventually becomes negative, and hence $\text{ind}\, T < 0$. Therefore in this case we have $\alpha(T) < \beta(T)$.

(ii) Let $q := q(T) < \infty$. Also here we can assume that $\alpha(T) < \infty$, otherwise there is nothing to prove. Consequently, as is easy to check, also $\beta(T^n) < \infty$ and by part (ii) of Lemma 3.2 $X = Y \oplus T(X)$ with $Y \subseteq \ker T^q$. From this it follows that $\beta(T) = \dim\, Y \le \alpha(T^q) < \infty$. If we use, with appropriate changes, the index argument used in the proof of part (i) then we obtain that $\beta(T) = \alpha(T)$ if $p(T) < \infty$, and $\beta(T) < \alpha(T)$ if $p(T) = \infty$.

(iii) It is clear from part (i) and part (ii).

(iv) This is an immediate consequence of the equality $\alpha(T^n) - \beta(T^n) = \text{ind}\, T^n = n \cdot \text{ind}\, T = 0$, valid for every $n \in \mathbb{N}$. ∎

The finiteness of $p(T)$ or $q(T)$ has also some remarkable consequences on $T|\, T^\infty(X)$, the restriction of T on $T^\infty(X)$. Recall that the hyper-range $T^\infty(X)$ is a T-invariant subspace of X.

Theorem 3.5. *Let T be a linear operator on the vector space X. We have:*

(i) *If either $p(T)$ or $q(T)$ is finite then $T\, |T^\infty(X)$ is surjective. To be precise*

$$T^\infty(X) = C(T),$$

where, as usual, $C(T)$ denotes the algebraic core of T;

(ii) *If either $\alpha(T) < \infty$ or $\beta(T) < \infty$ then*

$$p(T) < \infty \Leftrightarrow T|T^\infty(X) \text{ is injective.}$$

Proof (i) The assertion follows immediately from Lemma 1.9 because if $p = p(T) < \infty$ then by Lemma 3.2

$$\ker\, T \cap T^p(X) = \ker\, T \cap T^{p+k}(X) \quad \text{for all integers } k \ge 0;$$

whilst if $q = q(T) < \infty$ then

$$\ker\, T \cap T^q(X) = \ker\, T \cap T^{q+k}(X) \quad \text{for all integers } k \ge 0.$$

(ii) Assume that $p(T) < \infty$. We have $C(T) = T^\infty(X)$ and hence $T(T^\infty(X)) = T^\infty(X)$. Let $\widetilde{T} := T|T^\infty(X)$. Then $\widetilde{T}$ is surjective, thus $q(\widetilde{T}) = 0$. From our assumption and from the equality $\ker \widetilde{T}^n = \ker T^n \cap T^\infty(X)$ we also obtain $p(\widetilde{T}) < \infty$. From part (iii) of Theorem 3.4 we conclude that $p(\widetilde{T}) = q(\widetilde{T}) = 0$, and therefore the restriction $\widetilde{T}$ is injective.

Conversely, if $\widetilde{T}$ is injective then $\ker T \cap T^\infty(X) = \{0\}$. By assumption $\alpha(T) < \infty$ or $\beta(T) < \infty$, and this implies (see the proof of Theorem 1.10) that $\ker T \cap T^n(X) = \{0\}$ for some positive integer n. By Lemma 3.2 it then follows that $p(T) < \infty$. ∎

The finiteness of the ascent and the descent of a linear operator T is related to a certain decomposition of X.

Theorem 3.6. *Suppose that T is a linear operator on a vector space X. If $p = p(T) = q(T) < \infty$ then we have the decomposition*

$$X = T^p(X) \oplus \ker T^p.$$

Conversely, if for a natural number m we have the decomposition $X = T^m(X) \oplus \ker T^m$ then $p(T) = q(T) \leq m$. In this case $T|T^p(X)$ is bijective.

Proof If $p < \infty$ and we may assume that $p > 0$, then the decomposition $X = T^p(X) \oplus \ker T^p$ immediately follows from Lemma 3.2. Conversely, if $X = T^m(X) \oplus \ker T^m$ for some $m \in \mathbb{N}$ then $p(T), q(T) \leq m$, again by Lemma 3.2, and hence $p(T) = q(T) < \infty$ by Theorem 3.3.

To verify the last assertion observe that $T^\infty(X) = T^p(X)$, so from Theorem 3.5 $\widetilde{T} := T|T^p(X)$ is onto. On the other hand, $\ker \widetilde{T} \subseteq \ker T \subseteq \ker T^p$, but also $\ker \widetilde{T} \subseteq T^p(X)$, so the decomposition $X = T^p(X) \oplus \ker T^p$ entails that $\ker \widetilde{T} = \{0\}$. ∎

Remark 3.7. The following statements establish some other basic relationships between the ascent and the descent of a bounded operator $T \in L(X)$ on a Banach space X in the case of semi-Fredholm operators.

(a) If $T \in \Phi_\pm(X)$ then the chain lengths of T and its dual $T^\star$ are related by the equalities $p(T^\star) = q(T)$ and $p(T) = q(T^\star)$. This can be easily seen, because if $T \in \Phi_\pm(X)$ then $T^n \in \Phi_\pm(X)$, and hence the range of T^n is closed for all n. Analogously, also $T^{\star n}$ has closed range, and therefore for every $n \in \mathbb{N}$,

$$\ker T^{n\star} = T^n(X)^\perp, \quad \ker T^n = {}^\perp T^{n\star}(X^\star) = {}^\perp T^{\star n}(X^\star).$$

Obviously these equalities imply that $p(T^\star) = q(T)$ and $p(T) = q(T^\star)$. Note that these equalities hold in the Hilbert space sense: in the case of Hilbert space operators $T \in \Phi_\pm(H)$ the equalities $p(T^\star) = q(T)$ and $p(T) = q(T^\star)$ hold for the adjoint $T^\star$.

(b) The chain lengths of $(\lambda I - T)$ are intimately related to the poles of the resolvent $R(\lambda, T)$. In fact,

$$\lambda_0 \in \sigma(T) \text{ is a pole of } R(\lambda, T) \;\Leftrightarrow 0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty.$$

Moreover, if $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ then p is the order of the pole, every pole $\lambda_0 \in \sigma(T)$ is an eigenvalue of T, and if P_0 is the spectral projection associated with $\{\lambda_0\}$ then

$$P_0(X) = \ker\,(\lambda_0 I - T)^p, \quad \ker P_0 = (\lambda_0 I - T)^p(X),$$

see Heuser [**159**, Proposition 50.2].

(c) If $\lambda_0 \in \sigma(T)$ then $\lambda_0 I - T$ is a Fredholm operator having both ascent and descent finite if and only if λ_0 is an isolated spectral point of T and the corresponding spectral projection P_0 is finite-dimensional, see Heuser [**159**, Proposition 50.3].

The following theorem establishes a first relationship between the ascent and the descent of $\lambda_0 I - T$ and the SVEP at $\lambda_0 \in \mathbb{C}$.

Theorem 3.8. *For a bounded operator T on a Banach space X and $\lambda_0 \in \mathbb{C}$, the following implications hold:*

$$\begin{aligned} p(\lambda_0 I - T) < \infty \quad &\Rightarrow \quad \mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\} \\ &\Rightarrow \quad T \text{ has the SVEP at } \lambda_0, \end{aligned}$$

and

$$\begin{aligned} q(\lambda_0 I - T) < \infty \quad &\Rightarrow \quad X = \mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X) \\ &\Rightarrow \quad T^\star \text{ has the SVEP at } \lambda_0. \end{aligned}$$

Proof There is no loss of generality in assuming $\lambda_0 = 0$.

Assume that $p := p(T) < \infty$. Then $\mathcal{N}^\infty(T) = \ker T^p$, and therefore from Lemma 3.2 we obtain that

$$\mathcal{N}^\infty(T) \cap T^\infty(X) \subseteq \ker T^p \cap T^p(X) = \{0\}.$$

From Theorem 2.22 we then conclude that T has the SVEP at 0.

To show the second chain of implications suppose that $q := q(T) < \infty$. Then $T^\infty(X) = T^q(X)$ and

$$\mathcal{N}^\infty(T) + T^\infty(X) = \mathcal{N}^\infty(T) + T^q(X) \supseteq \ker T^q + T^q(X). \tag{55}$$

Now, the condition $q = q(T) < \infty$ yields that $T^{2q}(X) = T^q(X)$, so for every element $x \in X$ there exists $y \in T^q(X)$ such that $T^q y = T^q(x)$. Obviously $x - y \in \ker T^q$, and therefore $X = \ker T^q + T^q(X)$. From the inclusion (55) we conclude that $X = \mathcal{N}^\infty(T) + T^\infty(X)$, and therefore by Corollary 2.34 $T^\star$ has the SVEP at 0. ∎

The previous theorem indicates that the two notions of ascent and descent are quite useful for establishing the SVEP for some important classes of operators.

It is well known that if T is a normal operator on a Hilbert space H then

$$H = \ker(\lambda I - T) \oplus \overline{(\lambda I - T)(H)} \quad \text{for every } \lambda \in \mathbb{C},$$

(see [**159**, Proposition 70.3]), so by Lemma 3.2 we have for these operators

$$p(\lambda I - T) \le 1 \quad \text{for all } \lambda \in \mathbb{C}. \tag{56}$$

Consequently, every normal operator on a complex Hilbert space has the SVEP. Later we shall see that the condition (56) is also satisfied for every multiplier of a semi-prime Banach algebra.

The next example shows that the condition (56) is satisfied by classes of operators strictly larger than the class of all normal operators.

Example 3.9. A bounded operator T on a Banach space is said to be *paranormal* if

$$\|Tx\|^2 \le \|T^2 x\| \|x\| \quad \text{for all } x \in X\ .$$

It is easy to see that every paranormal operator T is *normaloid*, in the sense that $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T, or, equivalently, $\|T^n\| = \|T\|^n$ for every $n \in \mathbb{N}$. Obviously, if T is paranormal then the restriction of T to every closed invariant subspace is also paranormal, so T is *spectrally normaloid* in the sense of Heuser [**159**, §54].

An operator T on a Banach space X is said to be *totally paranormal* if $\lambda I - T$ is paranormal for every $\lambda \in \mathbb{C}$. Examples of totally paranormal operators are all the operators T on a Hilbert space H which are hyponormal. Recall that $T \in L(H)$ is said to be *hyponormal* if

$$\|T^\star x\| \leq \|Tx\| \quad \text{for every } x \in H.$$

Now, if T is hyponormal then

$$\|Tx\|^2 = \langle T^\star Tx, x\rangle \leq \|T^\star Tx\|\|x\| \leq \|T^2x\|\|x\| \quad \text{for all } x \in H,$$

so T is paranormal. Clearly T is totally paranormal, since every operator T is hyponormal if and only if $\lambda I - T$ is hyponormal for every $\lambda \in \mathbb{C}$.

It is immediate from the definition that every totally paranormal operator has ascent $p(\lambda I - T) \leq 1$ for every $\lambda \in \mathbb{C}$, so these operators have the SVEP. The SVEP of a totally paranormal operator T is also consequence of Theorem 2.31, once noted that for these operators we have

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \text{for all } \lambda \in \mathbb{C}.$$

To show this observe first that if T is totally paranormal then for every $x \in X$ and $\lambda \in \mathbb{C}$ we have

$$\|(\lambda I - T)^n x\|^{1/n} \geq \|(\lambda I - T)x\| \quad \text{for all } n \in \mathbb{N}.$$

If $x \in H_0(\lambda I - T)$ then $\|(\lambda I - T)^n x\|^{1/n} \to 0$ as $n \to \infty$, and consequently $(\lambda I - T)x = 0$, so $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)$. Since the reverse inclusion is always true for every operator it follows that $H_0(\lambda I - T) = \ker(\lambda I - T)$.

It should be noted that every totally paranormal operator has the property (C), see Laursen [**201**].

Example 3.10. Other examples of operators T for which the condition $p(\lambda I - T) < \infty$ is verified for all $\lambda \in \mathbb{C}$, and hence that have the SVEP, may be found amongst the class $\mathcal{P}_g(X)$ of operators on a Banach space X which satisfy a *polynomial growth condition*. An operator T satisfies this condition if there exists a $K > 0$, and a $\delta > 0$ for which

$$\|\exp(i\lambda T)\| \leq K(1 + |\lambda|^\delta) \quad \text{for all } \lambda \in \mathbb{R}, \tag{57}$$

Examples of operators which satisfy a polynomial growth condition are Hermitian operators on Hilbert spaces, nilpotent and projection operators, algebraic operators with real spectra, see Barnes [**61**] .

In Laursen and Neumann [**214**, Theorem 1.5.19] it is shown that $\mathcal{P}_g(X)$ coincides with the class of all generalized scalar operators having real spectra, see also Colojoară and Foiaş [**83**, Theorem 5.4.5].

To show the condition (56) we first note that the polynomial growth condition may be reformulated as follows: $T \in \mathcal{P}_g(X)$ if and only if $\sigma(T) \subseteq \mathbb{R}$ and there is a constant $K > 0$ and a $\delta > 0$ such that

$$\|(\lambda I - T)^{-1}\| \leq K(1 + |\text{Im } \lambda|^{-\delta}) \quad \text{for all } \lambda \in \mathbb{C} \text{ with Im } \lambda \neq 0, \tag{58}$$

see Theorem 1.5.19 of Laursen and Neumann [**214**], or Barnes [**61**].

We claim that if $T \in \mathcal{P}_g(X)$ and $c := \text{Im } \lambda > 0$ then

$$(\lambda I - T)^{-1} = -i \int_0^\infty e^{i\lambda t} e^{-itT} \ \mathrm{d}t. \tag{59}$$

Indeed, for every $s > 0$,

$$i(\lambda - T) \int_0^s e^{i(\lambda I - T)t} \ \mathrm{d}t = \int_0^s \frac{\mathrm{d}}{\mathrm{dt}} \left(e^{i(\lambda I - T)t} \right) \ \mathrm{d}t = e^{i(\lambda I - T)s} - I.$$

From the estimates

$$\|e^{i(\lambda I - T)s}\| = e^{-cs} \|e^{-isT}\| \le K e^{-cs}(1 + s\delta),$$

it follows that $\|e^{i(\lambda - T)s}\| \to 0$ as $s \to \infty$. From this we obtain that

$$i(\lambda I - T) \int_0^\infty e^{i(\lambda I - T)t} \ \mathrm{d}t = -I,$$

from which the equality (59) follows.

Now, in order to establish the finiteness of $p(\lambda I - T)$ we may only consider the case $\lambda \in \sigma(T)$. Of course, there is no loss of generality if we assume $\lambda = 0$. If δ is as in (58), put $m := [\delta] + 1$. Then

$$\lim_{t \to 0^+} (it)^m (itI - T)^{-1} = 0.$$

Suppose $p(T) > m$. Choose $x \in X$ and $f \in X^\star$ such that

$$T^{m+1}x = 0, \quad T^m x \neq 0, \quad \text{and} \quad f(T^m x) = 1.$$

Define a linear continuous functional ϕ on the Banach space $L(X)$ by the assignment

$$\phi(T) := f(Tx) \ \text{ for every } T \in L(X).$$

From (59) we have for all $t > 0$

$$(itI - T)^{-1} = -i \int_0^\infty e^{-tx} e^{-ixT} \ \mathrm{d}x \ ,$$

and therefore, for all $t > 0$,

$$\phi((itI - T)^{-1}) = -i \int_0^\infty e^{-tx} \left(\sum_{n=0}^\infty \frac{(-ix)^n}{n!} \phi(T^n) \right) \ \mathrm{d}x.$$

Clearly $\phi(T^n) = f(T^n x) = 0$ for all $n > m$, so for every $t > 0$ we have

$$\begin{aligned}
(it)^m \phi((itI - T)^{-1}) &= -(\mathrm{i})^{m+1} t^m \sum_{n=0}^m \frac{-i^n}{n!} \left[\sum_0^\infty x^n e^{-tx} \right] \\
&= -(\mathrm{i})^{m+1} t^m \sum_{n=0}^m \frac{(-i)^n}{n!} \left[\frac{(n+1)!}{t^{n+1}} \right] \phi(T^n) \\
= \quad & - \quad (\mathrm{i})^{2m+1} (m+1) t^{-1} \\
& + \quad \{ \text{ terms involving non-negative powers of t} \}.
\end{aligned}$$

This shows that $(it)^m\phi((it-T)^{-1})$ does not converge at 0 as $t \to 0^+$, a contradiction. Hence $p(\lambda I - T) \leq m$ for every $\lambda \in \mathbb{C}$.

It is easy to see that if $T \in \mathcal{P}_g(X)$ and $(\lambda_0 I - T)(X)$ has closed range for some $\lambda_0 \in \mathbb{C}$ then also $q(\lambda_0 I - T)$ is finite. In fact, if $T \in \mathcal{P}_g(X)$ then $T^\star \in \mathcal{P}(X^\star)$, so $p := p(\lambda_0 I - T) < \infty$. This means that

$$\ker(\lambda_0 I - T^\star)^p = \ker(\lambda_0 I - T^\star)^{p+k}$$

for every $k \in \mathbb{N}$, and hence

$$\overline{(\lambda_0 I - T)^p(X)} = \overline{(\lambda_0 I - T)^{p+k}(X)}, \quad k \in \mathbb{N}.$$

Finally, if $(\lambda_0 I - T)(X)$ is closed then $(\lambda_0 I - T)^k(X)$ is closed for every $k = 0, 1, \cdots$, thus $q(\lambda_0 I - T) < \infty$. It follows from part (c) of Remark 3.7 that if $\lambda_0 \in \sigma(T)$ is such that $\lambda_0 I - T$ has a closed range then λ_0 is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$.

The next example shows that T may have the SVEP, although the point spectrum $\sigma_p(T)$ is non empty.

Example 3.11. Let Ω denote a compact subset of $\mathbb{C}$ with a non-empty interior. Let $X := B(\Omega)$ denote the Banach space of all bounded complex-valued functions on Ω and consider the operator $T : X \to X$ defined by the assignment

$$(Tf)(\lambda) := \lambda f(\lambda) \quad \text{for all } f \in X \text{ and } \lambda \in \Omega.$$

It is easy to see that $p(\mu I - T)$ is less than or equal to 1 for every $\mu \in \mathbb{C}$, so T has the SVEP. Clearly $\sigma_p(T) = \Omega \neq \varnothing$.

As we observed in Theorem 5.4 and Theorem 2.31, each one of the two conditions $p(\lambda_0 I - T) < \infty$ or $H_0(\lambda_0 I - T)$ closed implies the SVEP at λ_0. In general these two conditions are not related. Indeed, the operator T defined in Example 2.32 has its quasi-nilpotent part $H_0(T)$ not closed whilst, being T injective, $p(T) = 0$.

In the following example we find an operator T which has a closed quasi-nilpotent part but ascent $p(T) = \infty$.

Example 3.12. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined

$$Tx := \left(\frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots\right), \text{ where } x = (x_n) \in \ell^2(\mathbb{N}).$$

It is easily seen that

$$\|T^k\| = \frac{1}{(k+1)!} \quad \text{for every } k = 0, 1, \dots$$

and from this it easily follows that the operator T is quasi-nilpotent and therefore $H_0(T) = \ell^2(\mathbb{N})$ by Theorem 1.68. Obviously $p(T) = \infty$.

Note that the operator T above shows that the reverse of the implication

$$p(\lambda_0 I - T) < \infty \Rightarrow T \quad \text{has the SVEP at } \lambda_0$$

established in Theorem 5.4 generally is not true. In fact, T has the SVEP at 0 since T is quasi-nilpotent, and $p(T) = \infty$.

2. The SVEP for operators of Kato type

In this section we shall characterize the SVEP at a point $\lambda_0 \in \mathbb{C}$ in the cases of operators of Kato type. In fact, we shall see that most of the conditions which ensure the SVEP at a point λ_0, established in Chapter 2 and in the previous section, are actually equivalences.

Recall that if $\lambda_0 I - T$ is of Kato type then also $\lambda_0 I^\star - T^\star$ is of Kato type. More precisely, the pair $(N^\perp, M^\perp)$ is a GKD for $\lambda_0 I^\star - T^\star$ with $\lambda_0 I^\star - T^\star | N^\perp$ semi-regular and $\lambda_0 I^\star - T^\star | M^\perp$ nilpotent.

Lemma 3.13. *Suppose that $\lambda_0 I - T$ has a GKD (M, N). Then*

$$\lambda_0 I - T|M \text{ is surjective } \Leftrightarrow \lambda_0 I^\star - T^\star | N^\perp \text{is injective.}$$

Proof We can assume $\lambda_0 = 0$. Suppose first that $T(M) = M$ and consider an arbitrary element $x^\star \in \ker T^\star | N^\perp = \ker T^\star \cap N^\perp$. For every $m \in M$ then there exists $m' \in M$ such that $Tm' = m$. Then we have

$$x^\star(m) = x^\star(Tm') = (T^\star x^\star)(m') = 0,$$

and therefore $x^\star \in M^\perp \cap N^\perp = \{0\}$.

Conversely, suppose that $T|M$ is not onto, i.e., $T(M) \subseteq M$ and $T(M) \neq M$. By assumption $T(M)$ is closed, since $T|M$ is semi-regular, and hence via the Hahn–Banach theorem there exists $z^\star \in X^\star$ such that $z^\star \in T(M)^\perp$ and $z^\star \notin M^\perp$.

Now, from the decomposition $X^\star = N^\perp \oplus M^\perp$ we have $z^\star = n^\star + m^\star$ for some $n^\star \in N^\perp$ and $m^\star \in M^\perp$. For every $m \in M$ we obtain

$$T^\star n^\star(m) = n^\star(Tm) = z^\star(Tm) - m^\star(Tm) = 0.$$

Hence $T^\star n^\star \in N^\perp \cap M^\perp = \{0\}$, and therefore $0 \neq n^\star \in \ker T^\star \cap N^\perp$. ∎

The first result shows that the SVEP at λ_0 of a bounded operator which admits a GKD (M, N) depends essentially on the behavior of $\lambda_0 I - T$ on the first subspace M.

Theorem 3.14. *Suppose that $\lambda_0 I - T \in L(X)$ admits a GKD (M, N). Then the following assertions are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) *$T\ |M$ has the SVEP at λ_0;*

(iii) *$(\lambda_0 I - T)\ |M$ is injective;*

(iv) *$H_0(\lambda_0 I - T) = N$;*

(v) *$H_0(\lambda_0 I - T)$ is closed;*

(vi) *$H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$;*

(vii) *$H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T)$ is closed.*

In particular, if $\lambda_0 I - T$ is semi-regular then the conditions (i)–(vii) *are equivalent to the following statement:*

(viii) $H_0(\lambda_0 I - T) = \{0\}$.

Proof Also here we shall consider the particular case $\lambda_0 = 0$.

The implication (i) $\Rightarrow$ (ii) is clear, since the SVEP at 0 of T is inherited by the restrictions on every closed invariant subspaces.

(ii) $\Rightarrow$ (iii) $T|M$ is semi-regular, so by Theorem 2.49 $T|M$ has the SVEP at 0 if and only if $T|M$ is injective.

(iii) $\Rightarrow$ (iv) If $T|M$ is injective, from Theorem 1.70 the semi-regularity of $T|M$ implies that $\overline{H_0(T|M)} = \overline{\mathcal{N}^\infty(T|M)} = \{0\}$, and hence

$$H_0(T) = H_0(T|M) \oplus H_0(T|N) = \{0\} \oplus N = N.$$

The implications (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are obvious, whilst the implications (v) $\Rightarrow$ (vi) and (vii) $\Rightarrow$ (i) have been proved in Theorem 2.31.

The last assertion is clear since the pair $M := X$ and $N := \{0\}$ is a GKD for every semi-regular operator. ∎

The next result shows that if the operator $\lambda_0 I - T$ admits a generalized Kato decomposition then all the implications of Theorem 2.33 are actually equivalences.

Theorem 3.15. *Suppose that $\lambda_0 I - T \in L(X)$ admits a GKD (M, N). Then the following assertions are equivalent:*

(i) *$T^\star$ has SVEP at λ_0;*

(ii) *$(\lambda_0 I - T)\ |M$ is surjective;*

(iii) *$K(\lambda_0 I - T) = M$;*

(iv) *$X = H_0(\lambda_0 I - T) + K(\lambda_0 I - T)$;*

(v) *$H_0(\lambda_0 I - T) + K(\lambda_0 I - T)$ is norm dense in X.*

In particular, if $\lambda_0 I - T$ is semi-regular then the conditions (i)–(v) *are equivalent to the following statement:*

(vi) *$K(\lambda_0 I - T) = X$.*

Proof Also we suppose $\lambda_0 = 0$ here.

(i) $\Leftrightarrow$ (ii) We know that the pair $(N^\perp, M^\perp)$ is a GKD for $T^\star$, and hence by Theorem 3.14 $T^\star$ has SVEP at 0 if and only if $T^\star\ |N^\perp$ is injective. By Lemma 3.13 $T^\star$ then has the SVEP at 0 if and only if $T\ |M$ is onto.

(ii) $\Rightarrow$ (iii) If $T|M$ is surjective then $M = K(T|M) = K(T)$, by Theorem 1.70.

(iii) $\Rightarrow$ (iv) By assumption $X = M \oplus N = K(T) \oplus N$, and therefore $X = H_0(T) + K(T)$, since $N = H_0(T|N) \subseteq H_0(T)$.

The implication (iv) $\Rightarrow$ (v) is obvious, whilst (v) $\Rightarrow$ (i) has been established in Theorem 2.33.

The last assertion is obvious since $M := X$ and $N := \{0\}$ provides a GKD for T. ∎

Now let us consider the case $\lambda_0 I - T$ is of Kato type. In this case, by Theorem 1.74 and Theorem 3.15 we have

$$T \text{ has the SVEP at } \lambda_0 \Leftrightarrow \mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\}.$$

The next result shows that in this case to the equivalent conditions (i)–(vii) of Theorem 3.14 we can add the condition $p(\lambda_0 I - T) < \infty$.

Theorem 3.16. *Let $T \in L(X)$, X a Banach space, and assume that $\lambda_0 I - T$ is of Kato type. Then the conditions* (i)–(vii) *of Theorem 3.14 are equivalent to the following assertions:*

(viii) $p(\lambda_0 I - T) < \infty$;

(ix) $\mathcal{N}^\infty(\lambda_0 I - T) \cap (\lambda_0 I - T)^\infty(X) = \{0\}$.

In this case, if $p := p(\lambda_0 I - T)$ then

$$H_0(\lambda_0 I - T) = \mathcal{N}^\infty(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p. \tag{60}$$

Proof Here we assume also that $\lambda_0 = 0$.

Let (M, N) be a GKD for which $T|N$ is nilpotent. Assume that one of the equivalent conditions (i)–(vii) of Theorem 3.14 holds, for instance the condition $H_0(T) = N$. We also have that $\ker T^n \subseteq \mathcal{N}^\infty(T) \subseteq H_0(T)$ for every $n \in \mathbb{N}$. On the other hand, from the nilpotency of $T|N$ we know that there exists a $k \in \mathbb{N}$ for which $(T|N)^k = 0$. Therefore $H_0(T) = N \subseteq \ker T^k$ and hence $H_0(T) = \mathcal{N}^\infty(T) = \ker T^k$. Obviously this implies that $p(T) \leq k$, so the equivalent conditions (i)–(vii) of Theorem 3.14 imply (viii).

The implication (viii) $\Rightarrow$ (ix) has been established in Theorem 5.4 and the condition (ix) implies the SVEP at 0, again by Theorem 5.4. The equalities (60) are clear because $\ker T^k = \ker T^p$. ∎

Theorem 3.17. *Let $Y \in L(X)$, X a Banach space, and assume that $\lambda_0 I - T$ is of Kato type. Then the conditions* (i)–(v) *of Theorem 3.15 are equivalent to the following conditions:*

(vi) $q(\lambda_0 I - T) < \infty$;

(vii) $X = \mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$;

(viii) $\mathcal{N}^\infty(\lambda_0 I - T) + (\lambda_0 I - T)^\infty(X)$ *is norm dense in X.*

In this case, if $q := q(\lambda_0 I - T)$ then

$$(\lambda_0 I - T)^\infty(X) = K(\lambda_o I - T) = (\lambda_0 I - T)^q(X).$$

Proof Assume that $\lambda_0 = 0$. Since T is of Kato type then $K(T) = T^\infty(X)$, by Theorem 1.42. Suppose that one of the equivalent conditions (i)–(v) of Theorem 3.15 holds, in particular suppose that $K(T) = M$. Then $M = T^\infty(X) \subseteq T^n(X)$ for every $n \in \mathbb{N}$.

On the other hand, by assumption there exists a positive integer k such that $(T|N)^k = 0$, so for all $n \geq k$ we have

$$T^n(X) \subseteq T^k(X) = T^k(M) \oplus T^k(N) = T^k(M) \subseteq M,$$

and hence $T^n(X) = M$ for all $n \geq k$. Therefore $q(T) < \infty$, so (vi) is proved.

The implication (vi) $\Rightarrow$ (vii) has been established in Theorem 5.4 whilst the implication (vii) $\Rightarrow$ (viii) is obvious. Finally, by Corollary 2.34 the condition (viii) implies the SVEP at 0 for $T^\star$, which is the condition (i) of Theorem 3.15.

The last assertion is clear since $T^q(X) = T^k(X)$. ∎

In the next result we consider the case where $\lambda_0 I - T$ is essentially semi-regular, namely N is finite-dimensional and M is finite-codimensional.

Theorem 3.18. *Suppose that $\lambda_0 I - T \in L(X)$ is a essentially semi-regular. Then the conditions* (i)–(vii) *of Theorem 3.14 and the conditions* (viii)–(ix) *of Theorem 3.16 are equivalent to the following condition:*

(a) *The quasi-nilpotent part $H_0(\lambda_0 I - T)$ is finite-dimensional.*

In particular, if T has the SVEP at λ_0 then $\lambda_0 I - T \in \Phi_+(X)$.

Again, the conditions (i)–(v) *of Theorem 3.15 and the conditions* (vi)–(viii) *of Theorem 3.17 are equivalent to the following condition:*

(b) *The analytical core $K(\lambda_0 I - T)$ is finite codimensional.*

In particular, if $T^\star$ has the SVEP at λ_0 then $\lambda_0 I - T \in \Phi_-(X)$.

Proof The condition (iv) of Theorem 3.14 implies (a) and this implies the condition (v) of Theorem 3.14. Analogously, the condition (iii) of Theorem 3.15 implies (b) whilst from (b) it follows that $(\lambda_0 I - T)^\infty(X) = K(\lambda_0 I - T)$ is finite-codimensional, see Theorem 1.42. Because $(\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)^q(X)$ for every $q \in \mathbb{N}$ we then may conclude that $q(\lambda_0 I - T) < \infty$, which is the condition (vi) of Theorem 3.17.

It remains to establish that (a) implies that $\lambda_0 I - T \in \Phi_+(X)$. Clearly if $H_0(\lambda_0 I - T)$ is finite- dimensional then its subspace $\ker(\lambda_0 I - T)$ is finite-dimensional. Moreover, if (M, N) is a GKD for $\lambda_0 I - T$ such that N is finite-dimensional then

$$(\lambda_0 I - T)(X) = (\lambda_0 I - T)(M) + (\lambda_0 I - T)(N)$$

is closed since it is the sum of the closed subspace $(\lambda_0 I - T)(M)$ and a finite-dimensional subspace of X. This shows that $\lambda_0 I - T \in \Phi_+(X)$.

Analogously, from the inclusion $K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)(X)$ we see that if $K(\lambda_0 I - T)$ finite codimensional then also $(\lambda_0 I - T)(X)$ is finite codimensional, so $\lambda_0 I - T \in \Phi_-(X)$. ∎

Corollary 3.19. *Let $T \in L(X)$, X a Banach space, and suppose that $\lambda_0 I - T \in \Phi_\pm(X)$. We have:*

(i) *If T has the SVEP at λ_0 then* ind $(\lambda_0 I - T) \leq 0$*;*

(ii) *If $T^\star$ has the SVEP at λ_0 then* ind $(\lambda_0 I - T) \geq 0$.

Consequently, if both the operators T and $T^\star$ have the SVEP at λ_0 then $\lambda_0 I - T$ has index 0.

Proof By Theorem 3.16 if T has the SVEP then $p(\lambda_0 I - T) < \infty$, and hence $\alpha(\lambda_0 I - T) \leq \beta(\lambda_0 I - T)$ by part (i) of Theorem 3.4. This shows the assertion (i). The assertion (ii) follows similarly from Theorem 3.17 and part (ii) of Theorem 3.4. The last assertion is clear. ∎

The following example shows that a Fredholm operator T having index less than 0 may be without the SVEP at 0.

Example 3.20. Let R and L denote the right shift operator and the left shift operator, respectively, on the Hilbert space $H := \ell^2(\mathbb{N})$, defined by

$$R(x) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x) := (x_2, x_3, \dots)$$

for all $(x) := (x_n) \in \ell_2(\mathbb{N})$. Clearly

$$\alpha(R) = \beta(L) = 0 \quad \text{and} \quad \alpha(L) = \beta(R) = 1,$$

so L and R are Fredholm. Let $e_n := (0, \cdots, 0, 1, 0, \dots) \in \ell^2(\mathbb{N})$, where 1 is the n-th term and all others are equal to 0. It is easily seen that $e_{n+1} \in \ker\ L^{n+1}$ whilst $e_{n+1} \notin \ker\ L^n$ for every $n \in \mathbb{N}$, so $p(L) = \infty$. Moreover, $p(R) = 0$ being R injective, and hence, since R and S are each one the adjoint of the other,

$$p(L) = q(R) = \infty \quad \text{and} \quad q(L) = p(R) = 0.$$

Consider the operator $L \oplus R \in L(H \times H)$ defined by

$$(L \oplus R)(x, y) := (Lx, Ry), \quad \text{with } x, y \in \ell_2(\mathbb{N}).$$

It is easy to verify that

$$\alpha(L \oplus R) = \alpha(L) = 1, \quad \beta(L \oplus R) = 1 \quad \text{and} \quad p(L \oplus R) = \infty.$$

Analogously, if $T := L \oplus R \oplus R \in L(H \times H \times H)$ then

$$\beta(T) = 2, \quad \alpha(T) = \alpha(L) = 1 \quad \text{and} \quad p(T) = \infty,$$

so T is a Fredholm operator having index ind $T < 0$ which by Theorem 5.4 does not have the SVEP at 0.

Corollary 3.21. *Let $\lambda_0 \in \sigma(T)$ and assume that $\lambda_0 I - T \in \Phi_\pm(X)$. Then the following statements are equivalent:*

(i) *T and $T^\star$ have the SVEP at λ_0;*

(ii) *$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$;*

(iii) *$H_0(\lambda_0 I - T)$ is closed and $K(\lambda_0 I - T)$ is finite-codimensional;*

(iv) *λ_0 is a pole of the resolvent $(\lambda I - T)^{-1}$, equivalently $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$;*

(v) *λ_0 is an isolated point of $\sigma(T)$.*

In particular, if any of the equivalent conditions (i)–(v) *holds and $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ then*

$$H_0(\lambda_0 I - T) = \mathcal{N}^\infty(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$K(\lambda_0 I - T) = (\lambda_0 I - T)^\infty(X) = (\lambda_0 I - T)^p(X).$$

Proof The equivalences of (i), (ii), (iii), and (iv) are obtained by combining all the results previously established. The implication (iv) $\Rightarrow$ (v) is obvious whilst the implication (v) $\Rightarrow$ (i) is an immediate consequence of the fact that both T and $T^\star$ have the SVEP at every isolated point of the spectrum $\sigma(T) = \sigma(T^\star)$. ∎

The next result shows that also the implications of Theorem 2.36 and Corollary 2.37 are actually equivalences if we assume that $\lambda_0 I - T$ is essentially semi-regular.

Theorem 3.22. *Suppose that $\lambda_0 I - T \in L(X)$, X a Banach space, is essentially semi-regular. Then T has the SVEP at λ_0 if and only if one of the following conditions hold:*

(c) $\mathcal{N}^\infty(\lambda_0 I - T^\star) + (\lambda_0 I - T^\star)(X^\star)$ *is weak $\star$-dense in $X^\star$;*

(d) $H_0(\lambda_0 I - T^\star) + (\lambda_0 I - T^\star)(X^\star)$ *is weak $\star$-dense in $X^\star$;*

(e) $H_0(\lambda_0 I - T^\star) + K(\lambda_0 I - T^\star)$ *is weak $\star$-dense in $X^\star$.*

Proof Assume $\lambda_0 = 0$. If one of the conditions (c), (d) and (e) holds then, by Theorem 2.36 and Corollary 2.37, T has the SVEP at 0.

Conversely, suppose that T has the SVEP at 0 or, equivalently by Theorem 2.22, that $K(T) \cap \ker T = \{0\}$. We know from Theorem 1.70 that if T is essentially semi-regular then ${}^\perp\mathcal{N}^\infty(T^\star) = K(T)$. Since $\ker T = {}^\perp T^\star(X^\star)$ we then obtain

$$\{0\} = K(T) \cap \ker T = {}^\perp \mathcal{N}^\infty(T^\star) \cap {}^\perp T^\star(X^\star),$$

which implies via the Hahn–Banach theorem that the sum $\mathcal{N}^\infty(T^\star) + T^\star(X^\star)$ is weak $\star$- dense in $X^\star$.

The proof that the SVEP of T at 0 implies each one of the conditions (d) and (e) is similar, and therefore is omitted. ∎

If $\lambda_0 I - T$ is of Kato type, the SVEP at λ_0 is simply characterized in terms of the approximate point spectrum as follows:

Theorem 3.23. *Suppose that $\lambda_0 I - T$, X a Banach space, is of Kato type. Then the following statements are equivalent:*

(i) *T has SVEP at λ_0;*

(ii) *$\sigma_{\mathrm{ap}}(T)$ does not cluster at λ_0.*

Proof If $\sigma_{\mathrm{ap}}(T)$ does not cluster at λ_0 then T has the SVEP at λ_0, see the observation just before Theorem 2.58. Hence we need only to prove the implication (i) $\Rightarrow$ (ii). We may suppose that $\lambda_0 = 0$.

Assume that T has SVEP at 0 and let (M, N) be a GKD for T. From Theorem 1.44 we know that there exists $\varepsilon > 0$ such that $\lambda I - T$ is semi-regular, and hence has closed range for every $0 < |\lambda| < \varepsilon$. If $\mathbb{D}_\varepsilon$ denotes the open disc centered at 0 with radius ε then $\lambda \in (\mathbb{D}_\varepsilon \setminus \{0\}) \cap \sigma_{\mathrm{ap}}(T)$ if and only if λ is an eigenvalue for T.

Now, from the inclusion $\ker(\lambda I - T) \subseteq T^\infty(X)$ for every $\lambda \neq 0$ we infer that every non-zero eigenvalue of T belongs to the spectrum of the restriction $T|T^\infty(X)$.

Finally, assume that 0 is a cluster point of $\sigma_{\mathrm{ap}}(T)$. Let (λ_n) be a sequence of non-zero eigenvalues which converges to 0. Then $\lambda_n \in \sigma(T|T^\infty(X))$ for every $n \in \mathbb{N}$, and hence $0 \in \sigma(T|T^\infty(X))$, since the spectrum of an operator is closed. But T has the SVEP at 0, so by Theorem 3.14 the restriction $T|M$ is injective, and hence by Theorem 1.41 $\{0\} = \ker T|M = \ker T \cap T^\infty(X)$. This shows that the restriction $T|T^\infty(X)$ is injective.

On the other hand, from the equality $T(T^\infty(X)) = T^\infty(X)$ we know that $T\ |T^\infty(X)$ is surjective, so $0 \notin \sigma(T\ |T^\infty(X))$; a contradiction. ∎

The result of Theorem 3.23 is quite useful for establishing the membership of cluster points of some distinguished parts of the spectrum to the Kato type spectrum $\sigma_{\mathrm{k}}(T)$.

A first application is given from the following result which improves a classical Putnam theorem about the non-isolated boundary points of the spectrum as a subset of the Fredholm spectrum.

Corollary 3.24. *If $T \in L(X)$, X a Banach space, every non-isolated boundary point of $\sigma(T)$ belongs to $\sigma_{\mathrm{k}}(T)$. In particular, every non-isolated boundary point of $\sigma(T)$ belongs to the Fredholm spectrum $\sigma_{\mathrm{f}}(T)$.*

Proof If $\lambda_0 \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$ then $\sigma_{\mathrm{ap}}(T)$ clusters at λ_0, since by Theorem 2.42 $\partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T)$. But T has the SVEP at every point of $\partial\sigma(T)$, so by Theorem 3.23 $\lambda_0 I - T$ is not of Kato type.

The last assertion is obvious since $\sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{f}}(T)$. ∎

Corollary 3.25. *Suppose that $T \in L(X)$, X a Banach space, has the SVEP. Then all cluster points of $\sigma_{\mathrm{ap}}(T)$ belong to $\sigma_{\mathrm{k}}(T)$ and in particular to $\sigma_{\mathrm{f}}(T)$.*

Proof Suppose that $\lambda_0 \notin \sigma_{\mathrm{k}}(T)$. Since T has the SVEP , and in particular has the SVEP at λ_0, by Theorem 3.23 it follows that $\sigma_{\mathrm{ap}}(T)$ does not cluster at λ_0. ∎

The next result gives a clear description of the points $\lambda_0 \notin \sigma_{\mathrm{k}}(T)$ which belong to the boundary of $\sigma(T)$.

Theorem 3.26. *Let $T \in L(X)$, X a Banach space, and suppose that $\lambda_0 \in \partial\sigma(T)$. Then $\lambda_0 I - T$ is of Kato type if and only if λ_0 is a pole of the resolvent $R(\lambda, T)$.*

Proof By Corollary 3.24 if $\lambda_0 I - T$ is of Kato type then λ_0 is isolated in $\sigma(T)$. Moreover T and $T^\star$ have the SVEP at $\lambda_0 \in \partial\sigma(T) = \partial\sigma(T^\star)$, so by Theorem 3.23 and Theorem 3.27 it follows that both $p(\lambda_0 I - T)$ and $q(\lambda_0 I - T)$ are finite,.

Conversely, assume that λ_0 is a pole of the resolvent $R(\lambda, T)$ or, equivalently, that $d := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$. Put $M := (\lambda_0 I - T)^d(X)$ and $N := \ker(\lambda_0 I - T)^d$. Then the restriction $\lambda_0 I - T|M$ is bijective by Theorem 3.6, and hence semi-regular. Obviously the restriction $\lambda_0 I - T|N$ is nilpotent. ∎

The following result is dual to that established in Theorem 3.23.

Theorem 3.27. *Suppose that $\lambda_0 I - T$, X a Banach space, is of Kato type. Then the following properties are equivalent:*

(i) *$T^\star$ has the SVEP at λ_0;*

(ii) *$\sigma_{\mathrm{su}}(T)$ does not cluster at λ_0.*

Proof The equivalence immediately follows from Theorem 3.23 since $\sigma_{\mathrm{ap}}(T^\star) = \sigma_{\mathrm{su}}(T)$. ∎

Corollary 3.28. *Suppose that for $T \in L(X)$, X a Banach space, $T^\star$ has the SVEP. Then all cluster points of $\sigma_{\mathrm{su}}(T)$ belong to $\sigma_{\mathrm{k}}(T)$.*

Proof Suppose that $\lambda_0 \notin \sigma_{\mathrm{k}}(T)$. Since $T^\star$ has the SVEP at λ_0 by Theorem 3.27 it follows that $\sigma_{\mathrm{su}}(T)$ does not cluster at λ_0. ∎

Since $\sigma_{\mathrm{se}}(T)$, $\sigma_{\mathrm{es}}(T)$ and $\sigma_{\mathrm{k}}(T)$ are subsets of $\sigma_{\mathrm{ap}}(T)$, one may ask if T has the SVEP at a point λ_0 whenever one of these spectra does not cluster at λ_0. Generally this is not true. To see this it suffices to consider the case that $\sigma_{\mathrm{se}}(T)$ does not cluster at λ_0, since $\sigma_{\mathrm{es}}(T)$ and $\sigma_{\mathrm{k}}(T)$ are subsets of $\sigma_{\mathrm{se}}(T)$.

Let $T \in L(X)$ be any non-injective semi-regular operator T. Then 0 is a point of the semi-regular resolvent $\rho_{\mathrm{se}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{se}}(T)$. This implies, $\rho_{\mathrm{se}}(T)$ being an open set of $\mathbb{C}$, that $\sigma_{\mathrm{se}}(T)$ does not cluster at 0. On the other hand, since T is not injective then by Theorem 2.49 T does not have the SVEP at 0.

Theorem 3.29. *Suppose that for $T \in L(X)$, X is a Banach space, the semi-regular spectrum $\sigma_{\mathrm{se}}(T)$ clusters at λ_0. Then $\lambda_0 \in \sigma_{\mathrm{sf}}(T)$.*

Proof Suppose that $\sigma_{\mathrm{se}}(T)$ clusters at λ_0 and $\lambda_0 I - T \in \Phi_\pm(X)$. Then $\lambda_0 \in \rho_{\mathrm{sf}}(T)$, where $\rho_{\mathrm{sf}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{sf}}(T)$ is the semi-Fredholm resolvent of T. Let Ω denote the component of the open set $\rho_{\mathrm{sf}}(T)$ which contains the point λ_0. If we set $\Gamma := \Omega \cap \sigma_{\mathrm{se}}(T)$ then $\Gamma \subseteq \sigma_{\mathrm{se}}(T) \setminus \sigma_{\mathrm{sf}}(T)$ and by Theorem 1.65 the last set is denumerable.

This shows that there exists a an open disc $\mathbb{D}(\lambda_0)$ centered at λ_0 such that $\lambda \notin \sigma_{\mathrm{se}}(T) \setminus \sigma_{\mathrm{sf}}(T)$ for every $\lambda \in \mathbb{D}(\lambda_0) \setminus \{\lambda_0\}$. But $\lambda I - T$ are semi-Fredholm for all $\lambda \in \Omega$, so $\lambda \notin \sigma_{\mathrm{se}}(T)$ for all $\lambda \in \mathbb{D}(\lambda_0) \setminus \{\lambda_0\}$; a contradiction. ∎

Example 3.30. Let $1 \le p < \infty$ and let T denote an arbitrary weighted right shift operator on $\ell^p(\mathbb{N})$. It has been already observed that

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : i(t) \le |\lambda| \le r(T)\},$$

where the quantity $i(T)$ is defined before Theorem 2.54. Since T has the SVEP, see Theorem 2.88, by part (i) of Theorem 2.45 we have that $\sigma_{\rm ap}(T) = \sigma_{\rm se}(T)$. We prove that

$$\sigma_{\rm f}(T) = \sigma_{\rm k}(T) = \sigma_{\rm es}(T) = \sigma_{\rm sf}(T) = \sigma_{\rm ap}(T).$$

The inclusion $\sigma_{\rm ap}(T) \subseteq \sigma_{\rm f}(T)$ follows from Corollary 3.25. We show now the opposite inclusion.

Assume that $\lambda \notin \sigma_{\rm ap}(T)$. In the case $|\lambda| > r(T)$ then $\lambda I - T$ is invertible, so $\lambda \notin \sigma_{\rm f}(T)$.

Assume the other case $|\lambda| < i(T)$ and therefore $|\lambda| < c(T)$, where the quantity $c(T)$ is defined as in the discussion after Example 2.60. To show that $\lambda I - T$ is Fredholm we need only to prove that $\beta(\lambda I - T) < \infty$, because by assumption $\alpha(\lambda I - T) = 0$. Now, $\lambda I - T$ is bounded below so $(\lambda I - T)(X)$ is closed and hence $\ker\,(\lambda I - T^\star) = [(\lambda I - T)(X)]^\perp$. The last space is canonically isomorphic to the quotient space $(X/(\lambda I - T)(X))^\star$ so, to show that $(\lambda I - T)(X)$ is finite-codimensional, it suffices to prove that $\ker\,(\lambda I - T^\star)$ is finite-dimensional. We prove that $\dim \ker\,(\lambda I^* - T^\star) = 1$.

Assume that $T^\star x = \lambda x$. Then $x_{n+1}\omega_n = \lambda x_n$ for every $n \in \mathbb{N}$, where $\{\omega_n\}$ is the sequence which defines T. If we impose the normalization $x_1 = 1$ and define the empty product to be 1, a simple recursive argument shows that this system has the unique solution

$$x = (x_n)_{n\in\mathbb{N}}, \quad \text{where} \quad x_n := \frac{\lambda^{n-1}}{\omega_1 \cdots \omega_{n-1}}.$$

The standard formula for the radius of convergence of a series then yields that the series $\sum_{n=1}^\infty e_n x_n$ converges for all $|\lambda| < c(T)$. This shows that $x = (x_n) \in \ell^q(\mathbb{N})$. Hence the solutions of the equation $T^\star x = \lambda x$ form a 1-dimensional subspace of $\ell^q(\mathbb{N})$, and hence

$$\beta(\lambda I - T)(X) = \alpha(\lambda I^\star - T^\star) = 1.$$

Therefore $\lambda \notin \sigma_{\rm f}(T)$, so the reverse inclusion $\sigma_{\rm f}(T) \subseteq \sigma_{\rm ap}(T)$ is proved. From this we then conclude that $\sigma_{\rm se}(T) = \sigma_{\rm ap}(T) = \sigma_{\rm f}(T)$, as desired.

The inclusions $\sigma_{\rm k}(T) \subseteq \sigma_{\rm ap}(T)$ and $\sigma_{sf}(T) \subseteq \sigma_{\rm ap}(T)$ are true for every operator. The opposite of these inclusions follows from Corollary 3.25 and Theorem 3.29, respectively. The equality of all these spectra with $\sigma_{\rm es}(T)$ is then clear, since $\sigma_{\rm k}(T) \subseteq \sigma_{\rm es}(T) \subseteq \sigma_{\rm f}(T)$ for every $T \in L(X)$.

3. The SVEP on the components of $\rho_{\rm k}(T)$

In this section we shall give a closer look at the connected components of resolvent sets associated with the various spectra introduced in the previous chapters. In particular, we shall obtain a classification of these components by using the equivalences between the SVEP at a point and the kernel type and range type conditions established in the previous section.

The results of Theorem 1.36 and Theorem 1.72 show that the mappings $\lambda \to K(\lambda I - T)$ and $\lambda \to \overline{H_0(\lambda I - T)}$ are constant as λ ranges through a connected component of the semi-regular resolvent $\rho_{\mathrm{se}}(T)$. Now, also the Kato type resolvent $\rho_{\mathrm{k}}(T)$ is an open subset of $\mathbb{C}$, so it may be decomposed in connected maximal components. The first result of this section shows the constancy of some mappings on the components of $\rho_{\mathrm{k}}(T)$.

Theorem 3.31. *Let $T \in L(X)$, X a Banach space, be of Kato type. Then there is an $\varepsilon > 0$ such that:*

(i) *$K(\lambda I - T) + H_0(\lambda I - T) = K(T) + H_0(T)$ for all $0 < |\lambda| < \varepsilon$, and*

(ii) *$K(\lambda I - T) \cap \overline{H_0(\lambda I - T)} = K(T) \cap \overline{H_0(T)}$ for all $0 < |\lambda| < \varepsilon$.*

Proof (i) Let (M, N) be a GKD for T such that $T\,|N$ is nilpotent. From the proof of part (i) of Theorem 1.74 we know that $K(T) + H_0(T) = K(T) + N$.

Now, from Theorem 1.44 there exists $\varepsilon > 0$ such that $\lambda I - T$ is semi-regular for all $0 < |\lambda| < \varepsilon$, and hence by Theorem 1.70,

$$H_0(\lambda I - T) \subseteq K(\lambda I - T) \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{61}$$

Clearly, the nilpotency of $T\,|N$ implies that $(\lambda I - T)^n(N) = N$ for all $\lambda \neq 0$ and $n \in \mathbb{N}$. From Theorem 1.24 it then follows that

$$\begin{aligned} K(\lambda I - T) &= (\lambda I - T)^{\infty}(X) = (\lambda I - T|M)^{\infty}(M) + (\lambda I - TN)^{\infty}(N) \\ &= K(\lambda I - T|M) + N, \end{aligned}$$

for all $\lambda \neq 0$. Since $T\,|M$ is semi-regular by Theorem 1.36 we may choose $\varepsilon > 0$ such that $K(\lambda I - T|M) = K(T|M) = K(T)$, and hence

$$K(\lambda I - T) = K(T) + N \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{62}$$

By Theorem 1.22 the equality $(\lambda I - T)(N) = N$ implies that

$$N \subseteq K(\lambda I - T) \ \text{ for all } \lambda \neq 0.$$

Finally, from (61) and (62) for all $0 < |\lambda| < \varepsilon$ we obtain that

$$H_0(\lambda I - T) + K(\lambda I - T) = K(\lambda I - T) = K(T) + N = K(T) + H_0(T),$$

so the first statement is proved.

(ii) Recall that from the proof of part (ii) of Theorem 1.74, if $\lambda I - T$ is of Kato type then

$$\overline{H_0((\lambda I - T)|M)} = \overline{H_0(\lambda I - T) \cap M} = \overline{H_0(\lambda I - T)} \cap M.$$

Now, by Theorem 1.44 we know that there is $\varepsilon > 0$ such that $\lambda I - T$ is semi-regular for all $0 < |\lambda| < \varepsilon$. Hence by Theorem 1.70 and Theorem 1.24, we have

$$\overline{H_0(\lambda I - T)} \subseteq \overline{K(\lambda I - T)} = K(\lambda I - T) \quad \text{for all } 0 < |\lambda| < \varepsilon,$$

and therefore

$$\overline{H_0(\lambda I - T)} = \overline{H_0(\lambda I - T)} \cap K(\lambda I - T) \quad \text{for all } 0 < |\lambda| < \varepsilon. \tag{63}$$

From Lemma 1.3 and Theorem 1.70 we also have

$$\overline{H_0(\lambda I - T)} = \overline{\mathcal{N}^{\infty}(\lambda I - T)} \subseteq T^{\infty}(X) = K(T) \subseteq M, \tag{64}$$

for all $0 < |\lambda| < \varepsilon$. By Theorem 1.70 the semi-regularity of $T|M$ yields that $\overline{H_0(T|M)} \subseteq K(T|M) = K(T)$, so

$$\begin{aligned}\overline{H_0(T)} \cap K(T) &= \overline{H_0(T)} \cap (M \cap K(T)) = \overline{H_0(T) \cap M} \cap K(T) \\ &= \overline{H_0(T|M)} \cap K(T) = \overline{H_0(T|M)}.\end{aligned}$$

Finally, by Theorem 1.72 we may chosen $\varepsilon > 0$ such that

$$\overline{H_0(T|M)} = \overline{H_0(\lambda I - T|M)} = \overline{H_0(\lambda I - T)} \cap M \quad \text{for all } |\lambda| < \varepsilon.$$

Using the inclusions (64) and (63) we then conclude that

$$\begin{aligned}\overline{H_0(T)} \cap K(T) &= \overline{H_0(T|M)} = \overline{H_0(\lambda I - T)} \cap M \\ &= \overline{H_0(\lambda I - T)} = \overline{H_0(\lambda I - T)} \cap K(\lambda I - T)\end{aligned}$$

for all $|\lambda| < \varepsilon$, so also (ii) is proved. ■

By using a compactness argument similar to that which has been used in the proof of Theorem 1.36 we obtain the following result:

Corollary 3.32. *Let $T \in L(X)$, X a Banach space. If Ω is a component of $\rho_{\rm k}(T)$ and $\lambda_0 \in \Omega$ is arbitrarily given, then*

$$K(\lambda I - T) + H_0(\lambda I - T) = K(\lambda_0 I - T) + H_0(\lambda_0 I - T)$$

and

$$\overline{H_0(\lambda I - T)} \cap K(\lambda I - T) = \overline{H_0(\lambda_0 I - T)} \cap K(\lambda_0 I - T)$$

for all $\lambda \in \Omega$. Therefore the mappings

$$\lambda \to K(\lambda I - T) + H_0(\lambda I - T)$$

and

$$\lambda \to \overline{H_0(\lambda I - T)} \cap K(\lambda I - T)$$

are constant on the connected components of $\rho_{\rm k}(T)$. ■

Remark 3.33. As an obvious consequence of Theorem 1.74 we obtain that the mappings

$$\lambda \to H_0(\lambda I - T) + K(\lambda I - T), \quad \lambda \to \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X),$$

and

$$\lambda \to \overline{H_0(\lambda I - T)} \cap K(\lambda I - T), \quad \lambda \to \overline{\mathcal{N}^\infty(\lambda I - T)} \cap (\lambda I - T)^\infty(X)$$

assume the same values on each component of $\rho_{\rm k}(T)$.

From Theorem 3.31 and the results established in the previous section we now obtain the following classification of the components of $\rho_{\rm k}(T)$.

Theorem 3.34. *Let $T \in L(X)$, X a Banach space, and Ω a component of the Kato type resolvent $\rho_{\mathrm{k}}(T)$. Then the following alternative holds:*

(i) *T has the SVEP for every point of Ω. In this case $p(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, $\sigma_{\mathrm{ap}}(T)$ does not have limit points in Ω; every point of Ω is not an eigenvalue of T, except a subset of Ω which consists of at most countably many isolated points.*

(ii) *T has the SVEP at no point of Ω. In this case $p(\lambda I - T) = \infty$ for all $\lambda \in \Omega$. Every point of Ω is an eigenvalue of T.*

Proof (i) Suppose that T has the SVEP at $\lambda_0 \in \Omega$. Then by Theorem 3.14 $H_0(\lambda I - T)$ is closed and

$$H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \overline{H_0(\lambda_0 I - T)} \cap K(\lambda_0 I - T) = \{0\}.$$

By Corollary 3.32 the mapping $\lambda \to \overline{H_0(\lambda I - T)} \cap K(\lambda I - T)$ are constant on Ω, so

$$\overline{H_0(\lambda I - T)} \cap K(\lambda I - T) = \{0\} \quad \text{for all } \lambda \in \Omega$$

and therefore, again by Theorem 3.14, T has the SVEP at every $\lambda \in \Omega$. This is equivalent by Theorem 3.16 to saying that $p(\lambda I - T) < \infty$ for all $\lambda \in \Omega$.

By Theorem 3.23 $\sigma_{\mathrm{ap}}(T)$ does not clusters in Ω, and consequently every point of Ω is not an eigenvalue of T, except a subset of Ω which consists of at most countably many isolated points.

(ii) It is clear, again by Theorem 3.16. ∎

Recall that $\lambda \in \mathbb{C}$ is said to be a *deficiency value* for if $\lambda I - T$ is not surjective.

Theorem 3.35. *Let $T \in L(X)$, X a Banach space, and Ω a component of $\rho_{\mathrm{k}}(T)$. Then the following alternative holds:*

(i) *$T^\star$ has the SVEP for every point of Ω. In this case $q(\lambda I - T) < \infty$ for all $\lambda \in \Omega$ and $\sigma_{\mathrm{su}}(T)$ does not have limit points in Ω and $\lambda I - T$ is not a deficiency value, except a subset of Ω which consists of at most countably many isolated points.*

(ii) *$T^\star$ has the SVEP at no point of Ω. In this case $q(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and every $\lambda \in \Omega$ is a deficiency value of T.*

Proof Proceed as in the proof of Theorem 3.34, combining the constancy of the mapping

$$\lambda \in \Omega \to K(\lambda I - T) + H_0(\lambda I - T)$$

with Theorem 3.16, Theorem 3.17 and Theorem 3.15. ∎

Let us consider the Fredholm resolvent $\rho_{\mathrm{sf}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{sf}}(T)$. Clearly

$$\rho_{\mathrm{sf}}(T) \subseteq \rho_{\mathrm{es}}(T) \subseteq \rho_{\mathrm{k}}(T),$$

and all these sets are open. It is natural to ask what happens for the components of $\rho_{\mathrm{sf}}(T)$ and the components of $\rho_{\mathrm{es}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{es}}(T)$. The classification of the components of $\rho_{\mathrm{es}}(T)$ may be easily obtained from Theorem 3.34 and Theorem 3.35, once observed, by Corollary 1.45, that the two sets $\rho_{\mathrm{es}}(T)$ and $\rho_{\mathrm{k}}(T)$ may be different only for a denumerable set.

A more interesting situation is that relative to the components of $\rho_{\mathrm{sf}}(T)$, since for semi-Fredholm operators we can consider the index. Clearly, by Theorem 3.34 and Theorem 3.35 T, as well as $T^\star$, has the SVEP either for every point or no point of a component Ω of $\rho_{\mathrm{sf}}(T)$.

We can classify the components of $\rho_{\mathrm{sf}}(T)$ as follows:

Theorem 3.36. *Let $T \in L(X)$, X a Banach space, and Ω a component of $\rho_{\mathrm{sf}}(T)$. For the SVEP, the index, the ascent and the descent on Ω, there are exactly the following four possibilities:*

(i) *Both T and $T^\star$ have the SVEP at every point of Ω. In this case we have* ind $(\lambda I - T) = 0$ *and $p(\lambda I - T) = q(\lambda I - T) < \infty$ for every $\lambda \in \Omega$. The eigenvalues and deficiency values do not have a limit point in Ω. This case occurs exactly when Ω intersects the resolvent $\rho(T)$;*

(ii) *T has the SVEP at the points of Ω, whilst $T^\star$ fails to have the SVEP at the points of Ω. In this case we have* ind $(\lambda I - T) < 0$, *$p(\lambda I - T) < \infty$, and $q(\lambda I - T) = \infty$ for every $\lambda \in \Omega$. The eigenvalues do not have a limit point in Ω, every point of Ω is a deficiency value;*

(iii) *$T^\star$ has the SVEP at the points of Ω, whilst T fails to have the SVEP at the points of Ω. In this case we have* ind $(\lambda I - T) > 0$, *$p(\lambda I - T) = \infty$, and $q(\lambda I - T) < \infty$ for every $\lambda \in \Omega$. The deficiency values do not have a limit point in Ω, whilst every point of Ω is an eigenvalue;*

(iv) *Neither T or $T^\star$ have the SVEP at the points of Ω. In this case we have $p(\lambda I - T) = q(\lambda I - T) = \infty$ for every $\lambda \in \Omega$. The index may assume every value in $\mathbb{Z}$; all the points of Ω are eigenvalues and deficiency values.*

Proof The case (i) is clear from the results established in the previous section, from Theorem 3.34 and Theorem 3.34. In the case (ii) the condition $p(\lambda I - T) < \infty$ implies that ind $(\lambda I - T) \leq 0$, by part (iii) of Theorem 3.4, whilst the condition $q(\lambda I - T) = \infty$ excludes that ind $(\lambda I - T) = 0$, again by Theorem 3.4, part (iv).
A similar argument shows in the case (iii) that ind $(\lambda I - T) > 0$.

The statements of (iv) are clear. ∎

The following corollary establishes that a very simple classification of the semi-Fredholm is obtained in the case that T or $T^\star$ has the SVEP. Note that the case that both T and $T^\star$ have the SVEP applies in particular to the decomposable operators, which will be studied later.

Corollary 3.37. *Let $T \in L(X)$, X a Banach space, and Ω any component of $\rho_{\mathrm{sf}}(T)$. If T has the SVEP then only the case* (i) *and* (ii) *of Theorem 3.36 are possible, whilst if $T^\star$ has the SVEP only the case* (i) *and* (iii) *are possible. Finally, if both T and $T^\star$ have the SVEP then only the case* (i) *is possible.* ∎

4. The Fredholm, Weyl, and Browder spectra

Two important classes of operators in Fredholm theory are given by the classes of semi-Fredholm operators which possess finite ascent or a finite descent. We shall distinguish two classes of operators. The class of all *upper semi-Browder operators* on a Banach space X that is defined by

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators* that is defined by

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* (known in the literature also as *Riesz Schauder operators*) is defined by

$$\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X) = \{T \in \Phi(X) : p(T),\ q(T) < \infty\}.$$

Clearly, from part (i) and part (ii) of Theorem 3.4 we have

$$T \in \mathcal{B}_+(X) \Rightarrow \text{ind } T \leq 0,$$

and

$$T \in \mathcal{B}_-(X) \Rightarrow \text{ind } T \geq 0,$$

so that

$$T \in \mathcal{B}(X) \Rightarrow \text{ind } T = 0.$$

From Remark 3.7 and Remark 1.54, we also obtain that

$$T \in \mathcal{B}_+(X) \Leftrightarrow T^\star \in \mathcal{B}_-(X^\star)$$

and, analogously,

$$T \in \mathcal{B}_-(X) \Leftrightarrow T^\star \in \mathcal{B}_+(X^\star).$$

A bounded operator $T \in L(X)$ is said to be a *Weyl operator* if T is a Fredholm operator having index 0. Denote by $\mathcal{W}(X)$ the class of all Weyl operators. Obviously $\mathcal{B}(X) \subseteq \mathcal{W}(X)$ and the inclusion is strict, see the operator $L \oplus R$ of Example 3.20. Combining Theorem 3.16, Theorem 3.17, and Theorem 3.4, we easily obtain for a Weyl operator T the following equivalence:

$$T \;\text{ has the the SVEP at } 0 \Leftrightarrow T^\star \text{ has the SVEP at } 0.$$

Moreover, if T or $T^\star$ has SVEP at 0 from Theorem 3.4 we deduce that

$$T \text{ is Weyl } \Leftrightarrow T \text{ is Browder.}$$

Note that by part (c) of Remark 3.7 if $0 \in \sigma(T)$ and T is Browder then 0 is an isolated point of $\sigma(T)$. Furthermore, from Theorem 3.16 and Theorem 3.17 it follows that

$$X = \mathcal{N}^\infty(T) \oplus T^\infty(X) = H_0(T) \oplus K(T) = \ker T^p \oplus T^p(X),$$

where $p := p(T) = q(T)$.

The classes of operators defined above motivate the definition of several spectra. The *upper semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

whilst the *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm b}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

Clearly

$$\sigma_{\rm b}(T) = \sigma_{\rm ub}(T) \cup \sigma_{\rm lb}(T).$$

The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}(X)\}.$$

Obviously

$$\sigma_{\rm f}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T). \tag{65}$$

Moreover, part (g) of Remark 1.54 ensures that if X is infinite-dimensional then $\sigma_{\rm f}(T)$ is non-empty, and consequently also $\sigma_{\rm w}(T)$ and $\sigma_{\rm b}(T)$ are non-empty.
It is clear that

$$\sigma_{\rm sf}(T) \subseteq \sigma_{\rm uf}(T) \subseteq \sigma_{\rm ub}(T) \subseteq \sigma_{\rm b}(T),$$

and

$$\sigma_{\rm sf}(T) \subseteq \sigma_{\rm lf}(T) \subseteq \sigma_{\rm lb}(T) \subseteq \sigma_{\rm b}(T).$$

It is easy to see that, in general, the inclusions (65) are proper. For instance, for the right shift operator R on $\ell_2(\mathbb{N})$ of Example 3.20 we have $0 \in \sigma_{\rm w}(T)$, whilst $0 \notin \sigma_{\rm f}(T)$. Furthermore, if $T := L \oplus R$ then $0 \notin \sigma_{\rm w}(T)$, whilst $0 \in \sigma_{\rm b}(T)$.

Remark 3.38. Recall that by $F(X)$ we denote the ideal in $L(X)$ of all finite-dimensional operators. A basic result of operator theory establishes that every finite-dimensional operator $T \in L(X)$ may be always represented in the form

$$Tx = \sum_{k=1}^{n} f_k(x)x_k,$$

where the vectors $x_1, \dots, x_n$ from X and the vectors $f_1, \dots, f_n$ from $X^\star$ are linearly independent, see Heuser[**159**, p. 81]. Clearly $T(X)$ is contained in the subspace Y generated by the vectors $x_1, \dots, x_n$.

Conversely, if $y := \lambda_1 x_1 + \ldots + \lambda_n x_n$ is an arbitrary element of Y we can choose $z_1, \dots, z_n$ in X such that $f_i(z_j) = \delta_{i,j}$, where $\delta_{i,j}$ denote the delta of Kronecker (a such choice is always possible, see Heuser [**159**, Proposition 15.1]). If we define $z := \sum_{k=1}^{n} \lambda_k z_k$ then

$$Tz = \sum_{k=1}^{n} f_k(z)x_k = \sum_{k=1}^{n} f_k\left(\sum_{k=1}^{n} \lambda_k z_k\right) x_k = \sum_{k=1}^{n} \lambda_k x_k = y.$$

This shows that the set $\{x_1, \dots, x_n\}$ forms a basis for the subspace $T(X)$.

Theorem 3.39. *For a bounded operator T on a Banach space X, the following assertions are equivalent:*

(i) $\lambda_0 I - T$ *is a Weyl operator;*

(ii) *There exists a finite-dimensional operator $K \in F(X)$ such that $\lambda_0 \notin \sigma(T+K)$;*

(iii) *There exists a compact operator $K \in K(X)$ such that $\lambda_0 \notin \sigma(T+K)$.*

Proof We can assume $\lambda_0 = 0$.
(i) $\Rightarrow$ (ii) Assume that T is a Fredholm operator having index ind $T = \alpha(T) - \beta(T) = 0$ and let $m := \alpha(T) = \beta(T)$. Let $P \in L(X)$ denote the projection of X onto the finite-dimensional space ker T. Obviously, ker $T \cap$ ker $P = \{0\}$ and according Remark 3.38 we can represent the finite-dimensional operator P in the form

$$Px = \sum_{i=1}^{m} f_i(x)x_i,$$

where the vectors $x_1, \dots, x_m$ from X, the vectors $f_1, \cdots, f_m$ from $X^\star$, are linearly independent. As observed in Remark 3.38, the set $\{x_1, \dots, x_m\}$ forms a basis of $P(X)$ and therefore $Px_i = x_i$ for every $i = 1, \dots, m$, from which we obtain that $f_i(x_k) = \delta_{i,k}$.

Denote by Y the topological complement of the finite-codimensional subspace $T(X)$. Then Y is finite-dimensional with dimension m, so we can choose a basis $\{y_1, \dots, y_m\}$ of Y. Let us define

$$Kx := \sum_{i=1}^{m} f_i(x)y_i$$

Clearly K is a finite-dimensional operator, so by part (f) of Remark 1.54 $S := T + K$ is a Fredholm operator, and from Remark 3.38 we know that $K(X) = Y$.

Finally, consider an element $x \in$ ker S. Then $Tx = Kx = 0$, and this easily implies that $f_i(x) = 0$ for all $i = 1, \dots, m$. From this it follows that $Px = 0$ and therefore $x \in \ker T \cap \ker P = \{0\}$, so S is injective.

In order to show that S is surjective observe first that

$$f_i(Px) = f_i\left(\sum_{k=1}^{m} f_k(x)x_k\right) = f_i(x).$$

From this we obtain that

$$KPx = \sum_{i=1}^{m} f_i(Px)y_i = \sum_{i=1}^{m} f_i(x)y_i = Kx. \tag{66}$$

Now, we have $X = T(X) \oplus Y = T(X) \oplus K(X)$, so every $z \in X$ may be represented in the form $z = Tu + Kv$, with $u, v \in X$. Set

$$u_1 := u - Pu \quad \text{and} \quad v_1 := Pv.$$

From (66) and from the equality $P(X) = \ker T$ we easily obtain that

$$Ku_1 = 0, \quad Tv_1 = 0, \quad Kv_1 = Kv \quad \text{and} \quad Tu_1 = Tu.$$

Therefore

$$S(u_1 + v_1) = (T + K)(u_1 + v_1) = Tu + Kv = z,$$

and hence S is surjective. Therefore $S = T + K$ is invertible.

(ii)$\Rightarrow$ (iii) Clear.

(iii)$\Rightarrow$ (i) Suppose $T + K = U$, where U is invertible and K is compact. Obviously U is a Fredholm operator having index 0, and hence by part (f) of Remark 1.54 we conclude that $T \in \mathcal{W}(X)$. ∎

Remark 3.40. By means of a modest modification of the proof of Theorem 3.39 we easily obtain the following equivalence:

(a) *The operator* $\lambda_0 I - T \in \Phi_+(X)$ *has* ind $(\lambda_0 I - T) \le 0$ *precisely when* $\lambda_0 \notin \sigma_{\mathrm{ap}}(T + K)$ *for some* $K \in K(X)$.

To show this equivalence take $m := \alpha(T)$ and proceed as in the proof of Theorem 3.39. The operator $S = T + K$ is then injective and has closed range, since $T + K \in \Phi_+(X)$, by part (f) of Remark 1.54.

Analogously we have:

(b) *The operator* $\lambda_0 I - T \in \Phi_-(X)$ *has* ind $(\lambda_0 I - T) \ge 0$ *precisely when* $\lambda_0 \notin \sigma_{\mathrm{su}}(T + K)$ *for some* $K \in K(X)$.

The proof of equivalence (b) is easily obtained taking $m := \beta(T)$ and proceeding as in the proof of Theorem 3.39.

Corollary 3.41. *Let $T \in L(X)$, X a Banach space. Then $\sigma_{\mathrm{w}}(T)$ is closed and*

$$\sigma_{\mathrm{w}}(T) = \bigcap_{K \in F(X)} \sigma(T + K) = \bigcap_{K \in K(X)} \sigma(T + K). \tag{67}$$

Proof Let $\rho_{\mathrm{w}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{w}}(T)$. The equality (67) may be restated, taking the complements, as follows

$$\rho_{\mathrm{w}}(T) = \bigcup_{K \in F(X)} \rho(T + K) = \bigcup_{K \in K(X)} \rho(T + K). \tag{68}$$

The equalities (68) are now immediate from Theorem 3.39. The last assertion is clear ∎

Lemma 3.42. *Suppose that $T \in L(X)$ and $K \in K(X)$ commute.*

(i) *If T is bounded below then $p(T - K) < \infty$;*

(ii) *If T is onto then $q(T - K) < \infty$.*

Proof We first establish the implication (ii). The implication (i) will follows then by duality.

(ii) Obviously T is lower semi-Fredholm, and hence by part (f) of Remark

1.54 we have $S := T - K \in \Phi_-(X)$. Consequently, from part (c) of Remark 1.54 also $S^k \in \Phi_-(X)$ for every $k \in \mathbb{N}$, and hence the range $S^k(X)$ is finite-codimensional.

Let us consider the map $\widehat{T} : X/S^k(X) \to X/S^k(X)$, defined canonically by

$$\widehat{T}\widehat{x} := \widehat{Tx} \quad \text{for all } \widehat{x} := x + S^k(X).$$

Since T is onto, for every $y \in X$ there exists an element $z \in X$ such that $y = Tz$, and therefore $\widehat{y} = \widehat{Tz} = \widehat{T}\widehat{z}$, thus $\widehat{T}$ is onto.

Since $X/S^k(X)$ is a finite-dimensional space then $\widehat{T}$ is also injective and this easily implies that $\ker T \subseteq S^k(X)$. The surjectivity of T also implies that $\gamma(T) > 0$, $\gamma(T)$ the minimal modulus of T, and

$$\|Tx\| \geq \gamma(T)\,\mathrm{dist}(x, \ker T) \quad \text{for all } x \in X.$$

Let $z \in S^k(X)$ be arbitrarily given. The equalities

$$T(S^k(X)) = (S^kT)(X) = S^k(X)$$

show that there is some $y \in S^k(X)$ for which $Ty = z$. For every $x \in X$ we have

$$\begin{aligned} \|Tx - z\| &= \|T(x-y)\| \geq \gamma(T)\,\mathrm{dist}(x-y, \ker T) \\ &\geq \gamma(T)\,\mathrm{dist}(x-y, S^k(X)), \end{aligned}$$

where the last inequality follows from the inclusion $\ker T \subseteq S^k(X)$. Consequently, for every $x \in X$ we obtain that

$$\|Tx - z\| \geq \gamma(T)\,\mathrm{dist}(x, S^k(X)) \quad \text{for all } z \in S^k(X),$$

and this implies that

$$\mathrm{dist}(Tx, S^k(X)) \geq \gamma(T)\,\mathrm{dist}(x, S^k(X)) \quad \text{for all } k \in \mathbb{N}.$$

Suppose that $q(S) = \infty$. Then there is a bounded sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in S^n(X)$ and $\mathrm{dist}(x_n, S^{n+1}(X)) \geq 1$ for every $n \in \mathbb{N}$. For $m > n$, m and $n \in \mathbb{N}$, we have

$$Kx_m - Kx_n = (Kx_m + (T-K)x_n) - Tx_n.$$

Now

$$Kx_m \in K(S^m(X)) = S^mK(X) \subseteq S^m(X),$$

and

$$(T_K)x_n \in (T-K)n + 1^(X) = S^{n+1}(X),$$

hence $w := Kx_m + (T-K)x_n \in S^{n+1}(X)$ for all $m > n$. Therefore

$$\begin{aligned} \|Kx_m - Kx_n\| &= \|w - Tx_n\| \geq \mathrm{dist}(Tx_n, S^{n+1}(X)) \\ &\geq \mathrm{dist}(x_n, S^{n+1}(X)) \geq \gamma(T), \end{aligned}$$

which contradicts the compactness of K. Therefore $S = T - K$ has finite descent.

(i) If K is a compact operator and T is bounded below then $K^\star$ is compact and $T^\star$ is onto, see Lemma 1.30. Moreover, by part (f) of Remark

1.54 the operators $T-K$ and $T^\star - K^\star$ are semi-Fredholm, hence $p(T-K) = q(T^\star - K^\star) < \infty$. ∎

Theorem 3.43. *Let $T \in L(X)$, $K \in L(X)$ be commuting operators on a Banach space X. If $K \in K(X)$ we have the following equivalences:*

(i) *If $T \in \Phi_+(X)$ then $p(T+K) < \infty$ if and only if $p(T) < \infty$;*

(ii) *If $T \in \Phi_-(X)$ then $q(T+K) < \infty$ if and only if $q(T) < \infty$.*

Proof Suppose first that $T \in \Phi_-(X)$ and $q := q(T) < \infty$. Then $T^q(X) = T^{q+1}(X)$ and $T^q(X)$ is a closed subspace of finite-codimension, since by Remark 1.54, part (c), also $T^q \in \Phi_-(X)$. Let $S := T + K$. We know from part (f) of Remark 1.54 that $S \in \Phi_-(X)$. The restriction of T to $T^q(X)$ is surjective, so by Lemma 3.42 the restriction of S to $T^q(X)$ has finite descent. From this it follows that there is a positive integer k for which

$$S^m(X) \supset (S^mT^q)(X) = (S^kT^q)(X) \quad \text{for all } m \geq k.$$

We have $S^kT^q \in \Phi_-(X)$, again from part (c) of Remark 1.54, thus the subspace $S^kT^q(X)$ has finite-codimension. From this we then conclude that $S = T + K$ has finite descent.

Conversely, assume that $q(T+K) < \infty$. Since $T + K \in \Phi_-(X)$, from the first part of the proof we obtain that $q(T) = q(T+K-K) < \infty$. Hence the equivalence (ii) is proved.

The equivalence (i) follows by duality from (ii), since T and $S = T + K$ are upper semi-Fredholm if and only if $T^\star$ and $S^\star = T^\star + K^\star$ are lower semi-Fredholm, respectively, and hence $p(T) = q(T^\star)$, $p(S) = q(S^\star)$, see part (a) of Remark 3.7. ∎

It should be noted that the equivalence (i) of Theorem 3.43 may be also deduced directly from the assertion (i) of Lemma 3.42.

We now give a characterization of semi-Browder operators by means of the SVEP.

Theorem 3.44. *For an operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) *$\lambda_0 I - T$ is essentially semi-regular and T has the SVEP at λ_0;*

(ii) *There exists a finite-dimensional operator $K \in L(X)$ such that $TK = KT$ and $\lambda_0 \notin \sigma_{\mathrm{ap}}(T+K)$;*

(iii) *There exists a compact operator $K \in L(X)$ such that $TK = KT$ and $\lambda_0 \notin \sigma_{\mathrm{ap}}(T+K)$;*

(iv) *$\lambda_0 I - T \in \mathcal{B}_+(X)$.*

Proof (i) $\Rightarrow$ (ii) Suppose that $\lambda_0 I - T$ is essentially semi-regular and that T has SVEP at λ_0. Let (M, N) be a GKD for $\lambda_0 I - T$, where $(\lambda_0 I - T)|N$ is nilpotent and N is finite-dimensional. Let P denote the finite-dimensional projection of X onto N along M. Clearly P commutes with T, because N and M reduce T. Since T has the SVEP at λ_0 it follows that $(\lambda_0 I - T)|M$

is injective, by Theorem 3.14 . Furthermore, the restriction $(\lambda_0 I - T - I)|N$ is bijective, since from the nilpotency of $(\lambda_0 I - T)|N$ we have $1 \notin \sigma((\lambda_0 I - T)|N)$. Therefore $(\lambda I_o - T - I)(N) = N$ and $\ker((\lambda_0 I - T - I)|N) = \{0\}$.

From this it follows that

$$\begin{aligned}
\ker(\lambda_0 I - T - P) &= \ker((\lambda_0 I - T - P)|M) \oplus \ker((\lambda_0 I - T - P)|N) \\
&= \ker((\lambda_0 I - T)|M) \oplus \ker((\lambda_0 I - T - I)|N) \\
&= \{0\}.
\end{aligned}$$

so that the operator $(\lambda_0 I - T - P)$ is injective.

On the other hand, the equalities

$$\begin{aligned}
(\lambda_0 I - T - P)(X) &= (\lambda_0 I - T - P)(M) \oplus (\lambda_0 I - T - P)(N) \\
&= (\lambda_0 I - T)(M) \oplus (\lambda_0 I - T - I)(N) \\
&= (\lambda_0 I - T)(M) \oplus N,
\end{aligned}$$

show that the subspace $(\lambda_0 I - T - P)(X)$ is closed, since it is the sum of the subspace $(\lambda_0 I - T)(M)$, which is closed by semi-regularity, and the finite-dimensional subspace N. This shows that $\lambda_0 \notin \sigma_{\mathrm{ap}}(T + P)$.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Suppose that there exists a commuting compact operator K such that $\lambda_0 I - (T + K)$ is bounded below, and therefore upper semi-Fredholm. The class $\Phi_+(X)$ is stable under compact perturbations, as noted in part (f) of Remark 1.54, and hence $\lambda_0 I - (T+K) - K = \lambda_0 I - T \in \Phi_+(X)$.

On the other hand, $p(\lambda_0 I - (T + K)) = 0$, and hence from Lemma 3.42 also $p(\lambda_0 I - T) = p(\lambda_0 I - (T + K) - K)$ is finite.

The implication (iv) $\Rightarrow$ (i) is clear from Theorem 3.14 since every semi-Fredholm operator is essentially semi-regular and hence of Kato type. ∎

Corollary 3.45. *Let $T \in L(X)$, X a Banach space. Then $\sigma_{\mathrm{ub}}(T)$ is closed and*

$$\sigma_{\mathrm{ub}}(T) = \bigcap_{K \in F(X),\, KT = TK} \sigma_{\mathrm{ap}}(T + K) = \bigcap_{K \in K(X),\, KT = TK} \sigma_{\mathrm{ap}}(T + K). \tag{69}$$

∎

Later we shall see that the Browder spectrum $\sigma_{\mathrm{b}}(T)$ is the intersection of all spectra of commuting compact perturbations of T. For this reason $\sigma_{\mathrm{ub}}(T)$ is sometimes called the *Browder essential approximate point spectrum* of $T \in L(X)$.

The next result is dual to that given in Theorem 3.44.

Theorem 3.46. *Let $T \in L(X)$, X a Banach space. Then the following properties are equivalent:*

(i) *$\lambda_0 I - T$ is essentially semi-regular and $T^\star$ has the SVEP at λ_0;*

(ii) *There exists a finite-dimensional operator $K \in L(X)$ such that $TK = KT$ and $\lambda_0 \notin \sigma_{\mathrm{su}}(T + K)$;*

(iii) *There exists a compact operator* $K \in L(X)$ *such that* $TK = KT$ *and* $\lambda_0 \notin \sigma_{\rm su}(T+K)$*;*

(iv) $\lambda_0 I - T \in \mathcal{B}_-(X)$.

Proof (i) $\Rightarrow$ (ii) Let $\lambda_0 I - T$ be essentially semi-regular and suppose that $T^\star$ has SVEP at λ_0. Let (M, N) be a GKD for $\lambda_0 I - T$, where $(\lambda_0 I - T)\,|N$ is nilpotent and N is finite-dimensional. Then $(N^\perp, M^\perp)$ is a GKD for $\lambda_0 I^\star - T^\star$, see Theorem 1.43.

Let P denote the finite rank projection of X onto N along M. Then P commutes with T, since N and M reduce T. Moreover, $(\lambda_0 I - T^\star)\,|N^\perp$ is injective by Theorem 3.14, and this implies that $(\lambda_0 I - T)\,|M$ is surjective, see Lemma 3.13. From the nilpotency of $(\lambda_0 I - T)\,|N$ we know that the restriction $(\lambda_0 I - T - I)\,|N$ is bijective, so we have

$$\begin{aligned}(\lambda_0 I - T - P)(X) &= (\lambda_0 I - T - P)(M) \oplus (\lambda_0 I - T - P)(N)\\ &= (\lambda_0 I - T)(M) \oplus (\lambda_0 I - T - I)(N)\\ &= M \oplus N = X.\end{aligned}$$

This shows that $\lambda_0 \notin \sigma_{\rm su}(T+P)$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Suppose that there exists a commuting compact operator K such that $\lambda_0 I - (T+K)$ is surjective and therefore lower semi-Fredholm. The class $\Phi_-(X)$ is stable under compact perturbations, so $\lambda_0 I - (T+K) - K = \lambda_0 I - T \in \Phi_-(X)$.

On the other hand, $q(\lambda_0 I - (T+K)) = 0$, and hence, again by Lemma 3.42, also $q(\lambda_0 I - T) = q(\lambda_0 I - (T+K) - K)$ is finite.

(iv) $\Rightarrow$ (i) This is clear from Theorem 3.15. ∎

Corollary 3.47. *Let* $T \in L(X)$*,* X *a Banach space. Then* $\sigma_{\rm lb}(T)$ *is closed and*

$$\sigma_{\rm lb}(T) = \bigcap_{K \in F(X),\, KT = TK} \sigma_{\rm su}(T+K) = \bigcap_{K \in K(X),\, KT = TK} \sigma_{\rm su}(T+K). \tag{70}$$

∎

The spectrum $\sigma_{\rm lb}(T)$ is sometime called the *Browder essential approximate defect spectrum* of $T \in L(X)$.

Combining Theorem 3.44 and Theorem 3.46 and recalling that $\sigma(T) = \sigma_{\rm ap}(T) \cup \sigma_{\rm su}(T)$ we readily obtain the following characterizations of Browder operators.

Theorem 3.48. *Let* $T \in L(X)$*,* X *a Banach space. Then the following properties are equivalent:*

(i) $\lambda_0 I - T$ *is essentially semi-regular,* T *and* $T^\star$ *have the SVEP at* λ_0*;*

(ii) *There exists a finite-dimensional operator* $K \in L(X)$ *such that* $TK = KT$ *and* $\lambda_0 \notin \sigma(T+K)$*;*

(iii) *There exists a compact operator* K *such that* $TK = KT$ *and* $\lambda_0 \notin \sigma(T+K)$;

(iv) $\lambda_0 I - T \in \mathcal{B}(X)$.

From the preceding characterizations of the Browder operators we readily obtain the following characterization of the Browder spectrum.

Corollary 3.49. *Let* $T \in L(X)$, X *a Banach space. Then* $\sigma_{\mathrm{b}}(T)$ *is closed and*

$$\sigma_{\mathrm{b}}(T) = \bigcap_{K \in F(X),\, KT=TK} \sigma(T+K) = \bigcap_{K \in K(X),\, KT=TK} \sigma(T+K).$$

■

Later we shall see that the ideal $F(X)$ or $K(X)$ in the intersections above may be replaced by any ideal of operators T in $L(X)$ for which $\lambda I - T$ is a Fredholm operator for every $\lambda \neq 0$. Of course, this is also true for the intersections (69) and (70).

The following corollary is an immediate consequence of Theorem 3.48, once observed that both the operators T and $T^\star$ have the SVEP at every boundary point of $\sigma(T)$.

Corollary 3.50. *Let* $T \in L(X)$, X *a Banach space, and suppose that* $\lambda_0 \in \partial\sigma(T)$. *Then* $\lambda_0 I - T$ *is essentially semi-regular if and only if* $\lambda_0 I - T$ *is semi-Fredholm, and this is the case if and only if* $\lambda_0 I - T$ *is Browder.* ■

It is immediate from Corollary 3.48 that if $T \in \Phi_\pm(X)$ and both operators T and $T^\star$ have the SVEP at 0, then T is a Browder operator.

An immediate consequence of Theorem 2.39 is given by the following result.

Corollary 3.51. *Suppose that* $T \in L(X)$ *is a bounded operator on a Banach space* X *and*

$$p(\lambda) := \prod_{i=1}^{k} (\lambda_i - \lambda)^{n_i}, \quad n_i \in \mathbb{N}.$$

Then $p(T) \in \mathcal{B}_+(X)$ *if and only if* $\lambda_i I - T \in \mathcal{B}_+(X)$ *for every* i. *A similar statement holds if we replace* $\mathcal{B}_+(X)$ *by* $\mathcal{B}_-(X)$, *or by* $\mathcal{B}(X)$.

Proof From Remark 1.54, part (c), we know that

$$p(T) \in \Phi_+(X) \Leftrightarrow \lambda_i I - T \in \Phi_+(X) \quad \text{for every } i = 1, \ldots, k.$$

From Theorem 2.39 we also have

$$p(T) \text{ has the SVEP at } 0 \Leftrightarrow T \text{ has the SVEP at } \lambda_i,\ i = 1, \ldots, k.$$

The equivalence is then a consequence of Theorem 3.44. The other statements may be obtained in a similar way from part (c) of Remark 1.54, Theorem 3.46 and Theorem 3.48, respectively. ■

The preceding result implies the spectral mapping theorem for the semi-Browder spectra and the Browder spectrum in the special case the analytic function is a polynomial. This result will be extended later to arbitrary functions analytic on an open neighborhood of $\sigma(T)$.

For an arbitrary operator $T \in L(X)$ let us consider the set

$$\Xi(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}.$$

The following theorem describes the relationships between the semi-Browdwer spectra, the spectrum $\sigma_{\mathrm{es}}(T)$ and the points where T, or $T^\star$, do not have the SVEP.

Theorem 3.52. *Let $T \in L(X)$, X a Banach space. Then*

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{es}}(T) \cup \Xi(T) = \sigma_{\mathrm{uf}}(T) \cup \Xi(T) \tag{71}$$

and

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{es}}(T) \cup \Xi(T^\star) = \sigma_{\mathrm{lf}}(T) \cup \Xi(T^\star). \tag{72}$$

Moreover,

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \Xi(T) = \sigma_{\mathrm{w}}(T) \cup \Xi(T^\star), \tag{73}$$

Again,

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{f}}(T) \cup \Xi(T) \cup \Xi(T^\star) \tag{74}$$

and

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) \cup \Xi(T^\star) = \sigma_{\mathrm{lb}}(T) \cup \Xi(T). \tag{75}$$

Proof If $\lambda_0 \notin \sigma_{\mathrm{ub}}(T)$ then $\lambda_0 I - T \in \Phi_+(X)$ and $p(\lambda_0 I - T) < \infty$, so, by Theorem 3.14, $\lambda_0 \notin \sigma_{\mathrm{es}}(T) \cup \Xi(T)$. Hence $\sigma_{\mathrm{es}}(T) \cup \Xi(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

Conversely, if $\lambda_0 \notin \sigma_{\mathrm{es}}(T) \cup \Xi(T)$ then $\lambda_0 I - T$ is upper semi-Browder, by Theorem 3.44. Hence the first equality in (71) is proved. The second equality of (71) follows from this, that if T has the SVEP at λ_0 then by Theorem 3.44 $\lambda_0 I - T$ is essentially semi-regular if and only if $\lambda_0 I - T \in \Phi_+(X)$.

The equality of (72) follows in a similar way from Theorem 3.27 and Theorem 3.46.

To prove the first equality of (73) it is sufficient to observe that if T fails to have the SVEP at λ_0 then $p(\lambda_0 I - T) = \infty$, by Theorem 5.4, so from the inclusion $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ we may conclude that $\sigma_{\mathrm{w}}(T) \cup \Xi(T) \subseteq \sigma_{\mathrm{b}}(T)$.

To show the opposite inclusion assume that $\lambda_0 \notin \sigma_{\mathrm{w}}(T) \cup \Xi(T)$. Then $\lambda_0 I - T$ is a Fredholm operator of index 0 and T has the SVEP at λ_0. From Theorem 3.16 we know that $p(\lambda_0 I - T) < \infty$ and this implies that also $q(\lambda_0 I - T) < \infty$, see part (iv) of Theorem 3.4. Hence $\lambda_0 \notin \sigma_{\mathrm{b}}(T)$.

The second equality of (73) can be proved in similar way by using Theorem 5.4 , Theorem 3.15, and Theorem 3.4. The equality (74) is a consequence of Theorem 3.48.

It remains only to prove the equalities (75). Clearly $\sigma_{\mathrm{ub}}(T) \subseteq (T)$ and if $T^\star$ fails to have the SVEP at λ_0 then $q(\lambda_0 I - T) = \infty$, again from Theorem

5.4, so the inclusion $\sigma_{\rm ub}(T) \cup \Xi(T^\star) \subseteq \sigma_{\rm b}(T)$ is verified.

On the other hand, the opposite inclusion readily follows from the property that if $\lambda_0 \notin \sigma_{\rm ub}(T) \cup \Xi(T^\star)$ then $\lambda_0 I - T$ is an upper semi-Fredholm operator having finite ascent. Since $T^\star$ has SVEP at λ_0 then $\lambda_0 I - T$ has finite descent by Theorem 3.15, so $\lambda_0 \notin \sigma_{\rm b}(T)$.

The second equality of (75) follows from a similar argument, by using Theorem 5.4 and Theorem 3.14. ∎

An useful consequence of the preceding result is that under the assumption of the SVEP for T, or for $T^\star$, various of the spectra considered above coalesce. This also generalizes some classical results on the spectra of normal operators on Hilbert spaces to operators having the SVEP.

Corollary 3.53. *Suppose that $T \in L(X)$, X a Banach space. We have*

(i) *If T has the SVEP then*

$$\sigma_{\rm es}(T) = \sigma_{\rm sf}(T) = \sigma_{\rm uf}(T) = \sigma_{\rm ub}(T), \tag{76}$$

and

$$\sigma_{\rm lb}(T) = \sigma_{\rm b}(T) = \sigma_{\rm w}(T); \tag{77}$$

(ii) *If $T^\star$ has the SVEP then*

$$\sigma_{\rm es}(T) = \sigma_{\rm sf}(T) = \sigma_{\rm lf}(T) = \sigma_{\rm lb}(T), \tag{78}$$

and

$$\sigma_{\rm ub}(T) = \sigma_{\rm b}(T) = \sigma_{\rm w}(T); \tag{79}$$

(iii) *If both T and $T^\star$ have the SVEP then all the spectra in (76), (77), (78) and (79) coincide and are equal to the Fredholm spectrum $\sigma_{\rm f}(T)$.*

Proof (i) For every bounded operator $T \in L(X)$ we have

$$\sigma_{\rm es}(T) \subseteq \sigma_{\rm sf}(T) \subseteq \sigma_{\rm uf}(T) \subseteq \sigma_{\rm ub}(T).$$

If T has SVEP then $\Xi(T) = \varnothing$, and hence from the equality (110) we obtain $\sigma_{\rm es}(T) = \sigma_{\rm ub}(T)$. The equalities (77) are obvious from (73) and (75) of Theorem 3.52.

(ii) The proof of (78) and (79) is similar to the proof of (76) and (77) of part (i).

(iii) Clearly, all the spectra in (76), (77), (78) coincide, by part (i) and part (ii). Moreover, these spectra coincide with $\sigma_{\rm f}(T)$ since $\sigma_{\rm f}(T) = \sigma_{\rm uf}(T) \cup \sigma_{\rm lf}(T)$. ∎

In the next theorem we consider a situation which occurs in some concrete cases.

Theorem 3.54. *Let $T \in L(X)$ be an operator for which $\sigma_{\rm ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then T has the SVEP and*

$$\sigma_{\rm ap}(T) = \sigma_{\rm k}(T) = \sigma_{\rm se}(T) = \sigma_{\rm es}(T)$$

Moreover, these spectra coincide with all the spectra of (76) of Corollary 3.53.

Proof By Corollary 3.24 we have $\sigma_{\mathrm{ap}}(T) = \partial\sigma(T) \subseteq \sigma_{\mathrm{k}}(T)$, whilst the inclusion $\sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ is true for every $T \in L(X)$. Hence $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{k}}(T)$.

Moreover, these last two spectra coincide with $\sigma_{\mathrm{se}}(T)$ and $\sigma_{\mathrm{es}}(T)$ because the inclusions $\sigma_{\mathrm{k}}(T) \subseteq \sigma_{\mathrm{es}}(T) \subseteq \sigma_{\mathrm{se}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$ are verified for every $T \in L(X)$. Finally, T has the SVEP at every point of the boundary as well as at every point λ which belongs to the remaining part of the spectrum, since $\sigma_{\mathrm{ap}}(T)$ does not cluster at λ.

Hence T has the SVEP, so all these spectra coincide with the spectra of (76). ∎

The proof of the following result is similar to that of Corollary 3.14, taking into account the equality (78) of Corollary 3.53, and that $T^\star$ has the SVEP at λ_0 whenever $\sigma_{\mathrm{su}}(T)$ does not cluster at λ_0.

Corollary 3.55. *Let $T \in L(X)$ be an operator for which $\sigma_{\mathrm{su}}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then $T^\star$ has the SVEP and*

$$\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{k}}(T) = \sigma_{\mathrm{se}}(T) = \sigma_{\mathrm{es}}(T).$$

Moreover, these spectra coincide with all the spectra of (78) of Corollary 3.53. ∎

5. Compressions

In this section we establish further characterizations of the semi-Browder spectra by means of compressions.

Let $\mathcal{P}(X)$ denote the set of all bounded projections $P \in L(X)$, where X is a Banach space, such that $\operatorname{codim} P(X) < \infty$. Let $T \in L(X)$ and $\mathcal{P}(X)$. The *compression* generated from P is the bounded linear operator $T_P : P(X) \to P(X)$ defined by

$$T_P y := PTy \quad \text{for every} \quad y \in P(X).$$

Lemma 3.56. *Let $P \in \mathcal{P}(X)$, where X is a Banach space. If T is semi-Fredholm then T_P is semi-Fredholm and* $\operatorname{ind} T_P = \operatorname{ind} T$.

Proof From the decomposition $X = \ker P \oplus P(X)$ we obtain that $\alpha(P) = \beta(P) < \infty$, so P is a Fredholm operator and hence PTP is semi-Fredholm. It is easily seen that

$$\alpha(T_P) = \alpha(PTP) - \alpha(P), \tag{80}$$

and

$$\beta(T_P) = \beta(PTP) - \beta(P). \tag{81}$$

Moreover, since

$$T = PT + (I - P)T = PTP + PT(I - P) + (I - P)T,$$

where $PT(I-P)+(I-P)T$ is a finite-dimensional operator, we conclude that $\operatorname{ind} T = \operatorname{ind} PTP$. By subtracting (80) and (81) we then conclude that $\operatorname{ind}(T_P) = \operatorname{ind}(T)$. ∎

Observe that if $TP = PT$ then $P(X)$ is T-invariant and T_P coincides with the restriction $T|P(X)$. In fact, for every $y = Px \in P(X)$ we have

$$T_P y = PTPx = TP^2x = TPx = Ty.$$

By Corollary 3.49 the Browder spectrum $\sigma_{\mathrm{b}}(T)$ is the intersection of the spectra of all *commuting* compact perturbations of T. The following result shows that $\sigma_{\mathrm{b}}(T)$ is the intersection of the spectra of all *commuting* compressions of T.

Theorem 3.57. *For every bounded operator $T \in L(X)$, where X is a Banach space, we have*

$$\sigma_{\mathrm{b}}(T) = \bigcap_{P\in\mathcal{P}(X),\, PT=TP} \sigma(T_P). \tag{82}$$

Proof Suppose that λ does not belong to the right hand side of (82). Then there is a projection $P \in L(X)$ commuting with T which is such that $P(X)$ is finite-codimensional and $\lambda I_P - T_P = (\lambda I - T)_P$ is invertible on $P(X)$. Since $PT = TP$ the compression $(\lambda I - T)_P$ coincides with the restriction of $\lambda I - T$ to $P(X)$, so that $(\lambda I - T)P(X)$ is closed.

From the decomposition $X = P(X) \oplus \ker P$, where $\ker P$ is finite-dimensional, we then obtain that

$$(\lambda I - T)(X) = (\lambda I - T)P(X) \oplus (\lambda I - T)(\ker P),$$

and hence $(\lambda I - T)(X)$ is closed since it is the sum of a closed subspace and a finite-dimensional subspace.

On the other hand, $\alpha((\lambda I - T)|\ker P) < \infty$ since $\ker P$ is finite-dimensional. Consequently

$$\begin{aligned}\alpha(\lambda I - T) &= \alpha((\lambda I - T)|P(X)) + \alpha((\lambda I - T)|\ker P)\\ &= \alpha((\lambda I - T)|\ker P) < \infty,\end{aligned}$$

so $\lambda I - T$ is an upper semi-Fredholm operator. By Lemma 3.56 we also have $\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda I - T)_P = 0$, and hence $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$. Therefore $\lambda I - T$ is a Fredholm operator having index 0.

Now we show that $\lambda \notin \sigma_{\mathrm{b}}(T)$. Obviously, if $\lambda I - T$ is invertible on X then $\lambda \notin \sigma_{\mathrm{b}}(T)$. Suppose the other case, that $\lambda I - T$ is not invertible on X.

Let $Q := I - P$. Clearly Q is a finite-dimensional operator which commutes with $\lambda I - T$. The restriction of $\lambda I - T$ to the invariant finite-dimensional subspace $Q(X)$ has ascent finite, whilst the restriction of $\lambda I - T$ on $\ker Q = P(X)$ has ascent zero, since $\lambda I - T|P(X)$ is invertible on $P(X)$. Therefore from the decomposition $X = Q(X) \oplus \ker Q$ we then infer that

$$\begin{aligned}p(\lambda I - T) &= p((\lambda I - T|Q(X)) + p((\lambda I - T|\ker Q)\\ &= p((\lambda I - T|Q(X)) < \infty.\end{aligned}$$

Since $\lambda I - T$ is a Fredholm operator having index 0, part (iv) of Theorem 3.4 entails that the descent $q(\lambda I - T)$ is finite. This shows that $\lambda \notin \sigma_{\mathrm{b}}(T)$. Consequently, $\sigma_{\mathrm{b}}(T)$ is contained in the right hand side of (82).

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{b}}(T)$. We have two possibilities: $\lambda I - T$ invertible or non-invertible. If $\lambda I - T$ is invertible in X then $\lambda \notin \sigma(T_P)$ for $P = I$. Suppose that $\lambda I - T$ is not invertible in X. Then λ is an isolated point of $\sigma(T)$ and $\lambda I - T$ is Fredholm. Let Q be the spectral projection associated with the spectral set λ. If $P := I - Q$ then P commutes with T and the subspace $P(X) = \ker Q$ is finite-codimensional. Hence $P \in \mathcal{P}(X)$. Moreover, $\lambda \notin \sigma(T_P) = \sigma(T|P(X)) = \sigma(T) \setminus \{\lambda\}$. Therefore the right-hand side of (82) is contained in $\sigma_{\mathrm{b}}(T)$, and this completes the proof. ■

We show now that a similar result holds for the upper and lower semi-Browder spectra. In this case the spectra of the compressions are replaced by the approximate point spectra and by the surjectivity spectra of the compressions, respectively.

Theorem 3.58. *For every bounded operator $T \in L(X)$ on a Banach space X we have*

$$\sigma_{\mathrm{ub}}(T) = \bigcap_{P \in \mathcal{P}(X),\, PT = TP} \sigma_{\mathrm{ap}}(T_P), \tag{83}$$

and

$$\sigma_{\mathrm{lb}}(T) = \bigcap_{P \in \mathcal{P}(X),\, PT = TP} \sigma_{\mathrm{su}}(T_P). \tag{84}$$

Proof To show the equality (83) suppose that λ does not belong to the right hand side of (83). Then there is a projection $P \in \mathcal{P}(X)$ such that $PT = TP$ and $\lambda I - T|P(X)$ is bounded below and hence upper semi-Fredholm. From the decomposition $X = P(X) \oplus \ker P$, since dim ker $P < \infty$ it follows that $\alpha(\lambda I - T) = \alpha((\lambda I - T)|\ker\, P)) < \infty$ and

$$(\lambda I - T)(X) = (\lambda I - T)(P(X)) \oplus (\lambda I - T)(\ker\, P)$$

is closed. Therefore $\lambda I - T \in \Phi_+(X)$. Moreover, $p((\lambda I - T)_P) = 0$ and hence

$$p(\lambda I - T) = p(\lambda I_P - T_P) + p((\lambda I - T)|\ker P) < \infty,$$

which implies that $\lambda \notin \sigma_{\mathrm{ub}}(T)$.

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Then $\lambda I - T$ is upper semi-Fredholm with finite ascent. According to Theorem 1.62 let (M, N) be a Kato decomposition for $\lambda I - T$ such that dim $N < \infty$, $\lambda I - T|M$ is semi-regular and $\lambda I - T|N$ is nilpotent. Let P be the projection of X onto M along N. Obviously $P \in \mathcal{P}(X)$ and commutes with T, because M and N reduce T. Since $\lambda I - T$ has finite ascent then by Theorem 3.16 $(\lambda I - T)_P = (\lambda I - T)|M$ is injective. Moreover, $(\lambda I - T)_P(P(X)) = (\lambda I - T)(M)$ and the last subspace is closed since $\lambda I - T|M$ is semi-regular. Therefore $\lambda I_P - T_P$ is bounded below and hence $\lambda \notin \sigma_{\mathrm{ap}}(T_P)$.

Analogously, to prove the equality (84) suppose that λ does not belong to the right hand side of (84). Then there is a projection $P \in \mathcal{P}(X)$ such that $PT = TP$ and $\lambda I - T$ is surjective on $P(X)$. This implies that $\lambda I - T$ is lower semi-Fredholm.

Moreover, $q((\lambda I_P - T_P) = 0$, and since $\ker P$ is finite-dimensional from the decomposition $X = P(X) \oplus \ker P$ we then obtain that

$$q(\lambda I - T) = q((\lambda I_P - T_P) + q((\lambda I - T)|\ker P) < \infty.$$

Hence $\lambda \notin \sigma_{\mathrm{lb}}(T)$.

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{lb}}(T)$. Then $\lambda I - T$ is lower semi-Fredholm with finite descent. Again, by Theorem 1.62 let (M, N) be a generalized Kato decomposition for $\lambda I - T$ such that $\dim N < \infty$. By Theorem 3.17 the condition $q(\lambda I - T) < \infty$ entails that $\lambda I - T|M$ is surjective. Hence, if P is the projection of X onto M, then $P \in \mathcal{P}(X)$, $PT = TP$ and

$$(\lambda I - T)_P(P(X) = (\lambda I - T)(M) = M = P(X).$$

Thus $\lambda \notin \sigma_{\mathrm{su}}(T_P)$. ∎

6. Some spectral mapping theorems

We wish now to show that for many of the spectra introduced in the previous sections, the spectral mapping theorem holds. We begin first with a preliminary result on the abstract setting of Banach algebras. This result provides an unifying tool in order to establish the spectral mapping theorem for some of the spectra previously introduced.

Let $\mathcal{A}$ be a complex Banach algebra with identity u and $\mathcal{J}$ a closed ideal two-sided ideal of $\mathcal{A}$. Let ϕ be the canonical homomorphism of $\mathcal{A}$ onto $\widehat{\mathcal{A}} := \mathcal{A}/\mathcal{J}$. Moreover, let us denote by $\widehat{\mathcal{G}}$ the group of all invertible elements in $\widehat{\mathcal{A}}$.

Definition 3.59. *An open semigroup $\mathcal{S}$ of $\mathcal{A}$ is said to be a Φ-semigroup if the following properties hold:*

(i) *If $a, b \in \mathcal{A}$ and $ab = ba \in \mathcal{S}$ then $a \in \mathcal{S}$ and $b \in \mathcal{S}$;*

(ii) *There exists a closed two-sided ideal $\mathcal{J}$ and an open semi-group $\widehat{\mathcal{R}}$ in $\widehat{\mathcal{A}} = \mathcal{A}/\mathcal{J}$ such that $\widehat{\mathcal{G}} \subseteq \widehat{\mathcal{R}}$, $\widehat{\mathcal{R}} \setminus \widehat{\mathcal{G}}$ is open and $\mathcal{S} = \phi^{-1}(\widehat{\mathcal{R}})$.*

Obviously a Φ-semi-group $\mathcal{S}$ contains all invertible elements of $\mathcal{A}$ and $\mathcal{S} + \mathcal{J} \subseteq \mathcal{S}$.
For every $a \in \mathcal{S}$ let $\mathcal{S}_a$ denote the component of $\mathcal{S}$ containing a. If $b \in \mathcal{J}$ and

$$\Lambda := \{a + tb : 0 \le t \le 1\}$$

is a path joining a to $a + b$ then the inclusion $\mathcal{S} + J \subseteq \mathcal{S}$ implies that $\Lambda \subseteq \mathcal{S}$. From this it follows that $a + \mathcal{J} \subseteq \mathcal{S}_a$. This also implies that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1$ and $\mathcal{S}_2$ open disjoint subsets of $\mathcal{S}$ then $\mathcal{S}_i + \mathcal{J} \subseteq \mathcal{S}_i$ for $i = 1, 2$.

An *index* on a Φ-semigroup $\mathcal{S}$ is defined as a locally constant homomorphism of $\mathcal{J}$ into $\mathbb{N}$. Evidently, if $i : \mathcal{S} \to \mathbb{N}$ is an index, then

$$i(a+b) = i(a) \quad \text{for all } a \in \mathcal{S},\ b \in \mathcal{J}.$$

By Remark 1.54 the sets $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ satisfy the condition (i) of Definition 3.59. The Atkinson characterization of Fredholm operators establishes that $\Phi(X) = \phi^{-1}(\widehat{\mathcal{G}})$, where $\widehat{\mathcal{G}}$ is the set of all invertible elements of $\widehat{\mathcal{A}} := L(X)/K(X)$, so $\Phi(X)$ satisfies also the condition (ii) and hence is Φ-semigroup of $\mathcal{A} = L(X)$. Clearly,

$$\Phi_+(X) \setminus \Phi(X) = \{T \in \Phi_+(X) : \operatorname{ind} T = -\infty\}$$

and

$$\Phi_-(X) \setminus \Phi(X) = \{T \in \Phi_-(X) : \operatorname{ind} T = +\infty\}$$

are open sets. Since the canonical homomorphism ϕ is an open mapping, it follows that also $\Phi_+(X)$ and $\Phi_-(X)$ are Φ-semigroups of $\mathcal{A} := L(X)/K(X)$.

For every $a \in \mathcal{A}$ and a Φ-semigroup $\mathcal{S}$ of $\mathcal{A}$ let

$$\sigma_{\mathcal{S}}(a) := \{\lambda \in \mathbb{C} : \lambda u - a \notin \mathcal{S}\}.$$

The following result establishes an abstract spectral mapping theorem for spectra generated by Φ-semigroups.

Theorem 3.60. *Let $\mathcal{A}$ be a Banach algebra with identity u and $\mathcal{S}$ any Φ-semigroup. Suppose that $i : \mathcal{S} \to \mathbb{N}$ is an index such that $i(b) = 0$ for all invertible elements $b \in \mathcal{A}$. If f is an analytic function on an open domain $\mathbb{D}$ containing $\sigma(a)$, $\sigma(a)$ the spectrum of a relative to $\mathcal{A}$, then the following statements hold:*

(i) *$f(a) \in \mathcal{S}$ if and only if $f(\lambda) \neq 0$ for all $\lambda \in \sigma_{\mathcal{S}}(a)$;*

(ii) *$\sigma_{\mathcal{S}}(f(a)) = f(\sigma_{\mathcal{S}}(a))$.*

Proof (i) Suppose that $f(a) \in \mathcal{S}$ and $f(\lambda_0) = 0$. Define on $\mathbb{D}$ the function $g(\lambda) := f(\lambda)(\lambda - \lambda_0)^{-1}$. Clearly g is analytic and $(a - \lambda_0 u)g(a) \in \mathcal{S}$. From property (i) of Definition 3.59 we have $a - \lambda_0 u \in \mathcal{A}$, and hence $\lambda_0 \notin \sigma_{\mathcal{S}}(a)$.

Conversely, assume that $f(\lambda) \neq 0$ on $\sigma_{\mathcal{S}}(a)$. Consider first the case that f does not vanish identically on any component of $\mathbb{D}$. Then f has only a finite number of zeros on $\sigma(a)$. Let $\lambda_1, \dots, \lambda_k$ be these zeros with multiplicities $n_1, \dots, n_k$, respectively. Since

$$g(\lambda) = f(\lambda) \prod_{i=1}^{k} (\lambda - \lambda_i)^{-n_i}$$

is invertible on $\sigma(a)$ then $g(a)$ is invertible in $\mathcal{A}$, thus $g(a) \in \mathcal{S}$. Moreover, from $\lambda_i \notin \sigma_{\mathcal{S}}(a)$, $i = 1, \dots, k$, we obtain that

$$f(a) = g(a) \prod_{i=1}^{k} (a - \lambda_i u)^{-n_i}$$

belongs to $\mathcal{S}$.

Suppose the case that f vanishes identically on at least one component Ω of $\mathbb{D}$. Let $\sigma := \sigma(a) \cap \Omega$. Clearly, σ is a spectral set so we can consider the idempotent associated with σ:

$$p := \frac{1}{2\pi i} \int_\Gamma (\lambda u - a)^{-1} \mathrm{d}\lambda,$$

where Γ is a simple closed Jordan curve in $\Omega \setminus \sigma$.

We claim that $p \in \mathcal{J}$, where $\mathcal{J}$ is the ideal given by (ii) of Definition 3.59. From our assumption on f we easily obtain that $\lambda u - a \in \mathcal{S}$ for $\lambda \in \sigma$, so that

$$\{\lambda u - a : \lambda \in \Omega \setminus \sigma\} \subseteq \mathcal{S}.$$

This implies that the set

$$\widehat{\mathcal{B}} := \{\lambda\phi(u) - \phi(a) : \lambda \in \Omega \setminus \sigma\} \subseteq \phi(\mathcal{S}) = \widehat{\mathcal{R}}.$$

Clearly $\widehat{\mathcal{B}}$ is connected, since Ω is connected and ϕ is continuous. Furthermore, for $\lambda_0 \in \Omega \setminus \sigma$ we have $\lambda_0\phi(u) - \phi(a) \in \widehat{\mathcal{B}} \cap \widehat{\mathcal{G}}$, so $\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{G}}$ by property (ii) of Definition 3.59.

Finally, let us consider the function $\lambda \in \Omega \to (\lambda\phi(u) - \phi(a))^{-1}$. This function is analytic on Ω and

$$\phi(p) = \frac{1}{2\pi i} \int_\Gamma (\lambda\phi(u) - \phi(a))^{-1} \mathrm{d}\lambda = 0 \in \mathcal{A}/\mathcal{J},$$

which proves that $p \in \mathcal{J}$.

To conclude the proof let

$$g(\lambda) := \begin{cases} f(\lambda) & \text{on components of } \mathbb{D} \text{ where } f \neq 0, \\ 1 & \text{on components of } \mathbb{D} \text{ where } f \equiv 0, \end{cases}$$

and let $h(\lambda) := g(\lambda) - f(\lambda)$. Then $h(a)$ is the finite sum of projections corresponding to the spectral sets where f vanishes identically and this implies that $h(a) \in \mathcal{J}$. Furthermore, $g(a) \in \mathcal{S}$ since g does not vanish identically on any component of $\mathbb{D}$. From this we then conclude that

$$f(a) = g(a) - h(a) \in \mathcal{S} + J \subseteq \mathcal{S},$$

which completes the proof of the equivalence (i).

(ii) Clearly $\mu \notin f(\sigma_{\mathcal{S}}(a))$ precisely when $\mu - f(\lambda)$ has no zeros on $\sigma_{\mathcal{S}}(a)$. From part (i) this is true if and only if $\mu - f(a) \in \mathcal{A}$, or equivalently, $\mu \notin \sigma_{\mathcal{S}}(f(a))$. ∎

As usual, let $\mathcal{H}(\sigma(T))$ denote the set of all analytic function defined on an open set containing $\sigma(T)$. Since $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ are Φ-semigroups, Theorem 3.60 has as a consequence that the spectral mapping theorem holds for the spectra related to these classes of operators:

Corollary 3.61. *Let $T \in L(X)$ be an arbitrary operator on a Banach space X. If $g \in \mathcal{H}(\sigma(T))$ then the following equalities hold:*

(i) $g(\sigma_{\mathrm{f}}(T)) = \sigma_{\mathrm{f}}(g(T))$;

(ii) $g(\sigma_{\mathrm{uf}}(T)) = \sigma_{\mathrm{uf}}(g(T))$;

(iii) $g(\sigma_{\mathrm{lf}}(T)) = \sigma_{\mathrm{lf}}(g(T))$. ∎

It should be noted that the equality $g(\sigma_{\mathrm{f}}(T)) = \sigma_{\mathrm{f}}(g(T))$ may be obtained from the ordinary spectral mapping theorem, since by the Atkinson characterization of Fredholm operators we have $\sigma_{\mathrm{f}}(T) = \sigma(\widehat{T})$, where $\widehat{T}$ is the element $T + K(X)$ of the Calkin algebra $L(X)/K(X)$.

Let $a \in \mathcal{A}$ and $\mathcal{S}$ a Φ-semigroup of $\mathcal{A}$ with an index i. For every $n \in \mathbb{N}$ let us define

$$\sigma_n := \{\lambda \in \sigma(T) : i(\lambda u - a) = n\}.$$

Lemma 3.62. *Suppose that $f(a) \in \mathcal{S}$ and let α_n be the number of zeros of f on σ_n, counted according to their multiplicities, ignoring components of σ_n where f is identically 0. Then*

$$i(f(a)) = \sum_n n\alpha_n. \tag{85}$$

Proof If $f(a) \in \mathcal{S}$ then f does not vanish on $\sigma_{\mathcal{S}}(a)$. If we define g and h as in the proof of Theorem 3.60 then

$$i(f(a)) = i(g(a) - h(a)) = i(g(a)). \tag{86}$$

The equalities (86) suggest that in order to prove (85) we may only consider a function f which has at most a finite number of zeros on $\sigma(a)$. For every $n \in \mathbb{N}$, let λ_{ni}, $i = 1, \cdots, k_n$, be the zero of the function f on $\sigma_n(a)$ with multiplicities α_{ni}. Define

$$q(\lambda) := f(\lambda) \prod_{n,i} (\lambda - \lambda_{ni})^{-\alpha_{ni}}.$$

Then q is invertible on $\sigma(a)$, thus $q(a)$ is invertible in $\mathcal{A}$ and $i(q(a) = 0$.

From this we obtain

$$\begin{aligned} i(f(a)) &= i\left(q(a) \prod_{n,i} (a - \lambda_{ni} u)^{-\alpha_{ni}}\right) \\ &= i(q(a) + \sum_{n,i} n\alpha_{ni} = \sum_n n\alpha_n, \end{aligned}$$

where $\alpha_n := \sum_{i=1}^{k_n} \alpha_{n,i}$. ∎

Theorem 3.63. *Let $T \in L(X)$ a bounded operator on the Banach space X. If $g \in \mathcal{H}(\sigma(T))$, then the following inclusions hold:*

(i) $g(\sigma_{\mathrm{sf}}(T)) \subseteq \sigma_{\mathrm{sf}}(g(T))$;

(ii) $g(\sigma_{\mathrm{w}}(T)) \supseteq \sigma_{\mathrm{w}}(g(T))$.

Proof (i) From Corollary 3.61 we have

$$\begin{aligned}\sigma_{\rm sf}(g(T)) &= \sigma_{\rm uf}(g(T)) \cap \sigma_{\rm uf}(g(T)) = g(\sigma_{\rm uf}(T)) \cap g(\sigma_{\rm lf}(T)) \\ &\supseteq g[\sigma_{\rm uf}(T) \cap (\sigma_{\rm lf}(T)] = g(\sigma_{\rm sf}(T)).\end{aligned}$$

(ii) For every $n \in \mathbb{N}$ let

$$\Phi_n(X) := \{T \in \Phi(X) : \text{ind } T = n\},$$

and

$$\Omega_n := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_n(X)\}.$$

Obviously

$$\sigma_{\rm w}(T) = \sigma_{\rm f}(T) \cup \left(\bigcup_{n \neq 0} \Omega_n\right). \tag{87}$$

Now, if $\mu \notin g(\sigma_{\rm w}(T))$ then $\mu - g(\lambda)$ has no zeros on $\sigma_{\rm w}(T)$, and in particular has no zero on $\sigma_{\rm f}(T)$. From part (i) of Theorem 3.60 applied to the Φ-semigroup $\Phi(X)$ we then conclude that $\mu - f(T) \in \Phi(X)$ and

$$\text{ind } (\mu I - g(T) = \sum_{n \neq 0} n\alpha_n,$$

where α_n is the number of isolated zeros of $\mu - g(\lambda)$ on Ω_n, counted according to their multiplicities. From the equality (87) we infer that $\alpha_n = 0$ for every $n \neq 0$. Hence ind $(\mu I - g(T)) = 0$ and consequently $\mu \notin \sigma_{\rm w}(g(T))$. ■

The next examples show that in general the inclusions (i) and (ii) of Theorem 3.63 are proper.

Example 3.64. Let us consider an operator $T \in L(X)$, X a Banach space, for which $(I + T)(X)$ is closed and that is such that

$$\alpha(I + T) < \infty, \quad \beta(I + T) = \infty, \quad \alpha(I - T) = \infty, \quad \beta(I - T) < \infty.$$

Clearly $I + T \in \Phi_+(X)$ and $I - T \in \Phi_-(X)$, so $\{1, -1\} \subseteq \sigma_{\rm sf}(T)$. Define

$$g(\lambda) := (1 + \lambda)(1 - \lambda).$$

Then $\alpha(g(T)) = \beta(g(T)) = \infty$, thus $0 \in \sigma_{\rm sf}(g(T))$. On the other hand, it is clear that $0 \notin g(\sigma_{\rm sf}(T))$. This shows that the equality (i) of Theorem 3.63 does not hold.

To show that the inclusion (ii) of Theorem 3.63 generally is proper, let us consider a bounded operator $T \in \Phi(X)$ such that ind $(\lambda I + T) = -1$ and ind $(\lambda I - T) = 1$. Then

$$\text{ind } [(\lambda I - T)(\lambda I + T)] = \text{ind } (\lambda I - T) + \text{ind } (\lambda I + T) = 0,$$

so that $0 \notin \sigma_{\rm w}(g(T))$, where as before $g(\lambda) := (1 + \lambda)(1 - \lambda)$.

On the other hand, $\{1, -1\} \subseteq \sigma_{\rm w}(T)$ and hence $0 \in g(\sigma_{\rm w}(T))$.

The *Weyl approximate point spectrum* is defined by

$$\sigma_{\mathrm{wa}}(T) := \bigcap_{K \in K(X)} \sigma_{\mathrm{ap}}(T+K),$$

whilst the *Weyl surjectivity spectrum* is defined by

$$\sigma_{\mathrm{ws}}(T) := \bigcap_{K \in K(X)} \sigma_{\mathrm{su}}(T+K).$$

These denominations are motivated by Corollary 3.41 and Corollary 3.47.

The next result shows some basic properties of $\sigma_{\mathrm{wa}}(T)$ and $\sigma_{\mathrm{ws}}(T)$. Denote by acc σ the set of all accumulation point of $\sigma \subseteq \mathbb{C}$.

Theorem 3.65. *For a bounded operator $T \in L(X)$, where X is a Banach space, the following statements hold:*

(i) *$\lambda \notin \sigma_{\mathrm{wa}}(T)$ if and only if $\lambda I - T \in \Phi_+(X)$ and* ind $(\lambda I - T) \leq 0$. *Dually, $\lambda \notin \sigma_{\mathrm{ws}}(T)$ if and only if $\lambda I - T \in \Phi_-(X)$ and* ind $(\lambda I - T) \geq 0$*;*

(ii) *$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ws}}(T^\star)$, $\sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{wa}}(T^\star)$ and*

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{ws}}(T);$$

(iii) *$\sigma_{\mathrm{wa}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{ws}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$;*

(iv) *If $\lambda \in \sigma_{\mathrm{ap}}(T)$ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ and $p(\lambda I - T) = \infty$ then $\lambda \in \sigma_{\mathrm{wa}}(T)$. If $\lambda \in \sigma_{\mathrm{su}}(T)$ is an isolated point of $\sigma_{\mathrm{su}}(T)$ and $q(\lambda I - T) = \infty$ then $\lambda \in \sigma_{\mathrm{ws}}(T)$;*

(v) *We have*

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T) \cup \operatorname{acc} \sigma_{\mathrm{ap}}(T), \tag{88}$$

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T) \cup \operatorname{acc} \sigma_{\mathrm{su}}(T), \tag{89}$$

and

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T). \tag{90}$$

Proof (i) See Remark 3.40.

(ii) Clear, from part (i).

(iii) Obvious, from Theorem 3.44 and Theorem 3.46.

(iv) If $\lambda \in \sigma_{\mathrm{ap}}(T)$ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ then T has SVEP at λ, by Theorem 3.23. Suppose that $p(\lambda I - T) = \infty$ and $\lambda \notin \sigma_{\mathrm{wa}}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and hence by Theorem 3.16 $p(\lambda I - T) < \infty$, a contradiction. The second statement may proved in a similar way.

(v) If $\lambda \notin \sigma_{\mathrm{wa}}(T) \cup \operatorname{acc} \sigma_{\mathrm{ap}}(T)$ then λ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ and $\lambda I - T \in \Phi_+(X)$, by part (i). From part (iv) we also have $p(\lambda I - T) < \infty$ and hence $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Conversely, if $\lambda \in \operatorname{acc} \sigma_{\mathrm{ap}}(T)$ then $\lambda \in \sigma_{\mathrm{wa}}(T)$ or $\lambda \notin \sigma_{\mathrm{wa}}(T)$. In the first case $\lambda \in \sigma_{\mathrm{ub}}(T)$, since $\sigma_{\mathrm{wa}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$. In the second case $\lambda I - T \in \Phi_+(X)$, so by Theorem 3.23 T does not have the SVEP at λ, and hence $p(\lambda I - T) = \infty$ by Theorem 3.16. From this we conclude that $\lambda \notin \sigma_{\rm ub}(T)$. Therefore the equality (88) is proved. The proof of the equality (89) is similar. The equality (90) follows combining (88) with (89) and taking into account that $\sigma(T) = \sigma_{\rm ap}(T) \cup \sigma_{\rm su}(T)$. ∎

By passing we mention that similar formulas to those established in Theorem 3.57 and Theorem 3.58 hold for the Weyl spectrum $\sigma_{\rm w}(T)$, the Weyl approximate point spectrum $\sigma_{\rm wa}(T)$, and the Weyl surjectivity spectrum $\sigma_{\rm ws}(T)$. Precisely,

$$\sigma_{\rm w}(T) = \bigcap_{P \in \mathcal{P}(X)} \sigma(T_P),$$

and

$$\sigma_{\rm wa}(T) = \bigcap_{P \in \mathcal{P}(X)} \sigma_{\rm ap}(T_P), \quad \sigma_{\rm ws}(T) = \bigcap_{P \in \mathcal{P}(X)} \sigma_{\rm su}(T_P).$$

The interested reader may be find the proofs of these equalities in Zemánek [**333**].

Theorem 3.66. *Suppose that for $T \in L(X)$, T or T^* has the SVEP. Then*

$$\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T) \quad \textit{and} \quad \sigma_{\rm ws}(T) = \sigma_{\rm lb}(T).$$

Proof Suppose first that T has the SVEP. By part (iv) of Theorem 3.65, to show that $\sigma_{\rm ub}(T) = \sigma_{\rm wa}(T)$, it suffices to prove that acc $\sigma_{\rm ap}(T) \subseteq \sigma_{\rm wa}(T)$. Suppose that $\lambda \notin \sigma_{\rm wa}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and the SVEP at λ ensures that $\sigma_{\rm ap}(T)$ does not cluster at λ, by Theorem 3.23. Hence $\lambda \notin$ acc $\sigma_{\rm ap}(T)$.

To prove the equality $\sigma_{\rm lb}(T) = \sigma_{\rm ws}(T)$ it suffices to show that $\sigma_{\rm lb}(T) \subseteq \sigma_{\rm ws}(T)$. Suppose that $\lambda \notin \sigma_{\rm ws}(T)$. Then $\lambda I - T \in \Phi_-(X)$ with $\beta(\lambda I - T) \leq \alpha(\lambda I - T)$. Again, the SVEP at λ gives $p(\lambda I - T) < \infty$, and hence by part (i) of Theorem 3.4 $\alpha(\lambda I - T) = \beta(\lambda I - T)$. At this point the finiteness of $p(\lambda I - T)$ implies by part (iv) of Theorem 3.4 that also $q(\lambda I - T)$ is finite, so $\lambda \notin \sigma_{\rm lb}(T)$. Therefore $\sigma_{\rm lb}(T) \subseteq \sigma_{\rm ws}(T)$, and the proof of the second equality is complete in the case that T has the SVEP.

Suppose now that T^* has SVEP. Then by the first part $\sigma_{\rm ub}(T^*) = \sigma_{\rm wa}(T^*)$ and $\sigma_{\rm lb}(T^*) = \sigma_{\rm ws}(T^*)$. By duality it follows that $\sigma_{\rm lb}(T) = \sigma_{\rm ws}(T)$ and $\sigma_{\rm ub}(T) = \sigma_{\rm wa}(T)$. ∎

Theorem 3.67. *Let $T \in L(X)$, X a Banach space, and let $f \in \mathcal{H}(\sigma(T))$. Then*

$$\sigma_{\rm wa}(f(T)) \subseteq f(\sigma_{\rm wa}(T))$$

and

$$\sigma_{\rm ws}(f(T)) \subseteq f(\sigma_{\rm ws}(T)).$$

Proof For every $n \in \mathbb{N}$ define

$$\Omega_n := \{\lambda \in \sigma(T) : \lambda I - T \in \Phi_+(X), \, {\rm ind}\,(\lambda I - T) = n\}.$$

Clearly

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{uf}}(T) \cup \left(\bigcup_{n \geq 1} \Omega_n \right). \tag{91}$$

Now, if $\mu \notin f(\sigma_{\mathrm{wa}}(T))$ then $\mu I - f(\lambda)$ has no zeros on $\sigma_{\mathrm{wa}}(T)$ and in particular has no zeros on $\sigma_{\mathrm{uf}}(T)$. From Theorem 3.60 and Lemma 3.62 we conclude that $\mu I - T \in \Phi_+(X)$ and

$$\operatorname{ind}(\mu - f(T)) = \sum_n n\alpha_n,$$

where, as in Lemma 3.62, α_n is the number of isolated zeros of $\mu - f(\lambda)$ on Ω_n, counted according to their multiplicities. From (91) we obtain that $\alpha_n = 0$ for $n \geq 1$ and hence $\operatorname{ind}(\mu - f(T)) \leq 0$. This implies that $\mu \notin \sigma_{\mathrm{wa}}(f(T))$.

The last inclusion is an obvious consequence of the equality $f(\sigma_{\mathrm{ws}}(T)) = f(\sigma_{\mathrm{wa}}(T^\star))$. ∎

Suppose that a Banach space X is the direct sum $X = M \oplus N$, where the closed subspaces M and N are T-invariant, and let P_M denote the projection of X onto M. Clearly, P_M commutes with T and

$$p(T) < \infty \Leftrightarrow p(T|M),\ p(T|N) < \infty$$

and

$$\ker T = \ker T|M \oplus \ker T|N, \quad T(X) = T(M) \oplus T(N).$$

Moreover, as observed before Lemma 2.47, $T(X)$ is closed if and only if $T(M)$ is closed in M and $T(N)$ is closed in N. Combining all these properties we obtain

$$T \in \mathcal{B}_+(X) \Leftrightarrow T|M \in \mathcal{B}_+(M),\ T|N \in \mathcal{B}_+(N),$$

and hence

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T|M) \cup \sigma_{\mathrm{ub}}(T|N).$$

Similar equivalences may be established for $\mathcal{B}_-(X)$ and $\mathcal{B}(X)$, so

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{lb}}(T|M) \cup \sigma_{\mathrm{lb}}(T|N),$$

and

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T|M) \cup \sigma_{\mathrm{b}}(T|N).$$

Lemma 3.68. *Let $T \in L(X)$, X a Banach space, and suppose that the function $f \in \mathcal{H}(\sigma(T))$ is constant on each connected component of an open set $\mathcal{U}$ containing $\sigma(T)$. Then*

$$\begin{aligned} f(\sigma_{\mathrm{ub}}(T)) &= f(\sigma_{\mathrm{lb}}(T)) = f(\sigma_{\mathrm{b}}(T)) = f(\sigma(T)) = \\ \sigma_{\mathrm{ub}}(f(T)) &= \sigma_{\mathrm{lb}}(f(T)) = \sigma_{\mathrm{b}}(f(T)) = \sigma(f(T)). \end{aligned}$$

Proof The proof is similar to that given in the proof of Lemma 2.47. ∎

In the proof of the following result we shall use the spectral mapping theorems for $\not<(T)$ and $\sigma_{\mathrm{ap}}(T)$. We have seen that for these spectra the spectral mapping theorem holds only in the cases that the analytic function

f is non-constant on the connected components of its domain of definition. For this reason we will consider in the proof two distinct cases. Lemma 3.68 will be then employed to prove the result also in the case that f is constant on some components of its domain.

Theorem 3.69. *Let $T \in L(X)$ be a bounded operator on a Banach space X and $f \in \mathcal{H}(\sigma(T))$. Then*

$$\sigma_{\rm ub}(f(T)) = f(\sigma_{\rm ub}(T)) \quad \textit{and} \quad \sigma_{\rm lb}(f(T)) = f(\sigma_{\rm lb}(T)). \tag{92}$$

Proof We show the inclusion $f(\sigma_{\rm ub}(T)) \subseteq \sigma_{\rm ub}(f(T))$.

Suppose first that f is non-constant on each connected component of $\mathcal{U}$. Then by Theorem 3.52, Theorem 1.78, and Theorem 2.39

$$\begin{aligned} f(\sigma_{\rm ub}(T)) &= f\left(\sigma_{\rm uf}(T) \cup \Xi(T)\right) \subseteq f(\sigma_{\rm uf}(T)) \cup f(\Xi(T)) \\ &= \sigma_{\rm uf}(f(T)) \cup \Xi(f(T)) = \sigma_{\rm ub}(f(T)). \end{aligned}$$

It remains to prove the opposite inclusion $f(\sigma_{\rm ub}(T)) \supseteq \sigma_{\rm ub}(f(T))$.

Suppose that $\lambda \in \sigma_{\rm ub}(f(T))$. We distinguish two cases.

First case: Suppose that $\lambda \in \sigma_{\rm wa}(f(T))$. In this case $\lambda \in f(\sigma_{\rm wa}(T))$, by Theorem 3.67, and from the inclusion $\sigma_{\rm wa}(T) \subseteq \sigma_{\rm ub}(T)$ it follows that $\lambda \in f(\sigma_{\rm ub}(T))$.

Second case: Suppose that $\lambda \notin \sigma_{\rm wa}(f(T))$. In this case by Theorem 3.65 λ is a limit point of $\sigma_{\rm ap}(f(T))$, so there is a sequence $\{\lambda_n\}$ of points of $\sigma_{\rm ap}(f(T))$ such that $\lambda_n \to \lambda$. From the equality $\sigma_{\rm ap}(f(T)) = f(\sigma_{\rm ap}(T))$, see Theorem 2.48, it follows that there is a sequence $\{\mu_n\} \subset \sigma_{\rm ap}(T)$ such that $f(\mu_n) = \lambda_n$. The sequence $\{\mu_n\}$ is bounded, hence there exists a convergent subsequence and we may assume that $\mu_n \to \mu \in \sigma_{\rm ap}(T)$. Then $\lambda = f(\mu) \in f(\text{acc}\, \sigma_{\rm ap}(T)$, so $\lambda \in f(\sigma_{\rm ub}(T))$ by part (v) of Theorem 3.65.

This completes the proof of the first equality of (92) in the case that f is non-constant on each connected component of $\mathcal{U}$.

Consider now the other possibility, i.e., f is constant on some components of $\mathcal{U}$. Proceeding as in the proof of Theorem 2.48 we can find two closed subspaces T-invariant M and N such that $X = M \oplus N$ and an open set Ω such that $\sigma(T|M) \subseteq \Omega$, $\sigma(T|N) \subseteq \mathcal{U} \setminus \Omega$ and the restriction g of f on Ω, as well as the restriction h of f onto $\mathcal{U} \setminus \Omega$ are analytic. Furthermore, g is constant on every connected component of Ω, h is non-constant on every connected component of $\mathcal{U} \setminus \Omega$. From $g(T|M) = f(T)\ |M$ and $h(T|N) = f(T)|N$, and taking into account Lemma 3.68 and the first part of the proof, we then obtain

$$\begin{aligned} \sigma_{\rm ub}(f(T)) &= \sigma_{\rm ub}(f(T)|M) \cup \sigma_{\rm ub}(f(T)|N) = \sigma_{\rm ub}(g(T|M)) \cup \sigma_{\rm ub}(h(T|N)) \\ &= g(\sigma_{\rm ub}(T|M)) \cup h(\sigma_{\rm ub}(T|N)) = f(\sigma_{\rm ub}(T|M)) \cup f(\sigma_{\rm ub}(T|N)) \\ &\qquad \text{(since } \sigma_{\rm ub}(T|M) \text{ and } \sigma_{\rm ub}(T|N) \text{ are disjoint)} \\ &= f(\sigma_{\rm ub}(T|M) \cup \sigma_{\rm ub}(T|N)) = f(\sigma_{\rm ub}(T)), \end{aligned}$$

which completes the proof of the first equality of (92).

The second equality of (92) easily follows from the equality $\sigma_{\mathrm{lb}}(f(T)) = \sigma_{\mathrm{ub}}(f(T^\star))$. ∎

We now establish the spectral mapping theorem for the Browder spectrum. Note that in the proof we use a argument similar to that used in the proof of preceding theorem.

Theorem 3.70. *Let $T \in L(X)$ be a bounded operator on a Banach space X and $f \in \mathcal{H}(\sigma(T))$. Then*

$$\sigma_{\mathrm{b}}(f(T)) = f(\sigma_{\mathrm{b}}(T))$$

Proof From Theorem 3.69 we have

$$\begin{aligned} f(\sigma_{\mathrm{b}}(T)) &= f(\sigma_{\mathrm{ub}}(T) \cup \sigma_{\mathrm{lb}}(T)) \subseteq f(\sigma_{\mathrm{ub}}(T)) \cup f(\sigma_{\mathrm{lb}}(T)) \\ &= \sigma_{\mathrm{ub}}(f(T)) \cup \sigma_{\mathrm{lb}}(f(T)) = \sigma_{\mathrm{b}}(f(T)). \end{aligned}$$

Conversely, let $\lambda \in \sigma_{\mathrm{b}}(f(T))$ and proceed as in the first part of the proof of Theorem 3.69. We have two possible alternatives: $\lambda \in f(\sigma_{\mathrm{w}}(T))$ or $\lambda \notin f(\sigma_{\mathrm{w}}(T))$.

If $\lambda \in f(\sigma_{\mathrm{w}}(T))$ then $\lambda \in f(\sigma_{\mathrm{b}}(T))$, since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Consider the other possibility, $\lambda \notin f(\sigma_{\mathrm{w}}(T))$. By part (v) of Theorem 3.65 there exists a sequence $\{\lambda_n\} \subseteq \sigma(f(T)) = f(\sigma(T))$ such that $\lambda_n \to \lambda$. Let $\mu_n \in \sigma(T)$ be such that $f(\mu_n) = \lambda_n$. The sequence μ_n is bounded so there exists a subsequence converging to a certain $\mu \in \sigma(T)$. We may assume that $\mu_n \to \mu$, so $\lambda = f(\mu)$ is a limit point of a sequence of $f(\sigma(T))$. From part (v) of Theorem 3.65 we then conclude that $\lambda \in f(\sigma_{\mathrm{b}}(T))$, which completes the proof. ∎

The spectral mapping theorem for the Browder spectrum may be proved in a different way, by showing that the Browder spectrum $\sigma_{\mathrm{b}}(T)$ is the ordinary spectrum of an element of a suitable Banach algebra. In fact, as noted by Gramsch and Lay in [**141**], see also Theorem 3.48, if $\mathcal{A}$ is the maximal commutative subalgebra of $L(X)$ containing T and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A} \cap K(X)$ is the canonical quotient homomorphism, then $\sigma_{\mathrm{b}}(T) = \sigma(\pi(T))$. Moreover, for every analytic function on an open set $\mathcal{U}$ containing $\sigma(T)$ we have $\sigma(\pi(f(T)) = f(\pi(T))$ and $f(T) \in \mathcal{A}$, so the result follows from the usual spectral mapping theorem in $\mathcal{A}/\mathcal{A} \cap K(X)$. The spectral mapping theorem for the spectra $\sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{lb}}(T)$ may be proved by using different methods. For instance, Oberai [**252**] proved this result by showing first that the mapping $T \to \sigma_{\mathrm{b}}(T)$ is upper semi-continuous. The spectral mapping theorem for the upper semi-Browder spectrum $\sigma_{\mathrm{ub}}(T)$ has been proved by Rakočević [**272**], through a similar argument, by proving that the mapping $T \to \sigma_{\mathrm{ub}}(T)$ is upper semi-continuous. Another proof may be also found in Schmoeger[**294**].

Theorem 3.71. *Let $T \in L(X)$ be a bounded operator on a Banach space X and $f \in \mathcal{H}(\sigma(T))$. If T or T^* has SVEP then*

$$\sigma_{\rm wa}(f(T)) = \sigma_{\rm ub}(f(T)), \quad \sigma_{\rm ws}(f(T)) = \sigma_{\rm lb}(f(T)), \tag{93}$$

and

$$\sigma_{\rm w}(f(T)) = \sigma_{\rm b}(f(T)). \tag{94}$$

Proof Suppose first that T has the SVEP. Then $f(T)$ has SVEP by Theorem 2.40, so the equalities of (93) follow from Theorem 3.66.
The proof in the case T^* has the SVEP follows by duality and Theorem 3.69. In fact, $f(T^*)$ has the SVEP so, from the first part of the proof, we have

$$\begin{aligned} \sigma_{\rm wa}(f(T)) &= \sigma_{\rm ws}(f(T)^*) = \sigma_{\rm ws}(f(T^*)) = \sigma_{\rm lb}(f(T^*)) \\ &= f(\sigma_{\rm lb}(T^*)) = f(\sigma_{\rm ub}(T)) = \sigma_{\rm ub}(f(T)). \end{aligned}$$

The proof of the equality $\sigma_{\rm ws}(f(T)) = \sigma_{\rm lb}(f(T))$ is analogous.
Finally,

$$\begin{aligned} \sigma_{\rm w}(f(T)) &= \sigma_{\rm wa}(f(T)) \cup \sigma_{\rm ws}(f(T)) \\ &= \sigma_{\rm ub}(f(T)) \cup \sigma_{\rm lb}(f(T)) = \sigma_{\rm b}(f(T)), \end{aligned}$$

so also (94) is proved. ∎

Corollary 3.72. *If T or T^* has the SVEP and $f \in \mathcal{H}(\sigma(T))$, then $f(\sigma_{\rm w}(T)) = \sigma_{\rm w}(f(T))$. Analogous equalities hold for $\sigma_{\rm wa}(T)$ and $\sigma_{\rm ws}(T)$. Moreover,*

$$\sigma_{\rm w}(f(T)) = \sigma_{\rm b}(f(T)) = f(\sigma_{\rm w}((T)) = f(\sigma_{\rm b}((T)),$$

for every $f \in \mathcal{H}(\sigma(T))$.

Proof By Theorem 3.71 and Theorem 3.70 we have $\sigma_{\rm w}(f(T)) = \sigma_{\rm b}(f(T)) = f(\sigma_{\rm b}(T))$ and $\sigma_{\rm b}(T) = \sigma_{\rm w}(T)$, by Corollary 3.53.
The assertions for $\sigma_{\rm wa}(T)$ and $\sigma_{\rm ws}(T)$ follow in a similar way. ∎

For an operator $T \in L(X)$, X a Banach space, we say that the *a-Browder's theorem* holds if $\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T)$. Note that if T satisfies *a*-Browder's theorem then $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$. Indeed, if $\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T)$ and $\lambda \notin \sigma_{\rm w}(T)$ from the inclusion $\sigma_{\rm wa}(T) \subseteq \sigma_{\rm w}(T)$ it follows that $\lambda \notin \sigma_{\rm wa}(T) = \sigma_{\rm ub}(T)$, and hence $p(\lambda I - T) < \infty$. But $\lambda I - T$ is Weyl, hence by Theorem 3.4 $q(\lambda I - T)$ is also finite, and consequently $\lambda \notin \sigma_{\rm b}(T)$. This shows that $\sigma_{\rm b}(T) \subseteq \sigma_{\rm w}(T)$, and since the opposite inclusion is always verified we conclude that $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$.

In literature a bounded operator on a Banach space for which the equality $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$ holds is said to satisfy *Browder's theorem.* Taking into account that $\sigma_{\rm wa}(T^*) = \sigma_{\rm ws}(T)$ and $\sigma_{\rm ub}(T^*) = \sigma_{\rm lb}(T)$, from Theorem 3.71 we immediately obtain the following result.

Corollary 3.73. *Let $T \in L(X)$ be a bounded operator on a Banach space X and $f \in \mathcal{H}(\sigma(T))$. If either T or T^* has SVEP then both $f(T)$ and $f(T^*)$ obey to a-Browder's theorem. In particular, T and T^* obey a-Browder's theorem.* ∎

7. Isolated points of the spectrum

In this section we shall take a closer look at the isolated points of the spectrum. Recall that from the functional calculus, if λ_0 is an isolated point of the spectrum and P_0 is the spectral projection associated with the spectral set $\{\lambda_0\}$, then the subspaces $P_0(X)$ and $\ker P_0$ are invariant under T and $\sigma(T\,|P_0(X)) = \{\lambda_0\}$, whilst $\sigma(T\,|\ker P_0) = \mathbb{C} \setminus \{\lambda_0\}$.

The first result of this section shows that for an isolated point λ_0 of $\sigma(T)$ the quasi-nilpotent part $H_0(\lambda_0 I - T)$ and the analytical core $K(\lambda_0 I - T)$ may be precisely described as a range or a kernel of a projection.

Theorem 3.74. *Let $T \in L(X)$, where X is a Banach space, and suppose that λ_0 is an isolated point of $\sigma(T)$. If P_0 is the spectral projection associated with $\{\lambda_0\}$, then:*

(i) $P_0(X) = H_0(\lambda_0 I - T)$*;*

(ii) $\ker P_0 = K(\lambda_0 I - T)$.

In particular, if $\{\lambda_0\}$ is a pole of the resolvent, or equivalently $p := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$, then

$$P_0(X) = H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$\ker P_0 = K(\lambda_0 I - T) = (\lambda_0 I - T)^p(X).$$

Proof (i) Since λ_0 is an isolated point of $\sigma(T)$ there exists a positively oriented circle $\Gamma := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \delta\}$ which separates λ_0 from the remaining part of the spectrum. We have

$$(\lambda_0 I - T)^n P_0 x = \frac{1}{2\pi i}\int_\Gamma (\lambda_0 I - T)^n (\lambda I - T)^{-1} x \, d\lambda \quad \text{for all } n = 0, 1, \cdots .$$

Now, assume that $x \in P_0(X)$. We have $P_0 x = x$ and it is easy to verify the following estimate:

$$\|(\lambda_0 I - T)^n x\| \leq \frac{1}{2\pi} 2\pi\delta^{n+1} \max_{\lambda\in\Gamma} \|(\lambda I - T)^{-1}\|\|x\|.$$

Obviously this estimate also holds for some $\delta_o < \delta$, and consequently

$$\limsup \|(\lambda_0 I - T)^n x\|^{1/n} < \delta. \tag{95}$$

This proves the inclusion $P_0(X) \subseteq H_0(\lambda_0 I - T)$.

Conversely, assume that $x \in H_0(\lambda_0 I - T)$ and hence that the inequality (95) holds. Let $S \in L(X)$ denote the operator

$$S := I - \frac{1}{\lambda_0 - \lambda}(\lambda_0 I - T).$$

Evidently the Neumann series

$$\sum_{n=0}^{\infty} S^n x = \sum_{n=0}^{\infty} \left(I - \frac{1}{\lambda_0 - \lambda}(\lambda_0 I - T)\right)^n x$$

converges for all $\lambda \in \Gamma$. If y_λ denotes its sum for every $\lambda \in \Gamma$, from a standard argument of functional analysis we obtain that $(I - S)y_\lambda = x$. A simple calculation also shows that

$$y_\lambda = (\lambda - \lambda_0)(\lambda I - T)^{-1} x$$

and therefore

$$(\lambda I - T)^{-1} x = -\sum_{n=0}^{\infty} \frac{\lambda_0 I - T)^n x}{(\lambda_0 - \lambda)^{n+1}} \quad \text{for all } \lambda \in \Gamma.$$

A term by term integration then yields

$$P_0 x = \frac{1}{2\pi i}\int_\Gamma (\lambda I - T)^{-1} x \, d\lambda = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{(\lambda_0 - \lambda)} x \, d\lambda = x,$$

and this proves the inclusion $H_0(\lambda_0 I - T) \subseteq P_0(X)$. This completes the proof of the equality (i).

(ii) There is no harm in assuming that $\lambda_0 = 0$. We have $\sigma(T|P_0(X)) = \{0\}$, and $0 \in \rho(T|\ker P_0)$. From the equality $T(\ker P_0) = \ker P_0$ we obtain $\ker P_0 \subseteq K(T)$, see Theorem 1.22.

It remains to prove the reverse inclusion $K(T) \subseteq \ker P_0$. To see this we first show that $H_0(T) \cap K(T) = \{0\}$. This is clear because $H_0(T) \cap K(T) = K(T|H_0(T))$, and the last subspace is $\{0\}$ since the restriction of T on the Banach space $H_0(T)$ is a quasi-nilpotent operator, see Corollary 2.28. Hence $H_0(T) \cap K(T) = \{0\}$. From this it then follows that

$$\begin{aligned} K(T) &\subseteq K(T) \cap X = K(T) \cap [\ker P_0 \oplus P_0(X)] \\ &= \ker P_0 + K(T) \cap H_0(T) = \ker P_0, \end{aligned}$$

so the desired inclusion is proved.

The last assertion is clear from Remark 3.7, part (b). ∎

Corollary 3.75. *Let $T \in L(X)$, where X is a Banach space. The following statements are equivalent:*

(i) *T is quasi-nilpotent;*

(ii) *$K(T) = \{0\}$ and 0 is an isolated point of the spectrum;*

(iii) *$\overline{H_0(T)} = X$ and 0 is an isolated point of the spectrum.*

Proof (i)⇔(ii) Obviously, if T is quasi-nilpotent then 0 is an isolated point of $\sigma(T)$ and $K(T) = \{0\}$, by Corollary 2.28.

Conversely, if 0 is isolated in $\sigma(T)$ then $P_0(X) = H_0(T) = X$ and $K(T) = \ker\, P_0 = \{0\}$. Obviously this implies that P_0 is the identity, so $H_0(T) = X$. From Theorem 1.68 we conclude that T is quasi-nilpotent.

(iii)⇒(i) If 0 is an isolated point of the spectrum then $H_0(T) = P_0(X)$ is closed and therefore $H_0(T) = X$. Again, by Theorem 1.68 we conclude that T is quasi-nilpotent.

(i)⇒(iii) Obvious. ∎

From Theorem 3.74 we infer that if λ_0 is isolated in $\sigma(T)$ then $X = K(\lambda_0 I - T) \oplus H_0(\lambda_0 I - T)$. The following result show that the reverse implication holds if we assume that $K(\lambda_0 I - T)$ is closed .

Theorem 3.76. *For a bounded operator $T \in L(X)$, where X is a Banach space, the following assertions are equivalent:*

(i) *λ_0 is an isolated point of $\sigma(T)$;*

(ii) *$K(\lambda_0 I - T)$ is closed and $X = K(\lambda_0 I - T) \oplus H_0(\lambda_0 I - T)$.*

Proof Also here we assume $\lambda_0 = 0$. Since $K(T)$ is closed, by Theorem 1.22 we know $T(K(T)) = K(T)$. Moreover, since $\ker T \subseteq H_0(T)$ the operator $\overline{T} : K(T) \to K(T)$ is invertible. Hence there exists a $\varepsilon > 0$ such that $\lambda I - T$ is invertible for every $|\lambda| < \varepsilon$. It follows that

$$(\lambda I - T)(K(T)) = K(T) \text{ for every } |\lambda| < \varepsilon. \tag{96}$$

Since ker $(\lambda I - T) \subseteq K(T)$ for all $\lambda \neq 0$, we have

$$\ker\, (\lambda I - T) = \{0\} \quad \text{for every } 0 < |\lambda| < \varepsilon. \tag{97}$$

By Theorem 3.74, we also have

$$H_0(T) \subseteq (\lambda I - T)(X) \quad \text{for every } \lambda \neq 0. \tag{98}$$

The equality (96) and the inclusion (97) imply

$$X = K(T) \oplus H_0(T) \subseteq (\lambda I - T)(X) \quad \text{for every } 0 < |\lambda| < \varepsilon.$$

Consequently

$$\{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon\} \subseteq \rho(T),$$

and hence 0 is an isolated point of $\sigma(T)$. ∎

Theorem 3.77. *Let λ_0 be an isolated point of $\sigma(T)$. Then the following assertions are equivalent:*

(i) *$\lambda_0 I - T \in \Phi_\pm(X)$;*

(ii) *$\lambda_0 I - T$ is Browder;*

(iii) *$H_o(\lambda_0 I - T)$ is finite-dimensional;*

(iv) *$K(\lambda_0 I - T)$ is finite-codimensional.*

Proof The equivalence (i) ⇔ (ii) follows from Corollary 3.50. The implication (ii) ⇒ (iii) is clear from Theorem 3.18 since T has the SVEP at every isolated point of $\sigma(T)$. The implication (iii) ⇒ (iv) is clear, since, as observed above, $X = H_o(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$.

(iv) ⇒ (i) We have $K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \subseteq (\lambda_0 I - T)(X)$, so the finite-codimensionality of $K(\lambda_0 I - T)$ implies that also $(\lambda_0 I - T)(X)$ is finite-codimensional and hence $\lambda_0 \in \Phi_-(X)$. ∎

Now, $\sigma(T) = \sigma_{\rm ap}(T) \cup \sigma_{\rm su}(T)$. In the next result, instead of isolated points of $\sigma(T)$ we shall consider, separately, spectral points isolated in $\sigma_{\rm ap}(T)$, or isolated in $\sigma_{\rm su}(T)$.

Theorem 3.78. *Let $T \in L(X)$, X a Banach space, and $\lambda_0 \in \mathbf{C}$. We have:*

(i) If λ_0 is isolated in $\sigma_{\rm ap}(T)$ then

$$\lambda_0 I - T \in \Phi_{\pm}(X) \Leftrightarrow \dim H_o(\lambda_0 I - T) < \infty \quad \textit{and } (\lambda_0 I - T)(X) \quad \textit{is closed.}$$

(ii) If λ_0 is isolated in $\sigma_{\rm su}(T)$, then

$$\lambda_0 I - T \in \Phi_{\pm}(X) \Leftrightarrow \operatorname{codim} K(\lambda_0 I - T) < \infty.$$

Proof (i) Let λ_0 be a spectral point isolated in $\sigma_{\rm ap}(T)$. Then T has the SVEP at λ_0, and hence by Theorem 3.44 $\lambda_0 I - T \in \Phi_{\pm}(X)$ precisely when $\lambda_0 I - T \in \mathcal{B}_+(X)$. In this case the range of $\lambda_0 I - T$ is closed, $H_o(\lambda_0 I - T)$ is finite dimensional and coincides with $\ker(\lambda_0 I - T)^p$, where $p := p(\lambda_0 I - T)$, by Theorem 3.16 and Theorem 3.18.

Conversely, if $H_o(\lambda_0 I - T)$ is finite-dimensional the inclusion $\ker\,(\lambda_0 I - T) \subseteq H_o(\lambda_0 I - T)$ entails that $\ker\,(\lambda_0 I - T)$ is finite-dimensional, so if $\lambda_0 I - T$ has closed range then $\lambda_0 I - T \in \Phi_+(X)$.

(ii) Let λ_0 be a spectral point isolated in $\sigma_{\rm su}(T)$. Then $T^\star$ has the SVEP at λ_0, and hence by Theorem 3.46 $\lambda_0 I - T \in \Phi_{\pm}(X)$ precisely when $\lambda_0 I - T \in \mathcal{B}_+(X)$. In this case $K(\lambda_0 I - T)$ is finite-codimensional and coincides with $(\lambda_0 I - T)^q(X)$, where $q := q(\lambda_0 I - T)$, again by Theorem 3.17 .

Conversely, if $K(\lambda_0 I - T)$ is finite-codimensional, from $K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)(X)$ we obtain that also $(\lambda_0 I - T)(X)$ is finite-codimensional, and hence $\lambda_0 \in \Phi_-(X)$. ∎

The following result gives a description of $\sigma_{\rm sf}(T)$ or $\sigma_{\rm f}(T)$ from the point of view of local spectral theory.

Theorem 3.79. *Let $T \in L(X)$ and $\lambda_0 \in \mathbb{C}$. Then we have:*

(i) *If T has the SVEP then $\lambda_0 \in \sigma_{\rm sf}(T)$ if and only if λ_0 is a cluster point of $\sigma_{\rm ap}(T)$ or λ_0 is an isolated point of $\sigma_{\rm ap}(T)$ for which either $H_o(\lambda_o I - T)$ is infinite-dimensional or $\lambda_o I - T$ has closed range.*

(ii) *If $T^\star$ has the SVEP then $\lambda_0 \in \sigma_{\rm sf}(T)$ if and only if λ_0 is a cluster point of $\sigma_{\rm su}(T)$ or λ_0 is an isolated point of $\sigma_{\rm su}(T)$ for which $K(\lambda_o I - T)$ is infinite-codimensional.*

(iii) *If both T and $T^\star$ have the SVEP then $\lambda_0 \in \sigma_{\rm f}(T)$ if and only if λ_0 is a cluster point of $\sigma(T)$ or λ_0 is an isolated point of $\sigma(T)$ for which $K(\lambda_o I - T)$ is infinite-codimensional, or equivalently, $H_o(\lambda_o I - T)$ is infinite-dimensional.*

Proof (i) If T has the SVEP and λ_0 is a cluster point of $\sigma_{\rm ap}(T)$ then $\lambda_0 \in \sigma_{\rm kt}(T) \subseteq \sigma_{\rm sf}(T)$, by Corollary 3.25. If λ_0 is isolated in $\sigma_{\rm ap}(T)$ and

$H_o(\lambda_o I - T)$ is infinite-dimensional, or $\lambda_o I - T$ has closed range, then $\lambda_0 \in \sigma_{\mathrm{sf}}(T)$, by part (i) of Theorem 3.78.

Conversely, suppose that $\lambda_0 \in \sigma_{\mathrm{sf}}(T)$. Obviously λ_0 is a cluster point of $\sigma_{\mathrm{ap}}(T)$ or is isolated in $\sigma_{\mathrm{ap}}(T)$. In the second case, again by part (i) of Theorem 3.78, either $H_o(\lambda_o I - T)$ is infinite-dimensional, or $\lambda_o I - T$ has closed range.

(ii) If $T^\star$ has the SVEP and λ_0 is a cluster point of $\sigma_{\mathrm{su}}(T)$ then $\lambda_0 \in \sigma_{\mathrm{kt}}(T) \subseteq \sigma_{\mathrm{sf}}(T)$ by Corollary 3.28. If λ_0 is isolated in $\sigma_{\mathrm{su}}(T)$ and $K(\lambda_o I - T)$ is infinite-codimensional then $\lambda_0 \in \sigma_{\mathrm{sf}}(T)$ by part (ii) of Theorem 3.78.

Conversely, if λ_0 is an isolated point of $\sigma_{\mathrm{su}}(T)$ then $K(\lambda_o I - T)$ is infinite-codimensional, again by part (ii) of Theorem 3.78.

(iii) If both T and $T^\star$ have the SVEP then by Corollary 3.53 $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T)$ and $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{sf}}(T)$. The assertion then follows from part (ii). ∎

It has been observed in Remark 2.25 that if an operator $T \in L(X)$ has the SVEP atλ_0, and if Y is a closed subspace of X such that $(\lambda_0 I - T)(Y) = Y$ then $\ker(\lambda_0 I - T) \cap Y = \{0\}$.

The following useful result shows that this result is even true whenever we assume that Y is complete with respect to a new norm and Y is continuously embedded in X.

Lemma 3.80. *Suppose that X is a Banach space and that the operator $T \in L(X)$ has the SVEP at λ_0. Let Y be a Banach space which is continuously embedded in X and satisfies $(\lambda_0 I - T)(Y) = Y$. Then $\ker(\lambda_0 I - T) \cap Y = \{0\}$.*

Proof It follows from the closed graph theorem that the restriction $T|Y$ is continuous with respect to the given norm $\|\cdot\|_1$ on Y. Moreover, since every analytic function $f : \mathcal{U} \to (Y, \|\cdot\|_1)$ on an open set $\mathcal{U} \subseteq \mathbb{C}$ remains analytic when considered as a function from $\mathcal{U}$ to X, it is clear that $T|Y$ inherits the SVEP at λ_0 from T. Hence Corollary 2.24 applies to $T|Y$ with respect to the norm $\|\cdot\|_1$. ∎

By Theorem 3.16, if $\lambda_0 I - T$ a semi-Fredholm operator T has the SVEP at λ_0 precisely when $p(\lambda_0 I - T) < \infty$. The next result shows that this equivalence holds also under the assumption that $q(\lambda_0 I - T) < \infty$.

Theorem 3.81. *Let $T \in L(X)$, X a Banach space, and suppose that $q(\lambda_0 I - T) < \infty$. Then the following conditions are equivalent:*

(i) *T has the SVEP at λ_0;*

(ii) *$p(\lambda_0 I - T) < \infty$;*

(iii) *λ_0 is a pole of the resolvent;*

(iv) *λ_0 is an isolated point of $\sigma(T)$.*

Proof There is no harm in assuming $\lambda_0 = 0$.

(i) $\Rightarrow$ (ii) Let $q := q(T)$ and $Y := T^q(X)$. Let us consider the map

$\widehat{T} : X/\ker T^q \to Y$ defined by $\widehat{T}(\widehat{x}) := Tx$ where $x \in \widehat{x}$. Clearly, since $\widehat{T}$ is continuous and bijective we can define in Y a new norm

$$\|y\|_1 := \inf\{\|x\| : T^q(x) = y\},$$

for which $(Y, \|\cdot\|_1)$ becomes a Banach space. Moreover, if $y = T^q(x)$ from the estimate

$$\|y\| = \|T^q(x)\| \leq \|T^q\|\|x\|$$

we deduce that Y can be continuously embedded in X. Since $T(T^q(X)) = T^{q+1}(X) = T^q(X)$, by Corollary 3.80 we conclude that $\ker T \cap T^q(X) = \{0\}$ and hence by Lemma 3.2 $p(T) < \infty$.

(ii) $\Rightarrow$ (iii) If $p := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$ then λ_0 is a pole of order p, see Remark 3.7, part (c).

(iii) $\Rightarrow$ (iv) Obvious.

(iii) $\Rightarrow$ (iv) This has been observed above. ∎

The preceding result is reminiscent of the equivalences established in Corollary 3.21 under the assumption that $\lambda_0 I - T$ is semi-Fredholm.

Theorem 3.82. *For a bounded operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) *λ_0 is a pole of the resolvent of T;*

(ii) *There exists $p \in \mathbb{N}$ such that* $\ker\,(\lambda_0 I - T)^p = H_0(\lambda_0 I - T)$ *and* $(\lambda_0 I - T)^p(X) = K(\lambda_0 I - T)$.

Proof Suppose that $\lambda_0 \in \sigma(T)$ is a pole of the resolvent of T. Then $p(\lambda_0 I - T)$ and $q(\lambda_0 I - T)$ are finite and hence equal, see Remark 3.7 and Theorem 3.3. Moreover, if $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ then $P_0(X) = \ker\,(\lambda I - T)^p$ and $\ker P_0 = (\lambda I - T)^p(X)$, where P_0 is the spectral projection associated with $\{\lambda_0\}$, so the assertion (ii) is true by Theorem 3.74.

Conversely, assume that (ii) is verified. We show that $p(\lambda_0 I - T)$ and $q(\lambda_0 I - T)$ are finite. From

$$\ker\,(\lambda_0 I - T)^{p+1} \subseteq H_0(\lambda_0 I - T) = \ker\,(\lambda_0 I - T)^p$$

we obtain that $\ker\,(\lambda_0 I - T)^{p+1} = \ker\,(\lambda_0 I - T)^p$, thus $p(T) \leq p$.

From the inclusion

$$(\lambda_0 I - T)^{p+1}(X) \supseteq (\lambda_0 I - T)^\infty(X) \supseteq K(\lambda_0 I - T) = (\lambda_0 I - T)^p(X)$$

we then conclude that $(\lambda_0 I - T)^{p+1}(X) = (\lambda_0 I - T)^p(X)$, thus also $q(T)$ is finite. Therefore λ_0 is a pole of $R(\lambda, T)$. ∎

Recall that $T \in L(X)$ is *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. The next result characterizes the operators having the Kato type of spectrum empty.

Theorem 3.83. *Let $T \in L(X)$, where X is a Banach space. Then the following statements are equivalent:*

(i) *$\sigma_{\rm kt}(T)$ is empty;*

(ii) *$\lambda I - T$ has finite descent for every $\lambda \in \mathbb{C}$;*

(iii) *$\lambda I - T$ has finite descent for every $\lambda \in \partial\sigma(T)$, where $\partial\sigma(T)$ is the topological boundary of $\sigma(T)$;*

(iv) *$\sigma(T)$ is a finite set of poles of $R(\lambda I, T)$;*

(v) *T is algebraic.*

Proof (i) $\Rightarrow$ (ii) Suppose that $\sigma_{\rm kt}(T) = \varnothing$. Then $\rho_{\rm kt}(T)$ has an unique component $\Omega = \mathbb{C}$ and therefore by Theorem 3.34 T has the SVEP at every point of $\mathbb{C}$ since T has the SVEP at the point of the resolvent $\rho(T)$.

On the other hand, if $\lambda I - T$ is of Kato type then also $\lambda I^\star - T^\star$ is of Kato type. Therefore $\mathbb{C} = \rho_{\rm kt}(T) = \rho_{\rm kt}(T^\star)$, and consequently by Theorem 3.35 also $T^\star$ has the SVEP. Since $\lambda I - T$ is of Kato type by Theorem 3.17 we then conclude that $q(\lambda I - T) < \infty$ for every $\lambda \in \mathbb{C}$.

(ii)$\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Since T has the SVEP at every $\lambda \in \partial\sigma(T)$ then the condition $q(\lambda I - T) < \infty$ entails that every $\lambda \in \partial\sigma(T)$ is a pole of $R(\lambda, T)$, by Theorem 3.81, and hence an isolated point of $\sigma(T)$. Clearly this implies that $\sigma(T) = \partial\sigma(T)$, so $\sigma(T)$ is a finite set of poles.

(iv) $\Rightarrow$ (i) It suffices to prove that $\lambda I - T$ is of Kato type for all $\lambda \in \sigma(T)$. Suppose that $\sigma(T)$ is a finite set of poles of $R(\lambda, T)$. If $\lambda \in \sigma(T)$ let P be the spectral projection associated with the singleton $\{\lambda\}$. Then $X = M \oplus N$, where $M := K(\lambda I - T) = \ker P$ and $N := H_0(\lambda I - T)$, by Theorem 3.74. Since $I - T$ has positive finite ascent and descent, if $p := p(\lambda_0 I - T) = q(\lambda I - T)$ then $N = \ker(\lambda I - T)^p$, see Remark 3.7, part (b). From the classical Riesz functional calculus we know that $\sigma(T|M) = \sigma(T) \setminus \{\lambda\}$, so that $(\lambda I - T)|M$ is bijective, whilst $(\lambda I - T|N)^p = 0$. Therefore $\lambda I - T$ is of Kato type for every $\lambda \in \mathbb{C}$.

(iv) $\Rightarrow$ (v) Assume that $\sigma(T)$ is a finite set of poles $\{\lambda_1, \dots, \lambda_n\}$, where for every $i = 1, \dots, n$ with p_i we denote the order of λ_i. Let $h(\lambda) := (\lambda_1 - \lambda)^{p_1} \dots (\lambda_n - \lambda)^{p_n}$. Then by Lemma 1.76

$$h(T)(X) = \bigcap_{i=1}^{n} (\lambda_i I - T)^{p_i}(X) = \bigcap_{i=1}^{n} K(\lambda_i I - T),$$

where the last equality follows since T has SVEP and $\lambda_i I - T$ is of Kato type, see Theorem 3.17. But the last intersection is $\{0\}$, because if $x \in K(\lambda_i I - T) \cap K(\lambda_j I - T)$, with $\lambda_i \neq \lambda_j$, then $\sigma_T(x) \subseteq \{\lambda_i\} \cap \{\lambda_j\} = \varnothing$ and hence $x = 0$, since T has the SVEP. Therefore $h(T) = 0$.

(v) $\Rightarrow$ (i) Let h be a polynomial such that $h(T) = 0$. From the spectral mapping theorem we easily deduce that $\sigma(T)$ is a finite set $\{\lambda_1, \dots, \lambda_n\}$. The points $\lambda_1, \dots, \lambda_n$ are zeros of finite multiplicities of h, say $k_1, \cdots, k_n$, respectively, so that $h(\lambda) = (\lambda_1 - \lambda)^{k_1} \dots (\lambda_n - \lambda)^{k_n}$ and hence by Lemma 1.76

$$X = \ker h(T) = \bigoplus_{i=1}^{n} \ker(\lambda_i I - T)^{k_i}.$$

Now suppose that $\lambda = \lambda_j$ for some j and define

$$h_0(\lambda) := \prod_{i \neq j} (\lambda_i - \lambda)^{k_i}.$$

We have

$$M := \ker h_0(T) = \bigoplus_{i \neq j} \ker(\lambda_i I - T)^{k_i},$$

and if $N := \ker(\lambda_j I - T)^{k_j}$ then $X = M \oplus N$ and M, N are invariant under $\lambda_j I - T$. From the inclusion $\ker(\lambda_j I - T) \subseteq \ker(\lambda_j I - T)^{k_j} = N$ we infer that the restriction of $\lambda_j I - T$ on M is injective. It is easily seen that

$$(\lambda_j I - T)(\ker(\lambda_i I - T)^{k_i}) = \ker(\lambda_i I - T)^{k_i}, \quad i \neq j,$$

so $(\lambda_j I - T)(M) = M$. Hence the restriction of $\lambda_j I - T$ on M is also surjective and therefore bijective. Obviously $(\lambda_j I - T)|N)^{k_j} = 0$, so $\lambda I - T$ is of Kato type also at the points of the spectrum, and the proof is complete. ∎

8. Weyl's theorem

In 1909 H. Weyl [**325**] studied the spectra of all compact perturbations $T + K$ of a Hermitian operator T acting on a Hilbert space and showed that $\lambda \in \mathbb{C}$ belongs to $\sigma_{\mathrm{w}}(T)$ precisely when λ is not an isolated point of finite multiplicity in $\sigma(T)$. Today this classical result is known as Weyl's theorem, and it has been extended to several classes of operators acting in Banach spaces. In this section Weyl's theorem will be related to the single-valued extension property. We shall emphasize the role of the quasi-nilpotent part $H_0(\lambda I - T)$ and shall see that the reason for which Weyl's theorem holds for many classes of operators essentially depends upon the form which the subspaces $H_0(\lambda I - T)$, $\lambda \in \mathbb{C}$, assume.

The first result of this section establishes several equivalences for bounded operators defined on Banach spaces. For a bounded operator $T \in L(X)$ we set

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\},$$

and

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

By part (b) of Remark 3.7 we have

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \tag{99}$$

Recall that the reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}.$$

Theorem 3.84. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $\pi_{00}(T) = p_{00}(T)$;

(ii) $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \varnothing$;

(iii) $\sigma_{\mathrm{sf}}(T) \cap \pi_{00}(T) = \varnothing$;

(iv) $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi_{00}(T)$;

(v) $H_0(\lambda I - T)$ *is finite-dimensional for all* $\lambda \in \pi_{00}(T)$;

(vi) $K(\lambda I - T)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;

(vii) $(\lambda I - T)^\infty(X)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;

(viii) $\beta(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$;

(ix) $q(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$;

(x) *The mapping* $\lambda \to \gamma(\lambda I - T)$ *is not continuous at each* $\lambda_0 \in \pi_{00}(T)$.

Proof (i) $\Rightarrow$ (ii) Evidently $p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, so

$$p_{00}(T) \cap \sigma_{\mathrm{b}}(T) = \pi_{00}(T) = \varnothing,$$

and since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ this implies that $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \varnothing$.

(ii) $\Rightarrow$ (iii) Obvious, since $\sigma_{\mathrm{sf}}(T) \subseteq \sigma_{\mathrm{w}}(T)$.

(iii) $\Rightarrow$ (iv) If $\lambda \in \pi_{00}(T)$ then $\lambda I - T$ is semi-Fredholm, so $(\lambda I - T)(X)$ is closed.

(iv) $\Rightarrow$ (v) If $(\lambda I - T)(X)$ is closed for all $\lambda \in \pi_{00}(T)$ then $\lambda_0 I - T \in \Phi_+(X)$. Since T has the SVEP at every isolated point of $\sigma(T)$, by Theorem 3.18 it follows that $H_0(\lambda I - T)$ is finite-dimensional.

(v) $\Rightarrow$ (vi) If λ is an isolated point of $\sigma(T)$ then the decomposition $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$ holds by Theorem 3.74. Consequently, if $H_0(\lambda I - T)$ is finite-dimensional then $K(\lambda I - T)$ is finite-codimensional.

(vi) $\Rightarrow$ (vii) Immediate, since $K(\lambda I - T) \subseteq (\lambda I - T)^\infty(X)$ for every $\lambda \in \mathbb{C}$.

(vii) $\Rightarrow$ (viii) It is obvious since $(\lambda I - T)^\infty(X) \subseteq (\lambda I - T)(X)$ for every $\lambda \in \mathbb{C}$.

(viii) $\Rightarrow$ (i) For every $\lambda \in \pi_{00}(T)$ we have $\alpha(\lambda I - T) < \infty$, so if $\beta(\lambda I - T) < \infty$ then $\lambda I - T \in \Phi(X)$. Since λ is an isolated point of $\sigma(T)$ the SVEP of T and T^* at λ ensures that $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite, by Theorem 3.16 and Theorem 3.17. Hence $\pi_{00}(T) \subseteq p_{00}(T)$ and since the opposite inclusion is satisfied by every operator it then follows that $\pi_{00}(T) = p_{00}(T)$.

(i) $\Rightarrow$ (ix) Clear.

(ix) $\Rightarrow$ (viii) This is immediate. In fact, by Theorem 3.4 if $q(\lambda I - T) < \infty$ then $\beta(\lambda I - T) \leq \alpha(\lambda I - T) < \infty$ for all $\lambda \in \pi_{00}(T)$.

(iv) $\Leftrightarrow$ (x) Observe first that if $\lambda_0 \in \pi_{00}(T)$ then there exists a punctured open disc $\mathbb{D}_0$ centered at λ_0 such that

$$\gamma(\lambda I - T) \leq |\lambda - \lambda_0| \quad \text{for all } \lambda \in \mathbb{D}_0. \tag{100}$$

In fact, if λ_0 is isolated in $\sigma(T)$ then $\lambda I - T$ is invertible, and hence has closed range in an open punctured disc $\mathbb{D}$ centered at λ_0. Take $0 \neq x \in \ker(\lambda_0 I - T)$. Then

$$\begin{aligned}\gamma(\lambda I - T) &\leq \frac{\|(\lambda I - T)x\|}{\text{dist }(x, \ker(\lambda I - T))} = \frac{\|(\lambda I - T)x\|}{\|x\|} \\ &= \frac{\|(\lambda I - T)x - (\lambda_0 I - T)x\|}{\|x\|} = |\lambda - \lambda_0|.\end{aligned}$$

Clearly, from the estimate (100) it follows that $\gamma(\lambda I - T) \to 0$ as $\lambda \to \lambda_0$, so the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at a point $\lambda_0 \in \pi_{00}(T)$ precisely when $\gamma(\lambda_0 I - T) > 0$, or, equivalenty, by Theorem 1.13 when $(\lambda_0 I - T)(X)$ is closed. Therefore the condition (iv) is equivalent to the condition (x). ■

Following Coburn [**81**] we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T).$$

Note that

$$\text{Weyl's theorem } \Rightarrow \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T).$$

Indeed, if $\lambda \in \sigma(T)$ does not belong to $\sigma_{\mathrm{w}}(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$, and hence λ is an isolated point of $\sigma(T)$. Since $\lambda I - T \in \Phi(X)$ the SVEP for T and T^* at λ ensures that $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite, by Theorem 3.16 and Theorem 3.17. Therefore, $\lambda \notin \sigma_{\mathrm{b}}(T)$ so $\sigma_{\mathrm{b}}(T) \subseteq \sigma_{\mathrm{w}}(T)$. The reverse inclusion holds for every $T \in L(X)$, and hence $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$.

Theorem 3.85. *Suppose that $T \in L(X)$ or T^* has the SVEP. Then Weyl's theorem holds for T if and only if one of the equivalent conditions* (i)–(x) *of Theorem 3.84 holds. If both T and T^* have the SVEP then Weyl's theorem holds for T if and only if $\sigma_{\mathrm{f}}(T) \cap \pi_{00}(T) = \varnothing$.*

Proof By Corollary 3.53 if T or T^* has the SVEP then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. If Weyl's theorem holds for T then

$$\pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T),$$

so the condition (i) of Theorem 3.84 is satisfied. Conversely, suppose that $\pi_{00}(T) = p_{00}(T)$. Then

$$\pi_{00}(T) = p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T).$$

Finally, if both T and T^* have the SVEP by Corollary 3.53 we know that $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{w}}(T)$. ■

Example 3.86. In general, we cannot expect that Weyl's theorem holds for an operator satisfying the SVEP. For instance, if $T \in L(\ell^2)(\mathbb{N})$ is defined by

$$T(x_0, x_1, \dots) := (\frac{1}{2}x_1, \frac{1}{3}x_2, \dots) \quad \text{for all } (x_n) \in \ell^2,$$

then T is quasi-nilpotent and hence has the SVEP. But T does not satisfy Weyl's theorem, since $\sigma(T) = \sigma_{\rm w}(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$.

Definition 3.87. *$T \in L(X)$, X a Banach space, is said to be* relatively regular *if there exists an operator $S \in L(X)$ for which*

$$T = TST \quad \textit{and} \quad STS = S.$$

There is no loss of generality if we require in the definition above only $T = TST$. In fact, if $T = TST$ holds then the operator $S' := STS$ will satisfy both the equalities

$$T = TS'T \quad \text{and} \quad S' = S'TS'.$$

We now establish a basic result.

Theorem 3.88. *A bounded operator $T \in L(X)$ is relatively regular if and only if $\ker T$ and $T(X)$ are complemented*

Proof If $T = TST$ and $STS = S$ then $P := TS$ and $Q := ST$ are idempotents, hence projections. Indeed

$$(TS)^2 = TSTS = TS \quad \text{and} \quad (ST)^2 = STST = ST.$$

Moreover, from the inclusions

$$T(X) = (TST)(X) \subseteq (TS)(X) \subseteq T(X),$$

and

$$\ker T \subseteq \ker(ST) \subseteq \ker(STS) = \ker T,$$

we obtain $P(X) = T(X)$ and $\ker Q = (I - Q)(X) = \ker T$.

Conversely, suppose that $\ker T$ and $T(X)$ are complemented. Write $X = \ker T \oplus U$ and $X = T(X) \oplus V$ and let us denote by P the projection of X onto $\ker T$ along U and by Q_0 the projection of Y onto $T(X)$ along V. Define $T_0 : U \to T(X)$ by $T_0 x = Tx$ for all $x \in X$. Clearly T_0 is bijective. Put $S := {T_0}^{-1} Q_0$. If we represent an arbitrary $x \in X$ in the form $x = y + z$, with $y \in \ker T$ and $z \in U$, we obtain

$$\begin{aligned} STx &= {T_0}^{-1} Q_0 T(y+z) = {T_0}^{-1} Q_0 Tz \\ &= {T_0}^{-1} Tz = z = x - y = x - Px. \end{aligned}$$

Similarly one obtains $TS = Q_0$. If $Q := I - Q_0$ then

$$ST = I_X - P \quad \text{and} \quad TS = I_Q. \tag{101}$$

If we multiply the first equation in (101) from the left by T we obtain $TST = T$, and analogously multiplying the second equation in (101) from the left by S we obtain $STS = S$. ∎

A bounded operator $T \in L(X)$ is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. A bounded operator $T \in L(X)$ is said to be *reguloid* if for every isolated point λ of $\sigma(T)$ the operator $\lambda I - T$ is relatively regular. Note that if T is reguloid then T is isoloid. To see this, suppose that T is reguloid and $\lambda \in \operatorname{iso} \sigma(T)$. If $\alpha(\lambda I - T) = 0$ then

$\lambda I - T \in \Phi_+(X)$, since $(\lambda I - T)(X)$ is closed. But T^* has the SVEP at λ, so by Theorem 3.17 we know that $q(\lambda I - T) < \infty$, and consequently by Theorem 3.3 $q(\lambda I - T) = p(\lambda I - T) = 0$. This implies that $\lambda \notin \sigma(T)$, a contradiction. Hence λ is an eigenvalue of T.

Lemma 3.89. *For every $T \in L(X)$, X a Banach space, and $f \in \mathcal{H}(\sigma(T))$ we have*

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) \subseteq f\left(\sigma(T) \setminus \pi_{00}(T)\right). \tag{102}$$

Furthermore, if T is isoloid then

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f\left(\sigma(T) \setminus \pi_{00}(T)\right). \tag{103}$$

Proof To show the inclusion (102) suppose that $\lambda_0 \in \sigma(f(T)) \setminus \pi_{00}(f(T))$. We distinguish two cases:

Case I. λ_0 is not an isolated point of $f(\sigma(T))$. In this case there exists a sequence $(\lambda_n) \subseteq f(\sigma(T))$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$. Since $f(\sigma(T)) = \sigma(f(T))$ there exists a sequence (μ_n) in $\sigma(T)$ such that $f(\mu_n) = \lambda_n \to \lambda_0$. The sequence (μ_n) contains a convergent subsequence and we may assume that $\lim_{n\to\infty} \mu_n = \mu_0$. Hence $\lambda_0 = \lim_{n\to\infty} f(\mu_n) = f(\mu_0)$. Since $\mu_0 \in \sigma(T) \setminus \pi_{00}(T)$ it then follows that $\lambda_0 \in f(\sigma(T) \setminus \pi_{00}(T))$.

Case II. λ_0 is an isolated point of $\sigma(f(T))$, so either λ_0 is not an eigenvalue of $f(T)$ or it is an eigenvalue for which $\alpha(\lambda_0 - f(T)) = \infty$. Put $g(\lambda) := \lambda_0 - f(\lambda)$. The function $g(\lambda)$ is analytic and admits only a finite number of zeros in $\sigma(T)$, say $\{\lambda_1, \dots, \lambda_k\}$. Write

$$g(\lambda) = p(\lambda)h(\lambda) \quad \text{with} \quad p(\lambda) := \prod_{i=1}^{k} (\lambda_i - \lambda)^{n_i},$$

where n_i is the multiplicity of λ_i for every $i = 1, \dots, k$. Clearly, $\lambda_0 I - f(T) = g(T) = p(T)h(T)$ and $h(T)$ is invertible.

Now, suppose that λ_0 is not an eigenvalue of $f(T)$. Then none of $\lambda_1, \dots, \lambda_k$ can be an eigenvalue of T, and hence $\lambda_0 \in f(\sigma(T) \setminus \pi_{00}(T))$.

Consider the other possibility, i.e., λ_0 is an eigenvalue of T of infinite multiplicity. Then at least one of $\lambda_1, \dots, \lambda_k$, say λ_1, is an eigenvalue of T of infinite multiplicity. Consequently $\lambda_1 \in \sigma(T) \setminus \pi_{00}(T)$ and $f(\lambda_1) = \lambda_0$, so $\lambda_0 \in f(\sigma(T) \setminus \pi_{00}(T))$. This concludes the proof of the inclusion (102).

To prove the equality (103) suppose that T is isoloid. We need only to prove the inclusion

$$f(\sigma(T) \setminus \pi_{00}(T)) \subseteq \sigma(f(T)) \setminus \pi_{00}(f(T)). \tag{104}$$

Let $\lambda_0 \in f(\sigma(T) \setminus \pi_{00}(T))$. From the equality $f(\sigma(T)) = \sigma(f(T))$ we know that $\lambda_0 \in \sigma(f(T))$. If possible let $\lambda_0 \in \pi_{00}(f(T))$, in particular, λ_0 is an isolated point of $\sigma(f(T))$. As above we can write $\lambda_0 I - f(T) = p(T)h(T)$, with

$$p(T) = \prod_{i=1}^{k} (\lambda_i I - T)^{n_i} \tag{105}$$

From the equality (105) it follows that any of $\lambda_1, \dots, \lambda_k$ must be an isolated point of $\sigma(T)$ and hence an eigenvalue of T, since by assumption T is isoloid. Moreover, since λ_0 is an eigenvalue of finite multiplicity any λ_i must also be an eigenvalue of finite multiplicity, and hence $\lambda_i \in \pi_{00}(T)$. This contradicts that $\lambda_0 \in f(\sigma(T) \setminus \pi_{00}(T))$. Therefore, $\lambda_0 \notin \pi_{00}(f(T)$, so the proof of the equality (103)is complete. ■

By Theorem 3.85 every reguloid operator T for which T or T^* has the SVEP obeys Weyl's theorem since the condition (iv) of Theorem 3.84 is satisfied. The next result shows that we have much more.

Theorem 3.90. *Suppose that $T \in L(X)$ is reguloid and let $f \in \mathcal{H}(\sigma(T))$. If either T or T^* has the SVEP then Weyl's theorem holds for $f(T)$.*

Proof Suppose that T is reguloid. Since T is isoloid by Lemma 3.89 we have $\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T))$. If either T or T^* has the SVEP then $f(\sigma_{\rm w}(T)) = \sigma_{\rm w}(f(T))$ by Corollary 3.72. As observed above the SVEP for T or T^* entails that Weyl's theorem holds for T, so $\sigma(T) \setminus \pi_{00}(T) = \sigma_{\rm w}(T)$. Hence

$$\sigma(f(T)) \setminus \sigma_{\rm w}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_{\rm w}(T)) = \sigma_{\rm w}(f(T)),$$

from which we see that Weyl's theorem holds for $f(T)$. ■

The condition (v) of Theorem 3.84 generally does not ensure that T has SVEP. However, if we let

$$\pi_{0f}(T) := \{\lambda \in \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$$

we have the following result.

Theorem 3.91. *If $T \in L(X)$ then the following statements hold:*

(i) *If $H_0(\lambda I - T)$ is closed for every $\lambda \in \sigma_{0f}(T)$ then $\sigma_{\rm b}(T) = \sigma_{\rm w}(T)$;*

(ii) *If $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \sigma_{0f}(T)$ then Weyl's theorem holds for T.*

Proof (i) Let $\lambda \in \sigma(T) \setminus \sigma_{\rm w}(T)$. Clearly $\lambda \in \pi_{0f}(T)$, and hence by assumption $H_0(\lambda I - T)$ is closed. Since $\lambda I - T$ is Weyl from Theorem 3.16 it follows that $p(\lambda I - T) < \infty$ and this implies, since $\alpha(\lambda I - T) = \beta(\lambda I - T)$, that $q(\lambda I - T) < \infty$. Therefore $\lambda \notin \sigma_{\rm b}(T)$, and consequently $\sigma_{\rm b}(T) \subseteq \sigma_{\rm w}(T)$. Since the opposite inclusion is verified for every operator we then conclude that $\sigma_{\rm b}(T) = \sigma_{\rm w}(T)$.

(ii) We have $\pi_{00}(T) \subseteq \pi_{0f}(T)$, so $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \pi_{00}(T)$. By Theorem 3.84 it then follows that $\pi_{00}(T) = p_{00}(T)$, and hence from the inclusion $\sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T)$ we obtain that $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\rm b}(T) \subseteq \sigma(T) \setminus \sigma_{\rm w}(T)$.

Conversely, to show the inclusion $\sigma(T) \setminus \sigma_{\rm w}(T) \subseteq \pi_{00}(T)$ assume that $\lambda \in \sigma(T) \setminus \sigma_{\rm w}(T)$. Then $0 < \alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \in \sigma_{0f}(T)$. Since $H_0(\lambda I - T)$ is finite-dimensional then, by Theorem 3.16, $p(\lambda I - T) < \infty$. The equality $\alpha(\lambda I - T) = \beta(\lambda I - T)$ then implies by Theorem 3.4 that

$q(\lambda I - T) < \infty$. Therefore $\lambda \in p_{00}(T) \subseteq \pi_{00}(T)$, and hence the equality $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$ is proved. ∎

The condition (v) of Theorem 3.84, is an useful tool in order to prove that Weyl's theorem holds for several important classes of operators. To see this we need to introduce a new class of operators.

Definition 3.92. *A bounded operator $T \in L(X)$ on a Banach space X is said to have the* property (H) *if*

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \textit{for all } \lambda \in \mathbb{C}.$$

Although the property (H) seems to be rather strong, the class of operators having property (H) is considerably large. In the next chapter we shall show that every multiplier of a semi-simple Banach algebra A has the property (H).

Example 3.93. In the sequel we give some other important classes of operators which satisfy this property.

(a) As observed in Example 3.9 every totally paranormal operator has the property (H), and in particular every hyponormal operator has the property (H). The class of totally paranormal operators includes also subnormal operators and quasi-normal operators. In fact, these operators are hyponormal, see Conway [**85**].

(b) A bounded operator $T \in L(X)$ is said to be *transaloid* if the spectral radius $r(\lambda I - T)$ is equal to $\|\lambda I - T\|$ for every $\lambda \in \mathbb{C}$. It is easy to check that every transaloid operator satisfies the property (H), see Lemma 2.3 and Lemma 2.4 of [**86**].

Recall that operator $S \in L(X)$ is said to be a quasi-affine transform of $T \in L(X)$ if there is $U \in L(X)$ injective with dense range such that $TU = US$. The next result shows that property (H) is preserved by quasi-affine transforms.

Theorem 3.94. *Suppose that $T \in L(X)$ has the property (H) and S is a quasi-affine transform of T. Then S has the property (H).*

Proof Suppose $TU = US$ with U injective, $\lambda \in \mathbb{C}$, and $x \in H_0(\lambda I - S)$. Then

$$\|(\lambda I - T)^n Ux\|^{1/n} = \|U(\lambda I - S)^n x\|^{1/n} \leq \|U\|^{1/n}\|(\lambda I - S)^n x\|^{1/n},$$

from which we obtain that $Ux \in H_0(\lambda I - T) = \ker\,(\lambda I - T)$. Hence

$$U(\lambda I - S)x = (\lambda I - T)Ux = 0,$$

and since U is injective this implies that $(\lambda I - S)x = 0$. Consequently $H_0(\lambda I - S) = \ker(\lambda I - S)$ for all $\lambda \in \mathbb{C}$. ∎

As a consequence of Theorem 3.94 we obtain some other examples of operators which satisfy the property (H). To see this, for any operator $T \in L(H)$, where H is a complex Hilbert space, let us denote by $T = W|T|$

the *polar decomposition* of T. Then $R := |T|^{1/2}W|T|^{1/2}$ is called the *Aluthge transform* of T, see Aluthge [**44**]. If $R = V|R|$ is the polar decomposition of R define

$$\widetilde{T} := |R|^{1/2}V|R|^{1/2}. \tag{106}$$

Example 3.95. An operator $T \in L(H)$ is said to be *log-hyponormal* if T is invertible and satisfies $\log\,(T^*T) \geq \log\,(TT^*)$. If T is log-hyponormal then the operator $\widetilde{T}$ defined in (106) is hyponormal and $T = K\widetilde{T}K^{-1}$, where $K := |R|^{1/2}|T|^{1/2}$, see Tanahashi [**303**] and Chō, Jeon and J.I. Lee [**80**]. Therefore T is similar to a hyponormal operator and consequently satisfies the property (H).

An operator $T \in L(H)$ is said to be be *p-hyponormal*, with $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$. Every invertible p-hyponormal T is quasi-similar to a log-hyponormal operator and consequently has the property (H), see Aluthge [**44**] and Duggal [**98**]).

Theorem 3.96. *Suppose that the operator $T \in L(X)$ has the property (H). Then T has the SVEP and $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$. Furthermore both T and T^* are reguloid.*

Proof From the inclusion

$$\ker(\lambda I - T)^n \subseteq H_0(\lambda I - T) = \ker(\lambda I - T) \quad \text{for all } n \in \mathbb{N}$$

we obtain that $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$. Note T has the SVEP by Theorem 5.4 or by Theorem 2.31.

To show that T is reguloid we prove that every isolated point λ of the spectrum $\sigma(T)$ is a simple pole of the resolvent. There is no loss of generality if we suppose that $\lambda = 0$. Let P denote the spectral projection associated with the spectral set $\{0\}$. By Theorem 3.74 we know that $P(X) = H_0(T) = \ker T$, and hence $TP = 0$. Now, 0 is an isolated in $\sigma(T)$ and hence a non-removable singularity of $(\lambda I - T)^{-1}$. Let us consider the Laurent expansion

$$(\lambda I - T)^{-1} = \sum_{n=1}^{\infty} \frac{P_n}{\lambda^n} + \sum_{n=0}^{\infty} \lambda^n Q_n$$

for every λ such that $0 < |\lambda| < \varepsilon$, with P_n, $Q_n \in L(X)$. Since $P_1 = P$ and $P_n = T^{n-1}P$ for all $n = 1, 2, \ldots$, the equality $TP = 0$ yields $P_n = 0$ for all $n \geq 2$. Hence 0 is a simple pole of the resolvent $(\lambda I - T)^{-1}$. This implies by part (b) of Remark 3.7 that $p(T) = q(T) = 1$ and 0 is an eigenvalue of T. Moreover, by Theorem 3.74 we have $\ker P = K(T) = T(X)$, so $\ker T$ and $T(X)$ are complemented, and consequently T is relatively regular. This shows that T is reguloid.

To show that T^* is reguloid let λ_0 be an isolated point of $\sigma(T^*) = \sigma(T)$. From the first part of the proof we know that λ_0 is a simple pole of $(\lambda I - T)^{-1}$. As above we have $X = \ker(\lambda_0 I - T) \oplus (\lambda_0 I - T)(X)$, and hence

$$X^* = \ker(\lambda_0 I - T)^{\perp} \oplus (\lambda_0 I - T)(X)^{\perp}.$$

On the other hand, $(\lambda_0 I - T)(X)$ is closed implies that $(\lambda_0 I^* - T^*)(X^*)$ is also closed, and from the equalities

$$(\lambda_0 I^* - T^*)(X^*) = \ker(\lambda_0 I - T)^\perp, \quad \ker(\lambda_0 I^* - T^*) = (\lambda_0 I - T)(X)^\perp$$

we may conclude that both $(\lambda_0 I^* - T^*)(X^*)$ and $\ker(\lambda_0 I^* - T^*)$ are complemented. Therefore $\lambda_0 I^* - T^*$ is relatively regular and hence T^* is reguloid. ∎

By Theorem 3.85 it is immediate that every operator T having the property (H) obeys Weyl's theorem. As a consequence of Theorem 3.96 and Theorem 3.90 we readily obtain the following result.

Corollary 3.97. *If $T \in L(X)$ has the property (H) and $f \in \mathcal{H}(\sigma(T))$ then Weyl's theorem holds for $f(T)$.* ∎

We have already observed that every invertible p-hyponormal T has the property (H), so Weyl's theorem holds for these operators. Actually, every p-hyponormal operator satisfies Weyl's theorem. To prove this result we first introduce a new property formally weaker than the property (H).

Definition 3.98. *A bounded operator satisfies the* property (H_0) *is for every $\lambda \in \mathbb{C}$ there exists an integer $p_\lambda \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^{p_\lambda} \quad \text{for all } \lambda \in \mathbb{C}.$$

The first result shows that the property (H_0) implies Weyl's theorem.

Theorem 3.99. *Let $T \in L(X)$ be a bounded operator on a Banach space X. Then the following statements hold:*

(i) *If T has the property (H_0) then Weyl's theorem holds for T and T^*;*

(ii) *If $T \in L(X)$ has the property (H_0) and S is a quasi-affine transform of T then S has property (H_0);*

(iii) *If $T \in L(X)$ has the property (H_0) and Y is a closed T-invariant subspace of X then $T|Y$ has the property (H_0).*

Proof (i) By Theorem 2.31 the property (H_0) implies the SVEP for T, and by Theorem 3.74 for every $\lambda \in \pi_{00}(T)$ we have $X = \ker(\lambda I - T)^p \oplus K(\lambda I - T)$, from which it follows that $(\lambda I - T)^p(X) = (\lambda I - T)(K(\lambda I - T)) = K(\lambda I - T)$. Consequently, λ is a pole of the resolvent and $q(\lambda I - T)$ is finite for all $\lambda \in \pi_{00}(T)$, so by Theorem 3.85 T satisfies Weyl's theorem.

To show that T^* satisfies Weyl's theorem, assume that $\lambda \in \pi_{00}(T^*)$. Evidently, $(\lambda I - T)^p(X)$ is closed, and hence $(\lambda I^* - T^*)^p(X^*)$ is closed. An inductive argument shows that $\ker(\lambda I^* - T^*)^p$ is finite-dimensional, so $(\lambda I^* - T^*)^p$, and hence also $\lambda I^* - T^*$, is semi-Fredholm. By Theorem 3.77 it follows that $H_0(\lambda I^* - T^*)$ is finite-dimensional, or equivalently we have $p(\lambda I^* - T^*) = q(\lambda I^* - T^*) < \infty$, so $\lambda \in p_{00}(T^*)$. Hence $\pi_{00}(T^*) = p_{00}(T^*)$, and by Theorem 3.85 this equality implies that Weyl's theorem holds for T^*.

(ii) The proof is analogous to that of Theorem 3.94.

(iii) If $H_0(\lambda I - T) = \ker(\lambda I - T)^{p_\lambda}$ then

$$H_0((\lambda I - T)|Y) \subseteq \ker(\lambda I - T)^{p_\lambda} \cap Y = \ker((\lambda I - T)|Y)^{p_\lambda},$$

from which we obtain $H_0((\lambda I - T)|Y) = \ker((\lambda I - T)|Y)^{p_\lambda}$. ■

Remark 3.100. It should be noted that Weyl's theorem is not generally inherited by the restriction on invariant subspaces, even if the operator has the SVEP. For instance, let T be the operator defined in Example 3.86 and let S be the unilateral left shift on $\ell^2(\mathbb{N})$. Define U on $X := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $U := T \oplus S$. Clearly T is quasi-nilpotent, and hence has SVEP, S has SVEP and its spectrum is the unit closed disc $\mathbf{D}(0,1)$, see Theorem 2.86. By Theorem 2.9 it follows that also U has the SVEP and, as it easy to see, $\sigma(U) = \mathbf{D}(0,1)$. Therefore $\sigma(U)$ does not have any isolated point, and this implies that Weyl's theorem holds for U. However, the restriction T of U does not satisfy Weyl's theorem.

Lemma 3.101. *Let $T \in L(X)$ and let p be a complex polynomial. If $p(\lambda_0) \neq 0$ then $H(\lambda_0 I - T) \cap \ker p(T)) = \{0\}$. If, additionaly, T has the SVEP then*

$$H_0(p(T)) = H_0(\lambda_1 I - T) \oplus H_0(\lambda_2 I - T) \cdots \oplus H_0(\lambda_n I - T),$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the distinct roots of p.

Proof Suppose that there is a non-zero element $x \in H(\lambda_0 I - T) \cap \ker p(T))$ and set $p(\lambda_0)I - p(T) = q(T)(\lambda_0 I - T)$, where q denotes a polynomial. Then $q(T)(\lambda_0 I - T)x = p(\lambda_0)x$ and hence $q(T)(\lambda_0 I - T)^n x = p(\lambda_0)^n x$. Therefore

$$|p(\lambda_0)| \|x\|^{1/n}\| \leq \|q(T)^n\|^{1/n} \|(\lambda_0 I - T)x\|^{1/n} \quad \text{for all } n \in \mathbb{N}.$$

Since x is a non-zero element of $H(\lambda_0 I - T)$ we then obtain $p(\lambda_0) = 0$, which is a contradiction.

To show the second assertion, let $x \in H_0(p(T))$. By Theorem 2.20 $H_0(p(T)) = \mathcal{X}_{p(T)}(\{0\})$, so there exists an analityc function f such that $x = (\mu I - p(T))f(\mu)$ for all $\mu \in \mathbb{C} \setminus \{0\}$. Hence for $\lambda \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$ we have

$$x = (p(\lambda)I - p(T))f(p(\lambda)) = (\lambda I - T)Q(T, \lambda)f(p(\lambda)),$$

where Q is a polynomial of T and λ. Consequently $\sigma_T(x) \subseteq \{\lambda_1, \dots, \lambda_n\}$, and hence

$$x \in X_T(\{\lambda_1, \dots, \lambda_n\}) = \bigoplus_{i=1}^{n} X_T(\{\lambda_i\}).$$

Since T has the SVEP, Theorem 2.20 also implies that $X_T(\{\lambda_i\}) = H_0(\lambda_i I - T)$ for all $i = 1, \dots, n$, and hence

$$H_0(p(T)) \subseteq \bigoplus_{i=1}^{n} H_0(\lambda_i I - T).$$

The opposite inclusion is clear, since each λ_i is a root of the polynomial p. ∎

Theorem 3.102. *For a bounded operator $T \in L(X)$ the following assertions are equivalent:*

(i) *T has the property (H_0);*

(ii) *$f(T)$ has the property (H_0) for every $f \in \mathcal{H}(\sigma(T))$;*

(iii) *There exists an analytic function h defined in an open neighbourhood $\mathcal{U}$ of $\sigma(T)$, not identically constant in any component of $\mathcal{U}$, such that $h(T)$ has the property (H_0).*

Proof (i) $\Rightarrow$ (ii) Suppose that T has the property (H_0). Let $\mu \in \mathbb{C}$ be arbitrarily given. If $\mu \notin f(\sigma(T)) = f(\sigma(T))$ then $\mu I - f(T)$ is invertible, and hence

$$H_0(\mu I - f(T)) = \ker(\mu I - f(T)) = \{0\}.$$

Therefore we may assume that $\mu \in f(\sigma(T))$. Let $g := f - \mu$ and suppose first that g has only finitely many zeros in $\sigma(T)$. Then $g(\lambda) = p(\lambda)h(\lambda)$, where h is analytic on $\mathcal{U}$ without zeros in $\sigma(T)$, p is a polynomial of the form $p(\lambda) = \Pi_{i=1}^{n}(\lambda_i - \lambda)^{n_i}$, with $\lambda_i \in \sigma(T)$ distinct roots of p. Then $g(T) = p(T)h(T)$ and $h(T)$ is invertible, so by Lemma 3.101

$$H_0(g(T)) = H_0(p(T)) = \bigoplus_{i=1}^{n} H_0(\lambda_i I - T).$$

On the other hand, since T has the property (H_0) we can choose an integer $d \in \mathbb{N}$ such that $H_0(\lambda_i I - T) = \ker(\lambda_i I - T)^d$ for all $i = 1, \dots, n$. Clearly the condition (H_0) entails that $\lambda I - T$ has finite ascent for all $\lambda \in \mathbb{C}$ and consequently

$$H_0(\lambda_i I - T) = \ker(\lambda_i I - T)^{n_i d} \quad \text{for all } i = 1, \dots, n.$$

From this it follows that

$$\begin{aligned} H_0(g(T)) &= \bigoplus_{i=1}^{n} \ker(\lambda_i I - T)^{n_i d} = \ker\,(\Pi_{i=1}^{n}(\lambda_i I - T)^{n_i d} \\ &= \ker\,(p(T))^d = \ker\,(g(T))^d, \end{aligned}$$

from which we conclude that $f(T)$ has the property (H_0).

Suppose now that g has infinitely many zeros on $\sigma(T)$. Then there are two disjoint open subsets $\mathcal{U}_1, \mathcal{U}_2$ such that $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, $g \equiv 0$ on $\mathcal{U}_1$ and g has only finitely many zeros on $\mathcal{U}_2$. It follows that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are two closed disjoint subsets of $\mathbb{C}$ and $\sigma_i \subseteq \mathcal{U}_i$ for $i = 1, 2$. Therefore, if P_1 and P_2 are the spectral projections associated with σ_1 and

σ_2, respectively, then $X = X_1 \oplus X_2$, $X_1 := P_1(X)$, $X_2 := P_2(X)$, $\sigma(T|X_i) = \sigma_i$; in particular $g(T)|X_1 = g(T|X_1) = 0$. Since $T|X_2$ has the property (H_0) by Teorem 3.99, and g has only finitely many zeros in $\sigma(T|X_2)$, the same argument as in the first part of the proof shows that $g(T|X_2)$ has the property (H_0), and consequently $H_0(\mu I - f(T)) = \ker(g(T)^k|X_2)$ for some $k \geq 1$. Finally,

$$\begin{aligned} H_0(\mu I - f(T)) &= H_0(g(T)) = X_1 \oplus \ker(g(T)^k|X_2) \\ &= \ker(g(T)^k) = \ker(\mu I - f(T))^k, \end{aligned}$$

which completes the proof.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Suppose that $\lambda_0 \in \sigma(T)$ and let $\mu := f(\lambda_0)$. Since f is non-constant on the components of $\mathcal{U}$ it follows that $\mu - f(\lambda) = (\lambda_0 - \lambda)^r p(\lambda) g(\lambda)$, where p is a polynomial such that $p(\lambda_0) \neq 0$ and g is an analytic function which does not vanish in $\sigma(T)$. Therefore

$$\mu I - f(T) = (\lambda_0 I - T)^r p(T) g(T)$$

with $g(T)$ invertible. On the other hand, by hypothesis there exists a natural $d \geq 1$ such that $H_0(\mu I - f(T)) = \ker(\mu I - f(T))^d$. Hence

$$H_0(\lambda_0 I - T) \subseteq H_0(\mu I - f(T)) = \ker(\lambda_0 I - T)^{dr} \oplus \ker p(T))^d.$$

From the inclusion $\ker(\lambda_0 I - T)^{dr} \subseteq H_0(\lambda_0 I - T)$ and since $H_0(\lambda_0 I - T) \cap \ker p(T) = \{0\}$ we may then conclude that $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^{dr}$, see Lemma 3.101; so the proof is complete. ∎

The class of operators having the property (H_0) is rather large. Obviously it contains every operator having the property (H). An operator $T \in L(X)$, X a Banach space, is said to be *generalized scalar* if there exists a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$, where $\mathcal{C}^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$, and Z denotes the identity function on $\mathbb{C}$. An operator similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*. The interested reader can find a well organized study of these operators in the Laursen and Neumann book [**214**]. Note that every quasi-nilpotent generalized scalar operator is nilpotent, [**214**, Proposition 1.5.10]. Moreover, every generalized scalar operator possesses Dunford property (C) since it is super-decomposable, see Theorem 1.5.4 of [**214**].

We show now that every subscalar operator T has the property (H_0). By Theorem 3.99 we may assume that T is generalized scalar. Consider a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$. Let $\lambda \in \mathbb{C}$. Since T has the property (C) we know from Theorem 2.77 that T has the SVEP and $H_0(\lambda I - T) = X_T(\{\lambda\})$ is closed. On the other hand, if $f \in \mathcal{C}^\infty(\mathbb{C})$ then $\Psi(f)(H_0(\lambda I - T)) \subseteq H_0(\lambda I - T)$

because $T = \Psi(Z)$ commutes with $\Psi(f)$. Define $\tilde{\Psi} : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ by

$$\tilde{\Psi}(f) = \psi(f)|H_0(\lambda I - T) \quad \text{for every } f \in \mathcal{C}^\infty(\mathbb{C}).$$

Clearly $T|H_0(\lambda I - T)$ is generalized scalar and quasi-nilpotent, so it is nilpotent. Thus there exists $d \geq 1$ for which $H_0(\lambda I - T) = \ker(\lambda I - T)^d$.

Example 3.103. We give some other examples of operators having the property (H_0).

A bounded operator T on a Hilbert space is said to be *M-hyponormal* if there is $M > 0$ such that $TT^* \leq MT^*T$. Every M-hyponormal operator is subscalar, see [**214**, Proposition 2.4.9], so it satisfies the property (H_0).

Every p-hyponormal operator is subscalar, see C. Lin, Y. Ruan and Z. Yan [**219**], so it satisfies the property (H_0). An operator $T \in L(H)$ is said *algebraically p-hyponormal* if there exists a non-constant polynomial q such that $q(T)$ is p-hyponormal. By Theorem 3.102 every algebraically p-hyponormal operator has the property (H_0). Evidently every hyponormal operator is algebraically hyponormal whilst the converse is not true. To see this set $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\ell^2 \oplus \ell^2$. It is easy to check that T^k is not hyponormal for every $k \in \mathbb{N}$, whereas $g(T) = 0$ with $g(\lambda) := (\lambda - 1)^2$.

Theorem 3.104. *Let $T \in L(X)$, X a Banach space, and suppose that there exists an analytic function h defined on an open neighbourhood of $\sigma(T)$, not identically constant in any component of $\mathcal{U}$, such that $h(T)$ has the property (H_0). Then Weyl's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T)$. In particular, if T has the property (H_0) then Weyl's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof By Theorem 3.102 $f(T)$ has the property (H_0) for every $f \in \mathcal{H}(\sigma(T)$. Therefore Weyl's theorem holds for both $f(T)$ and $f(T^*)$ by part (i) of Theorem 3.99. ∎

Define

$$\pi^a_{00}(T) := \{\lambda \in \text{iso}\,\sigma_{\text{ap}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Clearly, for every $T \in L(X)$ we have

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T).$$

Theorem 3.105. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $\sigma_{\text{ub}}(T) \cap \pi^a_{00}(T) = \varnothing$;

(ii) $\sigma_{\text{uf}}(T) \cap \pi^a_{00}(T) = \varnothing$;

(iii) $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi^a_{00}(T)$;

(iv) $H_0(\lambda I - T)$ *is finite-dimensional for all* $\lambda \in \pi_{00}(T)$ *and* $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi^a_{00}(T) \setminus \pi_{00}(T)$;

(v) *$q(\lambda I-T)<\infty$ for all $\lambda\in\pi_{00}(T)$ and $(\lambda I-T)(X)$ is closed for all $\lambda\in\pi^a_{00}(T)\setminus\pi_{00}(T)$;*

(vi) *The mapping $\lambda\to\gamma(\lambda I-T)$ is not continuous at each $\lambda_0\in\pi^a_{00}(T)$.*

Proof (i) $\Rightarrow$ (ii) is clear since $\sigma_{\mathrm{uf}}(T)\subseteq\sigma_{\mathrm{ub}}(T)$.

(ii) $\Rightarrow$ (iii) The implication follows since $\lambda I-T\in\Phi_+(X)$ for every $\lambda\in\pi^a_{00}(T)$.

(iii) $\Rightarrow$ (iv) We have $\pi_{00}(T)\subseteq\pi^a_{00}(T)$ so $\lambda I-T\in\Phi_+(X)$ for each $\lambda\in\pi_{00}(T)$. The SVEP at every $\lambda\in\pi_{00}(T)$ is equivalent by Theorem 3.18 to saying that $H_0(\lambda I-T)$ is finite-dimensional for all $\lambda\in\pi_{00}(T)$. The second assertion is obvious.

(iv) $\Rightarrow$ (v) By Theorem 3.74 we have $X=H_0(\lambda I-T)\oplus K(\lambda I-T)$ for every isolated point $\lambda\in\sigma(T)$, so $K(\lambda I-T)$ has finite-codimension for every $\lambda\in\pi_{00}(T)$. For every $\lambda\in\mathbb{C}$ the inclusion $K(\lambda I-T)\subseteq(\lambda I-T)(X)$ holds, and this obviously implies that $\beta(\lambda I-T)<\infty$. Therefore $\lambda I-T\in\Phi(X)$ for all $\lambda\in\pi_{00}(T)$. Finally, by Theorem 3.17 the SVEP of T^* at λ implies that $q(\lambda I-T)<\infty$ for all $\lambda\in\pi_{00}(T)$.

(v) $\Rightarrow$ (i) Suppose that $\lambda\in\pi^a_{00}(T)\setminus\pi_{00}(T)$. Then $(\lambda I-T)(X)$ is closed, and since $\alpha(\lambda I-T)<\infty$ this implies that $\lambda I-T\in\Phi_+(X)$. By assumption $\sigma_{\mathrm{ap}}(T)$ does not cluster at λ, hence from Theorem 3.23 we obtain $p(\lambda I-T)<\infty$. Therefore $\lambda\notin\sigma_{\mathrm{ub}}(T)$.

Suppose now the other case, $\lambda\in\pi_{00}(T)$. By Theorem 3.4 the condition $q(\lambda I-T)<\infty$ implies that $\beta(\lambda I-T)\le\alpha(\lambda I-T)<\infty$, and hence $\lambda I-T\in\Phi(X)$. Finally, since T has the SVEP at every isolated point of the spectrum, by Theorem 3.16 we may conclude that $p(\lambda I-T)<\infty$, and hence $\lambda\notin\sigma_{\mathrm{ub}}(T)$.

(vi) $\Leftrightarrow$ (iv) The proof is analogous to the proof of the equivalence (iv) $\Leftrightarrow$ (x) established in Theorem 3.84. ∎

Following Rakočević [**273**] we say that *a-Weyl's theorem* holds for $T\in L(X)$ if

$$\pi^a_{00}(T)=\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{wa}}(T).$$

An immediate example of operator which satisfies a-Weyl's theorem is given by any compact operator T on a Banach space having infinite spectrum. In fact, from the Riesz Schauder theory we know that $\sigma_{\mathrm{wa}}(T)=0$, $\sigma_{\mathrm{ap}}(T)=\sigma(T)$, and $\pi^a_{00}(T)=\sigma(T)\setminus\{0\}=\sigma_a(T)\setminus\sigma_{\mathrm{wa}}(T)$.

Theorem 3.106. *Suppose that $T\in L(X)$ satisfies a-Weyl's theorem. Then both Weyl's theorem and a-Browder's theorem hold for T.*

Proof Suppose that a-Weyl's theorem holds for T and let $\lambda\in\sigma(T)\setminus\sigma_{\mathrm{w}}(T)$. From the inclusion $\sigma_{\mathrm{wa}}(T)\subseteq\sigma_{\mathrm{w}}(T)$ it follows that $\lambda\notin\sigma_{\mathrm{wa}}(T)$, and since $\lambda I-T$ is Weyl we also have $0<\alpha(\lambda I-T)<\infty$, otherwise we would have $0=\alpha(\lambda I-T)=\beta(\lambda I-T)$ and hence $\lambda\notin\sigma(T)$. Therefore $\lambda\in\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{wa}}(T)=\pi^a_{00}(T)$. Hence λ is an isolated point of $\sigma_{\mathrm{ap}}(T)$ and this implies that T has the SVEP at λ. By Theorem 3.16 it then follows that

$p(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) = \beta(\lambda I - T)$, by Theorem 3.4 we then deduce that $q(\lambda I - T) < \infty$, and hence λ is an isolated point of $\sigma(T)$, see Remark 3.7, part (b). This shows that $\sigma(T) \setminus \sigma_{\rm w}(T) \subseteq \pi_{00}(T)$.

Conversely, suppose that $\lambda \in \pi_{00}(T)$. Obviously,

$$\lambda \in \pi^a_{00}(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T) \subseteq \sigma(T) \setminus \sigma_{\rm wa}(T).$$

Consequently, $\lambda I - T \in \Phi_+(X)$ with index less than or equal to 0. Since both T and T^* have the SVEP at every isolated point of $\sigma(T)$ then $p(\lambda I - T$ and $q(\lambda I - T$ are finite, again by Theorem 3.16 and Theorem 3.17. Hence by Theorem 3.4 we conclude that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \notin \sigma_{\rm w}(T)$.

This shows the inclusion $\pi_{00}(T) \subseteq \sigma(T) \setminus \sigma_{\rm w}(T)$, from which we conclude that Weyl's theorem holds for T.

To show that a-Browder's theorem holds suppose first that $\lambda \in \operatorname{acc} \sigma_{\rm ap}(T)$. If were $\lambda \notin \sigma_{\rm wa}(T)$ then $\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T) = \lambda \in \pi^a_{00}(T)$, so λ is an isolated point of $\sigma_{\rm ap}(T)$ and this is impossible. Therefore $\operatorname{acc} \sigma_{\rm ap}(T) \subseteq \sigma_{\rm wa}(T)$ and this implies by part (v) of Theorem 3.65 that $\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T)$. ∎

Theorem 3.107. *If T or T^* has the SVEP, then a-Weyl's theorem holds for T if and only if one of the equivalent conditions* (i)–(vi) *of Theorem 3.105 holds.*

Proof Observe first that both $\pi^a_{00}(T)$ and $\sigma_{\rm ub}(T)$ are subsets of $\sigma_{\rm ap}(T)$ and the condition $\sigma_{\rm ub}(T) \cap \pi^a_{00}(T) = \varnothing$ obviuously implies that

$$\pi^a_{00}(T) \subseteq \sigma_{\rm ap}(T) \setminus \sigma_{\rm ub}(T) \subseteq \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T).$$

To prove that a-Weyl's theorem holds for T we need to show the reverse inclusion $\sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T) \subseteq \pi^a_{00}(T)$.

Suppose that T has SVEP and $\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T)$. Then $\lambda \in \sigma_{\rm ap}(T)$ and $\lambda I - T \in \Phi_+(X)$. Clearly, since $(\lambda I - T)(X)$ is closed, $\lambda I - T$ is not injective and hence $0 < \alpha(\lambda I - T) < \infty$. On the other hand, $\lambda I - T \in \Phi_+(X)$ so by Theorem 3.23 the SVEP at λ is equivalent to saying that $\sigma_{\rm ap}(T)$ does not cluster at λ. Hence $\lambda \in \pi^a_{00}(T)$.

Finally, suppose that T^* has SVEP and $\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T)$. The SVEP for T^* implies by Corollary 3.53 and Theorem 3.66 that

$$\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T) = \sigma_{\rm w}(T) = \sigma_{\rm b}(T).$$

Moreover, by Corollary 2.45 we know that $\sigma_{\rm ap}(T) = \sigma(T)$, and hence

$$\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm b}(T) = \sigma(T) \setminus \sigma_{\rm b}(T) = p_{00}(T) \subseteq \pi^a_{00}(T),$$

so the proof is complete. ∎

Theorem 3.108. *If T^* has SVEP then the following statements are equivalent:*

(i) *Weyl's theorem holds for T;*

(ii) *a-Weyl's theorem holds for T.*

Proof We have only to show the implication (i) $\Rightarrow$ (ii). By Corollary 2.45 we know that if T^* has the SVEP then $\sigma_{\rm ap}(T) = \sigma(T)$, so $\pi^a_{00}(T) = \pi_{00}(T)$. On the other hand, by Corollary 3.53 the SVEP for T^* yields the equalities

$$\sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma_{\rm ub}(T) = \sigma_{\rm wa}(T).$$

If Weyl's theorem holds for T then

$$\pi^a_{00}(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_{\rm w}(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm wa}(T),$$

so a-Weyl's theorem holds for T. ■

Corollary 3.109. *Suppose that $T \in L(X)$ is reguloid and let $f \in \mathcal{H}(\sigma(T))$. If T^* has the SVEP then a-Weyl's theorem holds for $f(T)$. In particular, a-Weyl's theorem holds for $f(T)$ whenever T has property (H) and T^* has the SVEP.*

Proof By Theorem 2.40 we know that if T^* has the SVEP then $f(T^*) = f(T)^*$ has SVEP. Since by Theorem 3.90 Weyl's theorem holds for $f(T)$ we then conclude, by Theorem 3.108, that a-Weyl's theorem holds for $f(T)$. The second assertion follows from Corollary 3.97. ■

9. Riesz operators

We now introduce a class of operators T on Banach spaces for which every non-zero spectral point is a pole of $R(\lambda, T)$.

Definition 3.110. *A bounded operator $T \in L(X)$ on a Banach space X is said to be a* Riesz operator *if $\lambda I - T \in \Phi(X)$ for every $\lambda \in \mathbb{C} \setminus \{0\}$.*

The classical Riesz–Schauder theory of compact operators establishes that every compact operator is Riesz, see Heuser [**159**]. Other classes of Riesz operators will be investigated in Chapter 7. Clearly, if $T \in L(X)$ is Riesz then $\sigma_{\rm f}(T) \subseteq \{0\}$ and if X is infinite-dimensional then $\sigma_{\rm f}(T) = \{0\}$ since $\sigma_{\rm f}(T)$ is non-empty.

Theorem 3.111. *For a bounded operator T on a Banach space the following statements are equivalent:*

(i) *T is a Riesz operator;*

(ii) *$\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(iii) *$\lambda I - T \in \mathcal{W}(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(iv) *$\lambda I - T \in \mathcal{B}_+(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(v) *$\lambda I - T \in \mathcal{B}_-(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(vi) *$\lambda I - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(vii) *$\lambda I - T \in \Phi_-(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(viii) *$\lambda I - T$ is essentially semi-regular for all $\lambda \in \mathbb{C} \setminus \{0\}$;*

(ix) *Each spectral point $\lambda \neq 0$ is isolated and the spectral projection associated with $\{\lambda\}$ is finite-dimensional.*

Proof (i) $\Rightarrow$ (ii) If T is a Riesz operator the semi-Fredholm resolvent has a unique component $\mathbb{C} \setminus \{0\}$. From this it follows by Theorem 3.36 that both T, $T^\star$ have SVEP at every $\lambda \neq 0$. Therefore, again by Theorem 3.36, $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \neq 0$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are clear, so (i), (ii), and (iii) are equivalent. The implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (viii), (ii) $\Rightarrow$ (v) $\Rightarrow$ (viii) are evident, so in order to show that all these assertions are actually equivalences we need to show that (viii) $\Rightarrow$ (ii).

(viii) $\Rightarrow$ (ii) Suppose that (viii) holds. Then the Kato type of resolvent $\rho_{\rm k}(T)$ contains $\mathbb{C} \setminus \{0\}$, and hence, since T, $T^\star$ have the SVEP at every $\lambda \in \rho(T)$, both the operators T and $T^\star$ have the SVEP at every $\lambda \neq 0$, by Theorem 3.34 and Theorem 3.35. From Theorem 3.48 we then conclude that $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \neq 0$.

(i) $\Rightarrow$ (ix) As above, T and $T^\star$ have the SVEP at every $\lambda \neq 0$, so by Corollary 3.21 every non-zero spectral point λ is isolated in $\sigma(T)$. From Theorem 3.77 and Theorem 3.74 it then follows that the spectral projection associated with $\{\lambda\}$ is finite-dimensional.

(ix) $\Rightarrow$ (ii) If the spectral projection associated with the spectral set $\{\lambda\}$ is finite-dimensional then $H_0(\lambda I - T)$ is finite-dimensional, so by Theorem 3.77 $\lambda I - T$ is Browder. ∎

Since every non-zero spectral point of a Riesz operator T is isolated, the spectrum $\sigma(T)$ of a Riesz operator $T \in L(X)$ is a finite set or a sequence of eigenvalues which converges to 0. Moreover, since $\lambda I - T \in \mathcal{B}(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$, every spectral point $\lambda \neq 0$ is a pole of $R(\lambda, T)$. Clearly, if X is an infinite-dimensional complex space the spectrum of a Riesz operator T contains at least the point 0. In this case $T \in L(X)$ is a Riesz operator if and only if $\widehat{T} := T + K(X)$ is a quasi-nilpotent element in the Calkin algebra $\widehat{L} := L(X)/K(X)$. This result is a consequence of the Atkinson characterization of Fredholm operators and in the literature is known as the *Ruston characterization* of Riesz operators.

Generally, the sum and the product of Riesz operators $T, S \in L(X)$ need not to be Riesz. However, the next result shows that is true if we assume T and S commutes.

Theorem 3.112. *If $T, S \in L(X)$ on a Banach space X the following statements hold:*

(i) *If T and S are commuting Riesz operators then $T + S$ is a Riesz operator;*

(ii) *If S commutes with the Riesz operator T then the products TS and ST are Riesz operators;*

(iii) *The limit of uniformly convergent sequence of commuting Riesz operators is a Riesz operator;*

(iv) *If T is a Riesz operator and $K \in K(X)$ then $T + K$ is a Riesz operator.*

Proof If T, S commutes the equivalences classes $\widehat{T}, \widehat{S}$ commutes in $\widehat{L}$, so (i), (ii), and (iii) easily follow from Ruston characterization of Riesz operators and from the well known spectral radius formulas

$$r(\widehat{T} + \widehat{S}) \leq r(\widehat{T}) + r(\widehat{S}) \quad \text{and} \quad r(\widehat{T}\widehat{S}) \leq r(\widehat{T})r(\widehat{S}).$$

The assertion (iv) is obvious, by the Ruston characterization of Riesz operators. ∎

It should be noted that in part (i) and part (ii) of Theorem 3.112 the assumption that T and K commute may be relaxed into the weaker assumption that T, S commute modulo $K(X)$, i.e., $TS - ST \in K(X)$.

Theorem 3.113. *Let $T \in L(X)$, where X is a Banach space, and let f be an analytic function on a neighbourhood of $\sigma(T)$.*

(i) *If T is a Riesz operator and $f(0) = 0$ then $f(T)$ is a Riesz operator.*

(ii) *If $f(T)$ is a Riesz operator and $f \in \mathcal{H}(\sigma(T)$ does not vanish on $\sigma(T) \setminus \{0\}$ then T is a Riesz operator. In particular, if T^n is a Riesz operator for some $n \in \mathbb{N}$ then T is a Riesz operator.*

((iii) *If M is a closed T-invariant subspace of a Riesz operator T then the restriction $T|M$ is a Riesz operator.*

Proof (i) Suppose that T is a Riesz operator. Since $f(0) = 0$ there exists an analytic function g on a neighbourhood of $\sigma(T)$ such that $f(\lambda) = \lambda g(\lambda)$. Hence $f(T) = Tg(T)$ and since $T, g(T)$ commute it then follows by part (ii) of Theorem 3.112 that $f(T)$ is a Riesz operator.

(ii) Assume that $f(T)$ is a Riesz operator and f vanishes only at 0. Then there exist an analytic function g on a neighbourhood of $\sigma(T)$ and $n \in \mathbb{N}$ such that $f(\lambda) = \lambda^n g(\lambda)$ holds on the set of definition of f and $g(\lambda) \neq 0$. Hence $f(T) = T^n g(T)$ and $g(T)$ is invertible. The operators $f(T), g(T)^{-1}$ commute, so by part (ii) of Theorem 3.112 $T^n = f(T)g(T)^{-1}$ is a Riesz operator. Hence T^n is quasi-nilpotent modulo $K(X)$ and from this it easily follows that T is quasi-nilpotent modulo $K(X)$. By the Ruston characterization we then conclude that T is a Riesz operator.

(iii) We show first that $(\lambda I - T)(M) = M$ for all $\lambda \in \rho(T)$. The inclusion $(\lambda I - T)(M) \subseteq M$ is clear. Let $|\lambda| > r(T)$. If $R_\lambda := (\lambda I - T)^{-1}$ from the well known representation $R_\lambda = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ it follows that $R_\lambda(M) \subseteq M$. For every $x' \in M^\perp$ and $x \in M$ let us consider the analytic function $\lambda \in \rho(T) \to x'(R_\lambda x)$. This function vanishes outside the spectral disk of T, so since $\rho(T)$ is connected we infer from the identity theorem for analytic functions that $x'(R_\lambda x) = 0$ for all $\lambda \in \rho(T)$. Therefore $R_\lambda x \in M^{\perp\perp} = M$ and consequently $x = (\lambda I - T)R_\lambda x \in (\lambda I - T)(M)$. This shows that $M \subseteq (\lambda I - T)(M)$, and hence $(\lambda I - T)(M) = M$ for all $\lambda \in \rho(T)$.

Now, $\lambda I - T$ is injective for all $\lambda \in \rho(T)$, so if $\widetilde{T} := T|M$ then $\rho(T) \subseteq \rho(\widetilde{T})$

and hence $\sigma(\widetilde{T}) \subseteq \sigma(T)$. Let λ_0 be an isolated spectral point of T, and hence an isolated point of $\sigma(\widetilde{T})$. If P denotes the spectral projection associated with $\{\lambda_0\}$ and T and $\widetilde{P}$ denotes the spectral projection associated with $\{\lambda_0\}$, T and $\tilde{T}$ then, as is easy to verify, $Px = \widetilde{P}x$ for all $x \in M$. Hence $\widetilde{P}$ is the restriction of P to M, so that, since P is finite-dimensional, $\widetilde{P}$ is finite-dimensional. From part (ix) of Theorem 3.111 we then conclude that $\widetilde{T}$ is a Riesz operator. ∎

Corollary 3.114. *A bounded operator T of the complex Banach spaces X is a Riesz operator if and only if T^* is a Riesz operator.*

Proof By definition, if T is a Riesz operator then $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$. Therefore $\lambda I^* - T^* \in \Phi(X^*)$ for all $\lambda \neq 0$, so T^* is Riesz. Conversely, if T^* is a Riesz operator, by what we have just proved the bi-dual T^{**} is also a Riesz operator. Since the restriction of T^{**} to the closed subspace X of X^{**} is T, it follows from part (iii) of Theorem 3.113 that T itself must be a Riesz operator. ∎

If $T \in L(X)$ for any closed T-invariant subspace M of X, let $\tilde{x}$ denote the class rest $x + M$. Define $\widetilde{T}_M : X/M \to X/M$ as

$$\widetilde{T}_M \tilde{x} : \widetilde{T}x \quad \text{for each } x \in X.$$

Evidently $\widetilde{T}_M$ is well-defined. Moreover, $\widetilde{T}_M \in L(X/M)$ since it is the composition $Q_M T$, where Q_M is the canonical quotient map of X onto X/M.

Theorem 3.115. *If $T \in L(X)$ is a Riesz operator and M is a closed T-invariant subspace M of X then $\widetilde{T}_M$ is a Riesz operator.*

Proof By Corollary 3.114 T^* is a Riesz operator. The annihilator $M^\perp$ of M is a closed subspace of X^* invariant under T^*, so by part (iii) of Theorem 3.113 the restriction $T^*|M^\perp$ is a Riesz operator. Now, a standard argument shows that the dual of $\widetilde{T}_M$ may be identified with $T^*|M^\perp$, so by Corollary 3.114 we may conclude that $\widetilde{T}_M$ is a Riesz operator. ∎

Further insight into the classes of Riesz operators will be given in Chapter 7.

10. The spectra of some operators

In this section we shall describe in some concrete cases the various spectra studied before, and in particular, in the case of unilateral weighted right shifts. As in the preceeding chapter we shall first consider bounded operators T on a Banach space X which satisfy the abstract shift condition $T^\infty(X) = \{0\}$, where $T^\infty(X)$ as usual denotes the hyper-range . Recall that this condition entails that $0 \in \sigma(T)$ since T is not surjective.

Theorem 3.116. *Let $T \in L(X)$, where X an infinite-dimensional Banach space, and suppose that $T^\infty(X) = \{0\}$. Then we have:*

(i) $\sigma(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$*;*

(ii) $q(\lambda I - T) = \infty$ *for every* $\lambda \in \sigma(T) \setminus \{0\}$*;*

(iii) T *is nilpotent* $\Leftrightarrow q(T) < \infty$.

Proof (i) By Corollary 3.53 we have $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. We show that $\sigma_{\mathrm{b}}(T) = \sigma(T)$. The inclusion $\sigma_{\mathrm{b}}(T) \subseteq \sigma(T)$ is obvious, so it remains to establish that $\sigma(T) \subseteq \sigma_{\mathrm{b}}(T)$. Observe that if the spectral point $\lambda \in \mathbb{C}$ is not isolated in $\sigma(T)$ then $\lambda \in \sigma_{\mathrm{b}}(T)$.

Suppose first that T is quasi-nilpotent. Then $\sigma_{\mathrm{b}}(T) = \sigma(T) = \{0\}$ since $\sigma_{\mathrm{b}}(T)$ is non-empty whenever X is infinite-dimensional. Suppose that T is not quasi-nilpotent and let $0 \neq \lambda \in \sigma(T)$. Since $\sigma(T)$ is connected, by Theorem 2.82, and $0 \in \sigma(T)$, it follows that λ is not an isolated point in $\sigma(T)$. Hence $\sigma(T) \subseteq \sigma_{\mathrm{b}}(T)$.

(ii) Let $\lambda \in \sigma(T) \setminus \{0\}$ and suppose that $q(\lambda I - T) < \infty$. By Theorem 2.82 we have $p(\lambda I - T) = 0$ for every $0 \neq \lambda$, and hence by Theorem 3.3 $q(\lambda I - T) = p(\lambda I - T) = 0$, which implies $\lambda \in \rho(T)$, a contradiction.

(iii) Clearly the nilpotency of T implies that $q(T) < \infty$. Conversely, if $q := q(T) < \infty$ then $T^q(X) = T^\infty(X) = \{0\}$. ∎

As in the previous chapter we set

$$i(T) := \lim_{n\to\infty} k(T^n)^{1/n},$$

where $k(T)$ denotes the lower bound of T.

Theorem 3.117. *Suppose that $T \in L(X)$, X an infinite-dimensional Banach space, is non-invertible and $i(T) = r(T)$. Then*

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{su}}(T) = \sigma(T) = \mathbf{D}(0, r(T)), \tag{107}$$

and

$$\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{se}}(T) = \sigma_{\mathrm{es}}(T) = \partial\mathbf{D}(0, r(T)). \tag{108}$$

In particular, these equalities hold if $T^\infty(X) = \{0\}$ and $i(T) = r(T)$.

Proof If T is non-invertible and $i(T) = r(T)$ then, as noted in Theorem 2.55, $\sigma(T)$ is the whole closed disc $\mathbf{D}(0, r(T))$ and $\sigma_{\mathrm{ap}}(T)$ is the circle $\partial\mathbf{D}(0, r(T))$. Since T has the SVEP then $\sigma_{\mathrm{su}}(T) = \sigma(T)$ by Corollary 2.45, and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, by Corollary 3.53. Suppose first that $i(T) = r(T) = 0$. Then T is quasi-nilpotent. The equalities (107) and (108) are then trivially satisfied (note that since X is infinite-dimensional, $\sigma_{\mathrm{b}}(T)$ is non-empty and hence is $\{0\}$). Suppose then that $i(T) = r(T) > 0$. Also in this case $\sigma(T) = \sigma_{\mathrm{b}}(T)$, since every non-isolated point of the spectrum lies on $\sigma_{\mathrm{b}}(T)$. Therefore the equalities (107) are proved. The equalities (108) follow by part (i) of Corollary 3.53. ∎

Note that Theorem 3.117 applies to every unilateral weighted right shift T on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, which satisfies the equality $i(T) = r(T)$. In the next result we shall consider a very special situation.

Corollary 3.118. *Suppose that T is an unilateral weighted right shift on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, with weight sequence $(\omega_n)_{n\in\mathbb{N}}$. If*

$$c(T) = \lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n} = 0,$$

then

$$\begin{aligned}\sigma_{\rm su}(T) &= \sigma_{\rm ap}(T) = \sigma_{\rm se}(T) = \sigma_{\rm es}(T) \\ &= \sigma_{\rm sf}(T) = \sigma_{\rm f}(T) = \sigma_{\rm w}(T) \\ &= \sigma_{\rm b}(T) = \sigma(T) = \mathbf{D}(0, r(T)).\end{aligned}$$

Proof If $c(T) = 0$ then both T and T^* have SVEP . The equalities of the spectra then follow by combining part (iii) of Corollary 3.53, Theorem 2.86 and part (i) of Theorem 3.116. ■

Theorem 3.117 also applies to every non-invertible isometry T on a Banach space X since $i(T) = r(T) = 1$. Examples of non-invertible isometries are the *semi-shifts* on Banach spaces. These are defined as the isometries for which the condition $T^\infty(X) = \{0\}$ is satisfied. It should be noted that for Hilbert space operators the semi-shifts coincide with the isometries for which none of the restrictions to a non-trivial reducing subspace is unitary, see Chapter I of Conway [**85**].

An operator $T \in L(X)$ for which the equality $\sigma_T(x) = \sigma(T)$ holds for every $x \neq 0$ is said to have *fat local spectra*, see Neumann [**246**]. Clearly, by Corollary 2.84 an isometry T is a semi-shift if and only if T has fat local spectra.

Examples of semi-shifts are the unilateral right shift operators of arbitrary multiplicity on $\ell^p(\mathbb{N})$, as well as every right translation operator on $L^p([0,\infty))$. In Laursen and Neumann [**214**, Proposition 1.6.9] it shown that if X is the Banach space of all analytic functions on a connected open subset $\mathcal{U}$ of $\mathbb{C}$, f a non-constant analytic function on $\mathcal{U}$, and if $T_f \in L(X)$ denotes the pointwise multiplication operator by f, then the condition $\sigma(T_f) \subseteq \overline{f(\mathcal{U})}$ implies that T_f has local fat spectra. In particular, these conditions are verified by every multiplication operator T_f on the disc algebra $\mathcal{A}(\mathbb{D})$ of all complex-valued functions continuous on the closed unit disc of $\mathbf{C}$ and analytic on the open unit disc $\mathbb{D}$, where $f \in \mathcal{A}(\mathbb{D})$, and the same result holds for the Hardy algebra $H^\infty(\mathbb{D})$. If $f \in H^\infty(\mathbb{D})$ and $1 \leq p < \infty$ the analytic Toeplitz operator on $H^p(\mathbb{D})$ defined by the multiplication by f has also a local fat spectra.

The next result generalizes the result of Corollary 2.83, which corresponds to the case that T is bounded below.

Theorem 3.119. *Let $T \in \Phi_\pm(X)$, X a Banach space, and let Ω denote the connected component of $\rho_{\rm sf}(T)$ which contains 0. Then the following assertions are equivalent:*

(i) $T^\infty(X) = \{0\}$;

(ii) $p := p(T) < \infty$ *and*

$$\Omega \subseteq \bigcap_{x \notin \ker T^p} \sigma_T(x). \tag{109}$$

In this case the following assertions are valid:

(iii) $q(\lambda I - T) = \infty$ *for all* $\lambda \in \sigma(T)$;

(iv) $\beta(\lambda I - T) > 0$ *for all* $\lambda \in \sigma(T)$;

(v) $T \in \Phi_+(X)$ *and* $\mathrm{ind}\,(\lambda I - T) < 0$ *for all* $\lambda \in \Omega$;

(v) $T^\star$ *does not have the SVEP.*

Proof Suppose that $T^\infty(X) = \{0\}$. Then $K(T) = \{0\}$ and T has the SVEP by Theorem 2.82. But $T \in \Phi_\pm(X)$, so by Theorem 3.16 $p := p(T) < \infty$ and $H_0(T) = \mathcal{N}^\infty(T) = \ker T^p$.

Now, by Theorem 3.31 the mapping $\lambda \in \Omega \to H_0(\lambda I - T) + K(\lambda I - T)$ is constant on Ω, so

$$\ker T^p = H_0(T) + K(T) = H_0(\lambda I - T) + K(\lambda I - T)$$

for all $\lambda \in \Omega$. Again, from Theorem 2.82 we have $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$, so by Theorem 2.18

$$\ker T^p = K(\lambda I - T) = \{x \in X : \lambda \notin \sigma_T(x)\}.$$

for all $\lambda \in \Omega \setminus \{0\}$. Thus for $x \notin \ker T^p$ we have $\Omega \setminus \{0\} \subseteq \sigma_T(x)$, and since by Theorem 2.82 $0 \in \sigma_T(x)$ for all $x \neq 0$, we conclude that the inclusion (109) holds.

Conversely, suppose that $p = p(T) < \infty$ and that (109) holds. Then $0 \in \sigma_T(x)$ for every $x \notin \ker T^p$. Suppose that there exists $x \in T^\infty(X)$ with $x \neq 0$. By Theorem 5.4 then $x \notin \ker T^p$, so $0 \in \sigma_T(x)$. But this is impossible since

$$x \in T^\infty(X) = K(T) = \{x \in X : 0 \notin \sigma_T(x)\}.$$

Therefore $T^\infty(X) = \{0\}$.

(iii) By part (iii) of Theorem 3.116 we need only to prove that $q(T) = \infty$. Suppose $q(T) < \infty$. By part (iii) of Theorem 3.116 T is then quasi-nilpotent, and since $p(T) < \infty$ this implies that T is a Fredholm operator, so $\sigma_{\rm f}(T)$ is empty, which is impossible since X is infinite-dimensional.

To show that $\beta(\lambda I - T)$ for all $\lambda \in \sigma(T)$, recall that T is upper semi-Fredholm since the SVEP at λ ensures that $\alpha(T) \leq \beta(T)$, see Corollary 3.19. The SVEP at all the points $\lambda \in \Omega$ also implies, again by Corollary 3.19, that $\mathrm{ind}(\lambda I - T) \leq 0$ for all $\lambda \in \Omega \cap \sigma(T)$.

Finally, assume that $\mathrm{ind}(\mu I - T) = 0$ for some $\mu \in \Omega \cap \sigma(T)$. Then $\mu \notin$

$\sigma_{\mathrm{w}}(T) = \sigma(T)$ by Theorem 3.116; a contradiction. Hence $\operatorname{ind}(\lambda I - T) < 0$ for all $\lambda \in \Omega \cap \sigma(T)$.

(v) This is a consequence of Theorem 3.17, since $q(\lambda I - T) = \infty$ for all $\lambda \in \Omega \cap \sigma(T)$. ■

Clearly, every isometry $T \in L(X)$ on a Banach space satisfies the condition $i(T) = r(T) = 1$, so if $T^\infty(X) = \{0\}$ we have

$$\sigma_T(x) = \sigma(T) = \mathbf{D}(0,1)$$

by Corollary 2.83. Moreover, since an isometry is bounded below, and therefore upper semi-Fredholm, Theorem 3.119 applies to every isometry which satisfies the condition $T^\infty(X) = \{0\}$.

For every semi-Fredholm operator $T \in L(X)$ we have by Theorem 1.70

$$^\perp K(T^\star) = \overline{\mathcal{N}^\infty(T)} \quad \text{and} \quad K(T) = ^\perp \mathcal{N}^\infty(T^\star).$$

These equalities, together with the characterizations of the SVEP at a point established in this chapter concerning semi-Fredholm operators allow us to deduce the following dual result of Theorem 3.119. We shall state it omitting the proof.

Theorem 3.120. *Let $T \in \Phi_\pm(X)$, X a Banach space, and let Ω be the connected component of $\rho_{\mathrm{sf}}(T)$ which contains 0. Then the following assertions are equivalent:*

(i) $\overline{\mathcal{N}^\infty(T)} = X$;

(ii) $q := q(T) < \infty$ *and*

$$\Omega \subseteq \bigcap_{x^\star \notin \ker T^{\star q}} \sigma_{T^\star}(x^\star).$$

In this case the following assertions are valid:

(iii) $(\lambda I - T)(X) = X$ *for all* $\lambda \neq 0$;

(iv) $T^\star$ *has the SVEP ;*

(v) T *does not have the SVEP ;*

(vi) *For every* $x^\star \neq 0$, $0 \in \sigma_{T^\star}(x^\star)$ *and* $\sigma_{T^\star}(x^\star)$ *is connected;*

(vii) $\sigma(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ *is connected;*

(viii) $T \in \Phi_-(X)$ *and* $\operatorname{ind}(\lambda I - T) > 0$ *for all* $\lambda \in \rho_{\mathrm{sf}}(T) \cap \sigma(T)$;

(ix) $p(\lambda I - T) = \infty$ *and* $\alpha(\lambda I - T) > 0$ *for all* $\lambda \in \sigma(T)$.

Theorem 3.54 also applies to the *Césaro operator* C_p defined on the classical Hardy space $H_p(\mathbb{D})$, $\mathbb{D}$ the open unit disc and $1 < p < \infty$, by

$$(C_p f)(\lambda) := \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}\mathrm{d}\mu \quad \text{for all} \quad f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D}.$$

As noted by T. L. Miller, V. G. Miller, and Smith in [**237**], the spectrum of the operator C_p is the closed disc Γ_p centered at $p/2$ with radius $p/2$, and

$$\sigma_{\mathrm{f}}(C_p) \subseteq \sigma_{\mathrm{ap}}(C_p) = \partial\Gamma_p.$$

From Corollary 3.14 we also have $\sigma_{\mathrm{ap}}(C_p) = \sigma_{\mathrm{sf}}(C_p) \subseteq \sigma_{\mathrm{f}}(T)$ and therefore $\sigma_{\mathrm{ap}}(C_p) = \sigma_{\mathrm{sf}}(C_p) = \sigma_{\mathrm{f}}(T) = \partial\Gamma_p$. Moreover, since T has the SVEP by Corollary 3.14 we obtain that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. It should be noted that ${C_p}^\star$ does not have SVEP at every point of the interior of Γ_p. In fact, if $\lambda_0 \in C_p \setminus \partial\Gamma_p$ then $\lambda_0 I - C_p$ is Fredholm and the SVEP of ${C_p}^\star$ at λ_0 would be equivalent to saying that $q(\lambda_0 - C_p) < \infty$, by Theorem 3.17. But $\lambda_0 I - C_p$ is injective, so we would have $q(\lambda_0 - C_p) = q(\lambda_0 - C_p) = 0$ and hence $\lambda_0 \notin \sigma(C_p)$, which is impossible. This argument also shows that $\sigma_{\mathrm{w}}(C_p) = \sigma_{\mathrm{b}}(C_p) = \Gamma_p$.

We shall now consider Weyl's theorem for operators $T \in L(X)$ for which the condition $K(T) = \{0\}$ holds. Observe that Theorem 3.85 works for unilateral weighted left shift on $\ell^p(\mathbb{N})$. In fact, although these operators need not to have the SVEP, the adjoints T^* are left shifts and hence have the SVEP. Moreover, Theorem 3.85 applies also to bilateral weighted shifts on $\ell^p(\mathbb{Z})$, since by Theorem 2.91 at least one of the operators T and T^* has the SVEP.

Theorem 3.121. *Suppose that for a bounded operator $T \in L(X)$ on a Banach space X we have $K(T) = \{0\}$. If T is not quasi-nilpotent and $f \in \mathcal{H}(\sigma(T))$ then $f(T)$ and $f(T^*)$ obey Weyl's theorem.*

Proof By Theorem 2.82 the condition $K(T) = \{0\}$ entails that T has the SVEP. Moreover, see Theorem 3.116 and Theorem 2.82, $\sigma(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$ and $\sigma(T)$ is a connected set containing 0. In particular, $\sigma(T)$ does not have any isolated point because otherwise $\sigma(T) = \{0\}$.

Now, let f be an analytic function defined on an open neighbourhood of $\sigma(T)$. Since the identity operator obeys Weyl's theorem we may assume that f is non-constant. Hence $f(\sigma(T)) = \sigma(f(T)) = \sigma(f(T^*))$ is a connected subset of $\mathbb{C}$ without isolated points and therefore

$$\pi_{00}(f(T)) = \pi_{00}(f(T^*)) = \mathrm{iso}\ \sigma(f(T)) = \varnothing.$$

Moreover, by Corollary 3.72 we have also

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T)) = \sigma_{\mathrm{w}}(f(T^*)).$$

Consequently both $f(T)$ and $f(T^)$ obey Weyl's theorem. ∎

Corollary 3.122. *Let $T \in L(X)$ be a non quasi-nilpotent weighted unilateral right shift T on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with weight $\omega = (\omega_n)_{n\in\mathbb{N}}$. Then T obeys Weyl's theorem. Moreover, if*

$$c(T) := \lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n} = 0,$$

then T obeys a-Weyl's theorem.

Proof The first assertion is clear. If $c(T) = 0$ then T^* has SVEP, see Theorem 2.88, and hence Theorem 3.108 applies to T. ∎

Remark 3.123. The previous theorem cannot, in general, be reversed. In fact, let V be the quasi-nilpotent Volterra operator on the Banach space $X := C[0,1]$, defined in Example 2.35. V is quasi-nilpotent and $K(V) = \{0\}$. Since V is injective we have $\pi_{00}(V) = \varnothing = \sigma(V) \setminus \sigma_{\rm w}(V)$. We note in passing that this argument also shows that if $T \in L(X)$, $K(T) = \{0\}$ and T is injective then T obeys Weyl's theorem. Consequently every weighted right shift with none of the weights $\omega_n = 0$ obeys Weyl's theorem.

10.1. Comments. A relevant part of the material of this chapter is a sample of results of Aiena *et al.* [**31**] [**32**], [**32**], [**16**], [**33**], [**34**] and [**14**]. Some of these results, in the more special case of semi-Fredholm operators, have been found in Finch [**115**], Schmoeger [**295**], and Mbekhta [**229**], see also the book [**17**]. The property that the finiteness of the ascent $p(\lambda_0 I - T)$ implies that T has SVEP at λ_0 was first noted by Finch [**115**], see also Laursen [**201**]. However, Theorem 5.4 in the form stated here has been taken from Aiena and Monsalve [**32**].

The class of paranormal and hyponormal operators has been investigated in the literature by several authors, for instance in Heuser's book [**159**] or Istrătescu̧ [**172**]. A somewhat different notion of paranormal operator may be found in Albrecht [**39**] and Mbekhta [**226**], whilst an extension of the concept of hyponormal operators has been studied, in the framework of local spectral theory by Duggal and Djordjević [**101**], [**99**] and [**100**]. That every totally paranormal operator has the property (C) was first observed by Laursen [**201**], who studied, by using different methods, the class of all operators for which the ascent $p(\lambda I - T)$ is finite for every $\lambda \in \mathbb{C}$.

The material on the operators which satisfy a polynomial growth condition is modeled after Barnes [**61**]. An illuminating discussion of these operators may be found in Laursen and Neumann [**214**], see, in particular, Theorem 1.5.19, where the operators which satisfy a polynomial growth condition are characterized as the generalized scalar operators having a real spectrum. Further information may be found in Colojoară and Foiaş [**83**].

The section on the SVEP for operators of Kato type is modeled after Aiena and Monsalve [**32**], Aiena, Colasante , and González [**16**]. The part concerning the SVEP at the points where the approximate point spectrum, as well as the surjectivity spectrum, does not cluster is taken from Aiena and Rosas [**33**].

Theorem 3.32, owed to Mbekhta and Ouahab [**233**], generalizes to the components of $\rho_{\rm k}(T)$ the results obtained in [**253**] by Ó Searcóid and West, which showed the constancy of the mappings

$$\lambda \to \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X),\ \lambda \to \overline{\mathcal{N}^\infty(\lambda I - T)} \cap (\lambda I - T)^\infty(X)$$

on the connected components of the semi-Fredholm spectrum $\rho_{\rm sf}(T)$.

The work by Ó Searcóid and West [**253**] extended previous results established by Homer [**165**], by Goldman and Kračkovskiǐ [**143**], [**144**], and by Saphar [**284**], which have studied the continuity of the functions $\lambda \to \overline{\mathcal{N}^\infty(\lambda I - T)}$ and $\lambda \to (\lambda I - T)^\infty(X)$ on a connected component of the semi-Fredholm resolvent, except for the discrete subset of points for which $\lambda I - T$ is not semi-regular. In particular, the constancy of the mappings $\lambda \to \overline{\mathcal{N}^\infty(\lambda I - T)} \cap (\lambda I - T)^\infty(X)$ and $\lambda \to \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X)$ as λ ranges through a connected component of the semi-Fredholm resolvent was noted by West [**323**].

The classification of the components of the semi-Fredholm region given in Theorem 3.36 is classical. However, it should be noted that the methods used here, inspired by the work of Aiena and Villafãne, which extend this classification to the other resolvents and involve the SVEP seem to be new.

The classes of semi-Browder operators were introduced by Harte [**147**] and have been investigated in latter years by several authors, see Rakočević [**276**], V. Kordula, Müller, and Rakočević [**192**], Laursen [**204**], Aiena and Carpintero [**14**]. Lemma 3.42 and Theorem 3.43 are taken from Grabiner [**139**]. The equivalences (ii) $\Leftrightarrow$ (iii) of Theorem 3.44 and Theorem 3.46 were proved by Rakočević in [**272**], see also [**275**], whilst the equivalences (i) $\Leftrightarrow$ (ii) of Theorem 3.44 and Theorem 3.46 were observed in Aiena and Carpintero [**14**]. The interested reader may find results on Weyl and Browder spectra in Harte [**146**], Harte and Raubenheimer [**154**], Harte, Lee and Littlejohn [**153**]. The relationship between the Dunford property (C) and the Weyl and Browder spectra for some special classes of operators has also been investigated by Duggal and Djordjević [**101**], [**99**] and [**100**].

The characterizations of the Browder spectrum and the semi-Browder spectra of an operator by means of the compressions are owed to Zemánek [**333**].

The spectral mapping theorems for $\sigma_{\rm uf}(T)$, $\sigma_{\rm lf}(T)$ are modeled after Gramsch and Lay [**141**]. The subsequent material on the spectral mapping theorems for $\sigma_{\rm ub}(T)$, $\sigma_{\rm lb}(T)$ is inspired in part by the work of Rakočević in [**272**] and the work of Oberai [**252**]. The spectral mapping theorem for the Browder spectrum was first proved by Nussbaum [**250**], and successively proved, by using different methods, by Gramsch and Lay [**141**] and Oberai [**252**]. However, the methods used here, which involve the local spectral theory, are taken from Aiena and Biondi [**13**]. The interested reader may find further results on the spectral mapping theorem also in Schmoeger [**294**]. A modern and unifying approach to the axiomatic theory of the spectrum may also found in Kordula and Müller [**190**] and in Mbekhta and Müller [**232**].

Theorem 3.66 and Theorem 3.71 are taken from Aiena and Biondi [**13**], whilst Corollary 3.72 was established by Curto and Han [**86**]. The class of operators for which a-Browder's theorem holds has been studied recently by several authors, see, for instance Harte and Lee [**152**], S. V. Djordjević and Han [**91**], Curto and Han [**86**].

Theorem 3.81 and the subsequent material on isolated points of the spectrum are modeled after Schmoeger [**291**]. The section concerning Weyl's theorem and a-Weil's theorem contains ideas from S. V. Djordjević, I. H. Jeon, E. Ko [**92**], Curto and Han [**86**], Aiena and Carpintero [**15**], Aiena and Villafãne [**35**], Oudghiri [**254**]. Lemma 3.89 is owed to Oberai [**251**]. The class of operators which satisfy Weyl's theorem was introduced by Coburn [**81**] and later studied by Berberian [**65**] and [**66**]. Berberian also showed that every Toeplitz operator obeys Weyl's theorem. These authors extended to some other classes of operators a classical result obtained by Weyl [**325**] for selfadjoint operators on Hilbert spaces. The interested reader may find more recent results in Harte and W. Y. Lee [**152**], Harte, W. Y. Lee and Littlejohn [**153**], Schmoeger [**295**], Barnes [**58**], Han and W. Y. Lee [**145**], Rakočević [**273**], D. S. Djordjević, S. V. Djordjević [**90**], S. H. Lee and W. Y. Lee [**218**], B. P. Duggal, S. V. Djordjević [**101**]. The class of operators which satisfy a-Weyl's theorem was introduced by Rakočević [**273**]. The class of operators having property (H) has been studied by Aiena and Villafãne [**35**]. All the material concerning the property (H_0) is taken from Oudghiri [**254**]. The interested reader may find some generalizations of these results in Duggal and S. V. Djordjević [**102**].

The class of Riesz operators was introduced by Ruston [**283**] and has received a lot of attention in many articles, see Caradus [**79**], West [**321**], Aiena [**1**] [**4**], and standard books on Fredholm theory, see, for instance, Heuser [**159**], Dowson [**93**], Caradus, Pfaffenberger, and Yood [**76**]. However, the methods here adopted for the study of this class of operators by means of the SVEP seems to be new, and, in particular, the characterization (viii) of Theorem 3.111 is taken from Aiena and Mbektha [**29**]. The last section on isometries is a sample of results due to Schmoeger [**295**] and Neumann [**246**], see also Laursen and Mbekhta [**208**], T.L. Miller and V.G. Miller [**235**], Aiena and Biondi [**12**] for further results. Semi-shifts were introduced by Holub [**164**] and studied by Laursen and Vrbová [**216**], Neumann [**246**] which introduced the concept of fat local spectra [**246**].

CHAPTER 4

Multipliers of commutative Banach algebras

A significant sector of the development of spectral theory outside the classical area of Hilbert spaces may be found among multipliers defined on a complex commutative Banach algebra. In fact, the Fredholm theory and the local spectral theory developed in the previous chapters find a significant, and elegant, application to the study of the spectral properties of this class of operators.

Multipliers first appeared in harmonic analysis in connection with the theory of summability for Fourier series, but subsequently this notion has been employed in many other contexts, amongst which we mention the study of the Fourier transform, the investigation of homomorphisms of group algebras, Banach modules, and the general theory of Banach algebras. Our main concern will not be with the investigation of these applications, but we shall address primarily the spectral theory of multipliers of commutative semi-prime Banach algebras, in particular, to the Fredholm and local spectral theory of multipliers.

The concept of multiplier extends that of a multiplication operator on commutative Banach algebras, but one of the reasons of the interest in this class of operators is given by the reason that the concept of multiplier providing an abstract frame for studying another important class of operators which arises from harmonic analysisl, the class of all convolution operators of group algebras. Of course, a thorough study of multipliers of commutative Banach algebras requires the knowledge of the basic tools of the theory of Banach algebras, and, in particular, the machinery of the Gelfand representation theory of commutative Banach algebras.

In the first section we shall develop the elementary properties of multipliers of Banach algebras. For the most part we have restricted our attention to faithful commutative algebras, although some notions are given in the absence of commutativity. In the second section we see shall show that to every multiplier T of a semi-simple commutative Banach algebra A there corresponds an unique bounded continuous function, the *Helgason–Wang function* φ_T defined on the maximal ideal space $\Delta(A)$ of A. The space $\Delta(A)$ may be viewed as an open subset, always with respect to the Gelfand topology, of the maximal ideal space $\Delta(M(A)$ of $M(A)$. The Helgason–Wang function is the restriction of the Gelfand transform $\widehat{T}$ of the element $T \in M(A)$ to $\Delta(A)$ and this property leads, in the third section, to some descriptions of the various parts of the spectrum of multipliers in terms of the range of this function. We shall see that if T is a multiplier of a semi-prime Banach algebra then the ascent $p(\lambda I - T)$ is finite for every $\lambda \in \mathbb{C}$, so T has the SVEP and therefore all the results proved in Chapter 2 and Chapter 3 apply.

The subsequent sections of this chapter address the multiplier theory of some concrete Banach algebras, such as group algebras, Banach algebras with orthogonal basis, commutative C^* Banach algebras, commutative H^* algebras. For these algebras we shall also give a precise description of the ideal in $M(A)$ of all multipliers which are compact operators. These characterizations will be then used in the next Chapter for giving a precise description of multipliers which are Fredholm.

1. Definitions and elementary properties

The purpose of this section is to introduce the basic properties of multipliers of Banach algebras. The material here presented is not exhaustive and we shall limit ourselves essentially to the multiplier theory of faithful commutative Banach algebras.

Let A denote a complex Banach algebra (not necessarily commutative) with or without a unit.

Definition 4.1. *The mapping $T : A \to A$ is said to be a* multiplier *of A if*

$$x(Ty) = (Tx)y \quad \textit{for all } x, y \in A. \tag{110}$$

The set of all multipliers of A is denoted by $M(A)$.

An immediate example of a multiplier of a Banach algebra A is given by the multiplication operator $L_a : x \in A \to ax \in A$ by an element a which commutes with every $x \in A$. In the case in which A is a commutative Banach algebra with unit u the concept of multiplier reduces, trivially, to the multiplication operator by an element of A. To see this, given a multiplier $T \in M(A)$, let us consider the multiplication operator L_{Tu} by the element Tu. For each $x \in A$ we have

$$L_{Tu}x = (Tu)x = u(Tx) = Tx,$$

thus $T = L_{Tu}$. In this case we can identify A with $M(A)$.

Given a nonempty subset B of A, we recall that the *left annihilator* and the *right annihilator* of B are the sets $\mathrm{lan}(B)$, $\mathrm{ran}(B)$ defined by

$$\mathrm{lan}(B) := \{x \in A : xB = \{0\}\},$$

and

$$\mathrm{ran}(B) := \{x \in A : Bx = \{0\}\}.$$

The basic properties of multipliers that we develop are generally presented in the framework of a very large class of Banach algebras.

Definition 4.2. *The Banach algebra A is said to be* faithful *if* $\mathrm{ran}(A) = \{0\}$ *or* $\mathrm{lan}(A) = \{0\}$.

In the definition of faithful Banach algebras we follow the terminology of Johnson [**175**] and of the recent book of Laursen and Neumann [**214**]. Faithful Banach algebras was also called *proper* by Ambrose [**46**] and *without order* in the book of Larsen [**200**], as well as by several other authors. Trivially, any Banach algebra with an identity is faithful. More generally, every Banach algebra A with *approximate units* is faithful, where A is said to have approximate units if for every $x \in A$ and $\varepsilon > 0$ there is an element u for which $\|x - ux\| < \varepsilon$.

We recall that an algebra A is said to be *semi-prime* if $\{0\}$ is the only two-sided ideal J for which $J^2 = \{0\}$. It is easy to check that if J is a left (or right) ideal of a semi-prime algebra A for which $J^2 = \{0\}$ then $J = \{0\}$. In fact, $\mathrm{lan}(A)$ is a bi-ideal of A with

$$[\mathrm{lan}(A)]^2 \subseteq (\mathrm{lan}(A))A = \{0\},$$

and hence $\mathrm{lan}(A) = \{0\}$. Clearly JA is a bi-ideal of A and $(JA)(JA) \subseteq J^2A = \{0\}$. Therefore $JA = \{0\}$ so that $J \subseteq \mathrm{lan}(A) = \{0\}$. This also shows that any semi-prime algebra is faithful.

Note that if A is semi-prime and $xAx = \{0\}$, then $x = 0$. This follows immediately from the equalities $(Ax)^2 = AxAx = \{0\}$, which imply that $Ax = \{0\}$ and hence $x = 0$.

We recall that a left ideal J of a Banach algebra $\mathcal{A}$ is said to be *regular* (or also *modular*) if there exists an element $v \in \mathcal{A}$ such that $\mathcal{A}(1 - v) \subseteq J$, where

$$\mathcal{A}(1 - v) := \{x - xv : x \in \mathcal{A}\}.$$

Similar definitions apply to right regular ideals and regular ideals. It is clear that if $\mathcal{A}$ has a unit u then every ideal, left, right, or two-sided, is regular. A two-sided ideal J of $\mathcal{A}$ is called *primitive* if there exists a maximal regular left ideal L of $\mathcal{A}$ such that

$$J = \{x \in \mathcal{A} : x\mathcal{A} \subseteq L\}.$$

It is well known that J is a primitive ideal of $\mathcal{A}$ if and only if J is the kernel of an irreducible representation of $\mathcal{A}$, see Bonsall and Duncan [**72**, Proposition 24.12].

The (Jacobson) *radical* of an algebra is the intersection of the primitive ideals of A, or, equivalently, the intersection of the maximal regular left (right) ideals of A, see [**72**, Proposition 24.14]. An algebra A is said to be *semi-simple* if its radical $\mathrm{rad}\, A$ is equal to $\{0\}$. If $A = \mathrm{rad}\, A$ then A is said to be a *radical algebra.* Each semi-simple Banach algebra is semi-prime and therefore faithful [**72**, Proposition 30.5]. Note that in a commutative Banach algebra $\mathcal{A}$ the radical is the set of all quasi-nilpotent elements of $\mathcal{A}$, see Bonsall and Duncan [**72**, Corollary 17.7], and consequently $\mathcal{A}$ is semi-simple precisely when it contains non-zero quasi-nilpotent elements, whilst a commutative Banach algebra $\mathcal{A}$ is semi-prime if and only if it contains non-zero nilpotent elements.

The *weighted convolution* algebra $L_1(\mathbb{R}_+, \omega)$, where the weight ω is chosen so that $\omega^{1/t} \to 0$ as $t \to 0$, is an example of a semi-prime Banach algebra which is not semi-simple [**72**], so these two classes of Banach algebras are distinct.

A trivial example of a semi-primeBanach algebra is an *integral domain.* This is a Banach algebra A for which the product of two nonzero elements of A is always nonzero. On the other hand, it is easy to see that a nonzero Banach algebra A for which the multiplication of any two elements is zero will provide an example of a Banach algebra which is not faithful.

As usual, let $L(A)$ denote the Banach algebra of all linear bounded operators on A. Observe that in the definition of multipliers there are no assumptions of linearity or continuity.

The next theorem shows that if A is faithful Banach algebra the linearity and continuity of a multiplier are consequences of the definition (110). Observe first that if A is faithful and $x, y, z \in A$, for each $T \in M(A)$ we have

$$z[x(Ty)] = z[(Tx)y] = (Tz)(xy) = zT(xy),$$

thus

$$[x(Ty) - T(xy)] \in \operatorname{ran}(A) = \{0\}$$

and therefore the equalities

$$x(Ty) = (Tx)y = T(xy) \tag{111}$$

hold for each $x, y \in A$.

Theorem 4.3. *Let A be a faithful Banach algebra. Then:*

(i) *Every multiplier is a linear bounded operator on A. The multiplier algebra $M(A)$ is a closed commutative subalgebra of $L(A)$ which contains the identity I of $L(A)$;*

(ii) *For every $T \in M(A)$, $T(A)$ is a two-sided ideal in A;*

(iii) *If the Banach algebra A is commutative then every multiplication operator $L_a : x \in A \to ax \in A$, where $a \in A$, is a multiplier of A. The mapping $a \in A \to L_a \in M(A)$ is a continuous isomorphism of the algebra A onto the ideal $\{L_a : a \in A\}$ of $M(A)$.*

Proof (i) First we prove that any $T \in M(A)$ is linear. For any $x, y, z \in A$ and $\lambda, \mu \in \mathbb{C}$ we have

$$z[T(\lambda x + \mu y)] = (Tz)(\lambda x + \mu y) = \lambda z(Tx) + \mu z(Ty) = z[\lambda Tx + \mu Ty],$$

and since $\operatorname{ran}(A) = \{0\}$ this implies $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$.

To prove that T is bounded let $y, z \in A$ and $(y_n) \subset A$ be a sequence for which $\|yn - y\| \to 0$ and $\|Ty_n - z\| \to 0$. For each $x \in A$ we have then

$$\begin{aligned} \|xz - x(Ty)\| &\leq \|x\|\|z - Ty_n\| + \|(Tx)y_n - (Tx)y\| \\ &\leq \|x\|\|z - Ty_n\| + \|Tx\|\|y_n - y\| \to 0, \end{aligned}$$

which implies that $xz = x(Ty)$ and hence $(z - Ty) \in \mathrm{ran}(A)$. Therefore $z = Ty$ and by the closed graph theorem we then conclude that T is a bounded operator.

Next, we show that $M(A)$ is a commutative subalgebra of $L(A)$. It is easy to verify that the sum of multipliers is a multiplier. Moreover if $T, S \in M(A)$ and $x, y \in A$ we have

$$[(TS)x]y = [T(Sx)]y = T[(Sx)y] = T[x(Sy)] = x[TS(y)],$$

so $TS \in M(A)$. Furthermore, by applying (111) to the product TS we obtain that

$$\begin{aligned} x[(TS)y] &= (TS)(xy) = T[S(xy)] = T[(Sx)y] \\ &= (Sx)(Ty) = x[(ST)y] \end{aligned}$$

from which we conclude that $(TS)y = (ST)y$ for all $y \in A$. Therefore $M(A)$ is commutative Banach algebra.

Now, let $(T_n) \in M(A)$ be a sequence which converges to $T \in L(A)$. Then for every $x, y \in A$ we have

$$\begin{aligned} \|(Tx)y - x(Ty)\| &\leq \|(Tx)y - (Tnx)y\| + \|(T_nx)y - x(Ty)\| \\ &\leq 2\|T - T_n\|\|x\|\|y\|, \end{aligned}$$

and this implies that $(Tx)y = x(Ty)$. Hence $M(A)$ is closed.

(ii) Clearly, if $T \in M(A)$ the equalities (111) imply, since T is linear, that $T(A)$ is a two-sided ideal of A.

(iii) The proof is straightforward. ■

Theorem 4.4. *If A is semi-prime then $M(A)$ is semi-prime.*

Proof Suppose that J is a two-sided ideal of $M(A)$ with $J^2 = \{0\}$. If $T \in J$ then $T^2 = 0$ and, from the equality $(Tx)(Ty) = (T^2x)y = 0$ for all $x, y \in A$, it follows that $[T(A)]^2 = \{0\}$. Since $T(A)$ is a two-sided ideal of A and A is semi-prime then $T(A) = \{0\}$. Hence $T = 0$, so that $J = \{0\}$. ■

By Theorem 4.3 a commutative faithful Banach algebra A may be identified the subset $\{L_a : a \in A\}$ of the multiplier algebra $M(A)$. Clearly, for each $a \in A$ we have $\|L_a\| \leq \|a\|$. Generally, if A is regarded as a subset of $M(A)$, the norm of A and the operator norm of $M(A)$ are not equivalent. A straightforward consequence of the open mapping theorem is that these two norms are equivalent if and only if the ideal A is a closed in $M(A)$.

We give now an example of a commutative faithful Banach algebra A for which the two norms on A and $M(A)$ are not equivalent.

Example 4.5. Let $\omega := (\omega_n)$ be a sequence of real number such that $\omega_n \geq 1$ for all $n \in \mathbb{N}$ and denote by $c_0(\omega)$ the space of all complex sequences $x := (x_n)$ for which $x_n\omega_n \to 0$, as $n \to \infty$. The space $c_0(\omega)$ with respect the coordinatewise operations and equipped with weighted supremum norm given by

$$\|x\|_\omega := \sup_{n\in\mathbb{N}} |x_n\omega_n|$$

is a commutative Banach algebra. If we define $\phi_k(x) := x_k$ for all elements $x = (x_n) \in c_0(\omega)$ then ϕ_k is a multiplicative linear function on $c_0(\omega)$ and it is easily seen that $c_0(\omega)$ is semi-simple. If $e_k := (\delta_{jk})_{j\in\mathbb{N}}$ then $\|e_k\| = 1$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ let us consider the multiplication operator L_{e_k}. It is clear that $\|L_{e_k}\| = 1$ for all $k \in \mathbb{N}$. Therefore if we choose $\omega_n \to \infty$ as $n \to \infty$ the norm on $M(c_0(\omega))$ is not equivalent to the norm of $c_0(\omega)$, and consequently $c_0(\omega)$ is not closed in the norm of its multiplier algebra .

A Banach algebra A is said to have an *approximate identity* if there exists a net $(e_\lambda)_{\lambda\in\Lambda}$ such that $e_\lambda a \to a$ in the norm of A for every $a \in A$. If the net $(e_\lambda)_{\lambda\in\Lambda}$ is bounded we shall say that A possesses a *bounded approximate identity*. It is straightforward to verify that a Banach algebra with a bounded approximate identity is faithful. Moreover, it is easily seen that for a commutative Banach algebra with a bounded approximate identity the norm of A is equivalent to the operator norm of A regarded as subset of $M(A)$, so that A is a closed subset of $M(A)$.

We show now that, in the commutative case, the algebra $M(A)$ is the maximal commutative extension of A into $L(A)$.

Theorem 4.6. *Let A be a faithful Banach algebra. Then the following statements are equivalent:*

(i) *A is commutative;*

(ii) *$M(A)$ is a maximal commutative subalgebra of $L(A)$, i.e., there is no proper subalgebra of $L(A)$ containing properly $M(A)$.*

Proof The proof depends on Zorn's Lemma. Suppose that A is commutative. If $M(A)$ were not maximal in $L(A)$, by Zorn's Lemma there exists a maximal commutative subalgebra B of $L(A)$ containing properly $M(A)$. Suppose that $T \in B$. For each $x, y \in A$ we have

$$x(Ty) = L_x(Ty) = (L_xT)y = (TL_x)(y) = T(xy). \tag{112}$$

Obviously we may interchange in (112) x with y and obtain $y(Tx) = T(yx)$ and since $xy = yx$ from (112), it then follows that

$$x(Ty) = T(xy) = T(yx) = y(Tx) = (Tx)y.$$

Therefore $T \in M(A)$ for each $T \in B$, contradicting that $M(A)$ is properly contained in B.

Conversely, suppose that $M(A)$ is maximal subalgebra of $L(A)$ and consider an arbitrary element $a \in A$. If L_a is the left multiplication operator by a and $T \in M(A)$ then

$$(L_aT)x = a(Tx) = T(ax) = (TL_a)x$$

for all $x \in A$. From this it follows that $L_a T = T L_a$ for all $T \in M(A)$ and hence $L_a \in M(A)$, by the maximality of A. This implies, for every x, $y \in A$,

$$yxa = L_y(xa) = x(L_y a) = xya$$

for all $a \in A$. Consequently $yx - xy \in \operatorname{lan}(A) = \{0\}$, hence $yx = xy$ for all $x, y \in A$. Therefore A is commutative. ■

Observe that the definition of a multiplier reminds us in a certain sense of the definition of a symmetric operator T which is defined on a Hilbert space H by

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in H, \tag{113}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. The classical *Hellinger Toeplitz* theorem establishes that the definition (113) implies the continuity of T, exactly as the definition (110)) implies the continuity of a multiplier on a faithful Banach algebra.

Definition 4.7. *We shall say that two elements $x, y \in A$ are* orthogonal *whenever $xy = yx = 0$. Given a nonempty subset B of A the orthogonal of B is defined to be the set*

$$B^\top := \{x \in A : xy = yx = 0 \text{ for each } y \in B\}.$$

Trivially, $B^\top$ is a two-sided ideal of A, and by the continuity of the product it is also closed.

Definition 4.8. *A projection P is said to be* orthogonal *if P projects A onto a closed two-sided ideal B along its orthogonal $B^\top$, i.e.,*

$$A = B \oplus B^\top, \quad P(A) = B, \quad \text{and} \quad \ker P = (I - P)(A) = B^\top.$$

The following result is reminiscent of analogous properties of symmetric operators on Hilbert spaces.

Theorem 4.9. *Let A be a faithful Banach algebra. For each $T \in M(A)$, we have:*

(i) $\ker T = T(A)^\top$ *and* $\overline{T(A)} \subseteq \ker T^\top$*;*

(ii) *If T is surjective then T is injective;*

(iii) *Eigenvectors of T corresponding to distinct eigenvalues are orthogonal to each other.*

Proof (i) Let $x \in \ker T$ and $y = Tz$, where $z \in A$ Then $xy = x(Tz) = (Tx)z = 0$ and in the same way $yx = 0$.

Conversely, if $y \in T(A)^\top$, for every $x \in A$ we have $(Tx)y = x(Ty)$, and this implies, since A is faithful, $Ty = 0$. Therefore $T(A)^\top \subseteq \ker T$.

To prove the second inclusion let us consider $y \in \overline{T(A)}$ and suppose that $Ty_n \to y \in A$ as $n \to \infty$. For any $x \in \ker T$ and every $n \in \mathbb{N}$ we then have $0 = (Tx)y_n = x(Ty_n)$, which implies that $xy = 0$. Analogously $yx = 0$, so $y \in (\ker T)^\top$.

(ii) If $T(A) = A$ then by part (i) $\ker T = A^\top = \{0\}$.

(iii) Let $Tx = \lambda x$, $Ty = \mu y$ where $\lambda \neq \mu$ and $x, y \neq 0$. Then

$$\lambda xy = (Tx)y = x(Ty) = \mu xy.$$

Therefore $(\lambda - \mu)xy = 0$, thus $xy = 0$. Analogously we have $yx = 0$. ∎

The next theorem shows that the orthogonal projections are exactly all the projections which are multipliers. It also exhibits another strong analogy between symmetric projections (which are orthogonal with respect to the inner product) on a Hilbert space and multiplier projections on a faithful Banach algebra .

Theorem 4.10. *Let A be a faithful Banach algebra and $P \in L(A)$ a projection. Then P is orthogonal if and only if P is a multiplier.*

Proof Let be P an orthogonal projection and let us suppose that $A = M \oplus M^\top$, M a closed two-sided ideal of A, is the corresponding orthogonal decomposition, i.e. $P(A) = M$ and $\ker P = M^\top$. Then for each $z, v \in A$ we have $z = x + y$, $v = u + w$, where $u, v \in M$, $y, w \in M^\top$. From these decompositions it follows that

$$(Pz)v = P(x + y)v = xv = x(u + w) = xu = (x + y)u = z(Pv)$$

for each $z, v \in A$, hence P is a multiplier of A.

Conversely, let $P \in M(A)$ be a projection. Then $A = P(A) \oplus (I - P)(A)$. Let us consider the elements $Px \in P(A)$ and $(I - P)y \in (I - P)(A)$. We have

$$P(x)[(I - P)y] = x[P(I - P)y] = 0$$

and, analogously, $[(I - P)y](Px) = 0$. Therefore $P(A) \subseteq (I - P)(A)]^\top$.

On the other hand, if we assume that $z \in A$ is orthogonal to $(I - P)(A)$, because $I - P$ is a multiplier then

$$z[(I - P)y] = (I - P)(zy) = 0 \quad \text{for each } y \in A,$$

which gives

$$zy = P(zy) = (Pz)y \quad \text{for each } y \in A.$$

Therefore $z - Pz \in A^\top = 0$ and hence $z = Pz$. This proves also that the inclusion $[(I - P)(A)]^\top \subseteq P(A)$ holds, so $[(I - P)(A)]^\top = P(A)$. ∎

Let B denote a closed two-sided ideal of A and $T \in M(A)$. We shall say that B *reduces* T if $A = B \oplus B^\top$ and B, $B^\top$ are both invariant under T.

Theorem 4.11. *Let A be a faithful Banach algebra. We have:*

(i) *If B is a closed two-sided ideal invariant under $T \in M(A)$, then also $B^\top$ is invariant under T;*

(ii) *If A admits the orthogonal decomposition $A = B \oplus B^\top$, where B a closed two-sided ideal, then B reduces any $T \in M(A)$ under which B is invariant.*

Proof (i) Let $y \in B^\top$ and let z be any element of B. We have $0 = y(Tz) = (Ty)z$ and $0 = (Tz)y = z(Ty)$, thus $Ty \in B^\top$.

(ii) Suppose $A = B \oplus B^\top$, B a closed two-sided ideal in A, and let P denote the corresponding orthogonal projection of A onto B. Then by Theorem 4.10 $P \in M(A)$ and the commutativity of $M(A)$ entails that $PT = TP$ for each $T \in M(A)$. This is equivalent to the property that the two ideals B and $B^\top$ are both invariant under T. ∎

It is easily seen that every closed ideal in a Banach algebra with an approximate identity is invariant under any $T \in M(A)$.

In a Hilbert space a symmetric operator T on a complex inner product space is uniquely determined by its quadratic form $\langle Tx, x\rangle$, see Heuser [**159**, §68]. Similarly, a multiplier T of a faithful commutative Banach algebra may be characterized by means of the products $(Tx)x$, $x \in A$. In fact, we have:

Theorem 4.12. *Let A be a faithful commutative Banach algebra . Then $T \in M(A)$ if and only if $Tx^2 = x(Tx) = (Tx)x$ holds for each $x \in A$.*

Proof Clearly we have only to show that if $Tx^2 = x(Tx)$ for each $x \in A$ then $T \in M(A)$. To see this let $x, z \in A$ be arbitrary. Then

$$\begin{aligned} (Tx)x \quad + &\quad 2\, T(xz) + (Tz)z \\ = &\quad T(x^2 + 2xz + z^2) \\ = &\quad T(x+z)^2 = (x+z)T(x+z) \\ = &\quad (Tx)x + (Tz)x + (Tx)z + (Tz)z, \end{aligned}$$

and hence

$$2T(xz) = (Tz)x + (Tx)z = x(Tz) + (Tx)z.$$

From this we obtain that

$$(2TL_x - L_xT)z = L_{Tx}z \quad \text{for every } z \in A.$$

This implies that

$$(2TL_x - L_xT) = L_{Tx} \in M(A),$$

and hence because $M(A)$ is commutative $2TL_x - L_xT$ commutes with L_z. Consequently,

$$2TL_xL_z - L_xTL_z = 2L_zTL_x - L_zL_xT = 2L_zTL_x - L_xL_zT. \tag{114}$$

Applying each side of the (114) to z, using the equality $Tz^2 = z(Tz)$, and simplifying we then obtain

$$T(xz^2) = zT(xz) \quad \text{for every } x, z \in A.$$

In particular, $T[y(x+z)^2] = (x+z)T(x+z)y$, and hence

$$\begin{aligned} Tx^2y + 2T(xzy) + Tz^2y &= T[(x+z)^2y] \\ &= T[y(x+z)^2] = (x+z)T(x+z)y \\ &= x(T(xy) + xT(zy) + zT(xy) + zT(zy). \end{aligned}$$

Therefore

$$2TL_xL_z = L_xTL_z + L_zTL_x. \tag{115}$$

By substituting (119) into (114) and simplifying we then obtain $L_zTL_x = L_zL_xT$, which gives $zT(xy) = z[x(Ty)]$ for all $z \in A$. Since A is faithful we then conclude that $T(xy) = x(Ty)$ for each $x, y \in A$, so $T \in M(A)$. ∎

In the sequel we shall need the following result which shows that if A has an approximate identity then every multiplier is the limit in the strong topology of multiplication operators.

Theorem 4.13. *Let A be a faithful commutative Banach algebra. Then the following properties are equivalent:*

(i) *A has an approximate identity;*

(ii) *The set $\{L_x : x \in A\}$ is dense in $M(A)$ in the strong operator topology.*

Proof (i) $\Rightarrow$ (ii) Suppose that (u_β) is an approximate identity of A. Then the net $(L_{T_{u_\beta}})$ converges strongly to the operator T for each $T \in M(A)$.

(ii) $\Rightarrow$ (i) Conversely if (ii) is satisfied the net (L_{u_β}) converges in the strong topology to the identity $I \in M(A)$, and this obviously implies that (u_β) is an approximate identity of A. ∎

2. The Helgason–Wang function

We first recall some basic elements of the classical Gelfand theory of commutative Banach algebras. The reader will find more information in Bonsall and Duncan [**72**] and Rickart [**279**].

Let $\Delta(A)$ denote the set of all maximal regular ideals of a commutative Banach algebra A and let A^* denote the dual of A. Recall that a *multiplicative linear functional* on a complex Banach algebra A is a non-zero linear functional $m \in A^*$ such that $m(xy) = m(x)m(y)$ for all $x, y \in A$, i.e., m is a non-zero homomorphism of A into $\mathbb{C}$. It is well known that if A is commutative the maximal regular ideals are precisely the kernels of the multiplicative linear functionals. Hence $\Delta(A)$ can be identified with the subset of the unit ball of A^* consisting of the multiplicative linear functionals on A. Consequently if A is a commutative Banach algebra A then

$$\text{rad } A = \bigcup_{m \in \Delta(A)} \ker m,$$

and from this it follows that rad A coincides with the set of all quasi-nilpotent elements of A, see [**72**, Corollary 17.7].

In $\Delta(A)$ it is possible to consider the so called *Gelfand topology*. This is the weak* topology on the unit ball of the dual restricted to $\Delta(A)$. The set $\Delta(A)$ provided with this topology is a locally compact Hausdorff space called the *regular maximal space* of A. Note that $\Delta(A)$ is compact with the

Gelfand topology if the Banach algebra A has a unit, see Bonsall Duncan [**72**, §17].

Now let $\widehat{x}$ denote the *Gelfand transform* of $x \in A$ defined by

$$\widehat{x}(m) := m(x) \quad \text{for each} \quad m \in \Delta(A),$$

and denote by $C_0(A)$ the Banach algebra of all continuous complex valued functions which vanish at infinity, provided with the sup-norm. The correspondence $x \in A \to \widehat{x} \in C_0(A)$ defines a continuous homomorphism of A onto a subalgebra of $C_0(A)$ which is called the *Gelfand representation* of A. If A has a unit the mapping $x \to \widehat{x}$ is a continuous homomorphism of A onto a subalgebra of $C(\Delta)$, where $C(\Delta)$ denotes the Banach algebra of all continuous complex valued functions provided with the sup-norm, since $\Delta(A)$ is compact in this case.

Finally, A is semi-simple if and only if the Gelfand representation is injective or equivalently that $\Delta(A)$ separates the points of A, i.e. for every non-zero element of A there exists some $m \in \Delta(A)$ such that $\widehat{x}(m) \neq 0$. Obviously the last condition implies that if $\widehat{x}(m)$ is zero for all $m \in \Delta(A)$ then $x = 0$.

The following result, owed to Wang [**314**], plays a fundamental role in the theory of multipliers of commutative semi-simple Banach algebras. It shows that every multiplier of A may be represented as a bounded continuous complex function on the locally compact Hausdorff space $\Delta(A)$.

Theorem 4.14. *Let A be a semi-simple commutative Banach algebra. Then for each $T \in M(A)$ there exists a unique bounded continuous function φ_T on $\Delta(A)$ such that the equation*

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{x}(m) \tag{116}$$

holds for all $x \in A$ and all $m \in \Delta(A)$. Moreover,

$$\|\varphi_T\|_\infty \leq \|T\| \quad \textit{for all } T \in M(A).$$

Proof For each $m \in \Delta(A)$ take $x \in A$ such that $\widehat{x}(m) \neq 0$ and define

$$\varphi_T(m) := \frac{\widehat{Tx}(m)}{\widehat{x}(m)}.$$

We show first that the definition of φ_T is independent of x. Indeed, let y be any element of A such that $\widehat{y}(m) \neq 0$. Then because $(Tx)y = x(Ty)$ we have $\widehat{Tx}(m)/\widehat{x}(m) = \widehat{Ty}(m)/\widehat{y}(m)$, and hence the function φ_T is well defined. Moreover, φ_T is a continuous function on $\Delta(A)$, so to prove the equation (120) for all $x \in A$ and $m \in \Delta(A)$ we only need to show that it is still satisfied whenever $\widehat{x}(m) = 0$.

Suppose $\widehat{x}(m) = 0$ and let $y \in A$ such that $\widehat{y}(m) \neq 0$. Then we have

$$\widehat{Tx}(m)\widehat{y}(m) = \widehat{x}(m)\widehat{Ty}(m) = 0,$$

which implies $\widehat{Tx}(m) = 0$. Hence $\widehat{Tx}(m) = \varphi_T(m)\widehat{x}(m)$ for all $x \in A$ and $m \in \Delta(A)$.

To prove the uniqueness of φ_T denote by ψ a second complex-valued function defined on $\Delta(A)$ for which $\widehat{Tx} = \psi\widehat{x}$. Then

$$(\varphi_T(m) - \psi(m))\widehat{x}(m) = 0 \quad \text{for all } x \in A,$$

and this trivially implies $\varphi_T(m) = \psi(m)$.

To prove the last assertion let us denote

$$\|m\| := \sup\{|\widehat{x}(m)| : \|x\| = 1\}.$$

Because $0 < \|m\| \leq 1$, for each $x \in A$ we have

$$|\varphi_T(m)\widehat{x}(m)| = |\widehat{Tx}(m)| \leq ||m\|\|Tx\|\|x\|.$$

In particular, considering those elements x having norm 1 we obtain

$$|\varphi_T(m)| \leq \inf_{\|x\|=1} \frac{\|m\|\|T\|}{|\widehat{x}(m)|} = \frac{\|m\|\|Tx\|}{\sup_{\|x\|=1}|\widehat{x}(m)|} = \|T\|,$$

so φ_T is bounded and $\|\varphi_T\|_\infty \leq \|T\|$. ∎

The function φ_T which corresponds by the previous theorem to a multiplier T will be called the *Helgason–Wang function* of T.

Clearly, for a multiplication operator L_a we have

$$\varphi_{L_a} = \widehat{a} \quad \text{for every } a \in A.$$

Let us denote by $\mathcal{M}(A)$ the normed subalgebra of $C(\Delta)$ defined as follows

$$\mathcal{M}(A) := \{\varphi \in C(\Delta) : \varphi \text{ is bounded and } \varphi\widehat{A} \subseteq \widehat{A}\},$$

where $\widehat{A}$ denotes the set $\{\widehat{x} : x \in A\}$. It is evident that the equation $\widehat{Tx}(m) = \varphi_T(m)\widehat{x}(m)$ defines a continuous isomorphism $T \to \varphi_T$ of $M(A)$ onto the algebra $\mathcal{M}(A)$.

In the next theorem we give another characterization of multipliers of commutative semi-simple Banach algebras.

Theorem 4.15. *Let A be a commutative semi-simple Banach algebra and T be a bounded operator on A. Then the following statements are equivalent:*

(i) $T \in M(A)$*;*

(ii) *Each maximal regular ideal of A is invariant under T, or, equivalently,* $T(\ker m) \subseteq \ker m$ *for each $m \in \Delta(A)$.*

Proof Let $T \in M(A)$ and $m \in \Delta(A)$. If $x \notin \ker m$, for every $y \in \ker m$ we have

$$\widehat{Ty}(m)\widehat{x}(m) = \varphi_T(m)\widehat{x}(m) = 0.$$

Since $\widehat{x}(m) \neq 0$ this implies $\widehat{Ty}(m) = m(Ty) = 0$, and hence $Ty \in \ker m$.

Conversely, suppose that (ii) holds and consider two elements x, y of A. Fix an arbitrary multiplicative linear functional $m \in \Delta(A)$. Suppose first the case that either x or y belongs to the maximal regular ideal $M := \ker m$, say $x \in M$. From $Tx \in \ker m$ we then obtain

$$[x(\widehat{Ty) - (T}x)y](m) = 0. \tag{117}$$

Consider the other case in which both x, y do not belong to the maximal regular ideal M. Since M has codimension 1 there then exist $\lambda, \mu \in \mathbb{C}$, an element $z \notin M$, and elements $u, v \in M$ such that $x = \lambda z + u$, $y = \mu z + v$. Taking into account that both the elements Tu and Tv belong to $M = \ker m$ we then have that

$$\widehat{x(Ty)}(m) = \widehat{Tx)}y(m) = \lambda\widehat{z}(m)\mu\widehat{Tz}(m).$$

Hence also in this second case the equality $[x(\widehat{Ty) - (T}x)y](m) = 0$ is satisfied. Therefore the equality (117) is verified for each $m \in \Delta(A)$ and for all $x, y \in A$, so from the semi-simplicity of A we may conclude that $x(Ty) = (Tx)y$ for all $x, y \in A$, which shows that $T \in M(A)$. ∎

Since $M(A)$ is a commutative Banach algebra we can consider the maximal regular ideal space $\Delta(M(A))$. As usual, by $\widehat{T}$ we shall denote the Gelfand transform of T as an elemente of the commutative Banach algebra $M(A)$. Observe that, since $M(A)$ has a unit, $\Delta(M(A))$ is compact with respect to the Gelfand topology.

Next we want to exhibit for a commutative faithful Banach algebra the relationships between $\Delta(A)$ and $\Delta(M(A))$. We recall that by Theorem 4.3 A can be algebraically identified with the ideal $\{L_x : x \in A\}$ of $M(A)$. In symbols we shall write $A = \{L_x : x \in A\}$.

Theorem 4.16. *Let A be a commutative faithful Banach algebra. To each $m \in \Delta(A)$ there corresponds an unique $\phi := \phi(m) \in \Delta(M(A))$ such that the restriction $\phi \mid A$ of ϕ on A coincides with m. Moreover, if $\nu \in \Delta(M(A))$ then either $\nu = 0$ on A or there is a unique $m \in \Delta(A)$ such that $\nu \mid A = m$.*

Proof Suppose that $m \in \Delta(A)$ and let x be an element of A such that $m(x) \neq 0$. Let ϕ be defined as follows

$$\phi(T) := \frac{m(Tx)}{m(x)} \quad \text{for any } T \in M(A).$$

Observe that the definition of ϕ does not depend on the choice of x since if $m(y) \neq 0$ then

$$\phi(T)m(y) = \frac{m(Tx)m(y)}{m(x)} = m((Tx)y) = \frac{m(x(Ty))}{m(x)} = m(Ty).$$

Evidently ϕ is a linear functional on $M(A)$ and

$$\phi(TS) = \frac{m((TS)x)}{m(x)} = \frac{\phi(T)m(Sx)}{m(x)} = \phi(T)\phi(S),$$

so $\phi \in \Delta(M(A))$. Moreover, for all $y \in A$ we have

$$\phi(L_y) = \frac{m(L_y x)}{m(x)} = \frac{m(yx)}{m(x)} = m(y),$$

which shows that $\phi \mid A = m$.

To prove that ϕ is unique let us consider another multiplicative linear functional $\mu \in \Delta M(A)$ such that $\mu(L_y) = m(y)$ for all $y \in A$. Choosing any $x \in A$ such that $m(x) \neq 0$, if $T \in M(A)$ we then have

$$\mu(T)m(x) = \mu(TL_x) = \phi(L_{Tx}) = m(Tx) = \phi(T)m(x),$$

and this obviously implies that $\mu = \phi$.

To prove the last part of the theorem let us consider a multiplicative linear functional ν on $M(A)$ such that $\nu(L_y) \neq 0$ holds for some $y \in A$. The equation $m(x) = \nu(L_x)$, where $x \in A$, defines a multiplicative linear functional m on A, and by the first part of the proof ν coincides with the unique $\phi \in \Delta(M(A))$ for which $\phi \mid A = m$, so the proof is complete. ■

Let Ω be the set defined by

$$\Omega := \{\phi \in \Delta(M(A)) : \phi(L_x) \neq 0 \text{ for some } x \in A\}.$$

As a consequence of Theorem 4.16 we know that there exists a bijective mapping

$$\phi : m \in \Delta(A) \to \phi = \phi(m) \in \Omega \tag{118}$$

of $\Delta(A)$ onto the subset Ω of $\Delta(M(A))$.

Definition 4.17. *Given an ideal J of a commutative Banach algebra $\mathcal{B}$, the* hull *of J is defined to be the set*

$$h_{\mathcal{B}}(J) := \{m \in \Delta(\mathcal{B}) : \ker m \subseteq J\}.$$

If E is a subset of $\Delta(\mathcal{B})$ the kernel *of E is defined to be the ideal of $\mathcal{B}$*

$$k_{\mathcal{B}}(E) := \bigcap_{m \in E} \ker m = \{x \in \mathcal{B} : \widehat{x}(m) = m(x) = 0 \quad \textit{for all } m \in E\}.$$

It is easily seen that $h_{\mathcal{B}}(J)$ and $k_{\mathcal{B}}(E)$ are closed subsets of $\Delta(\mathcal{B})$ and $\mathcal{B}$, respectively.
Now denote by $h_{M(A)}(A)$ the hull of the ideal $A = \{L_x : x \in A\}$ in $M(A)$:

$$h_{M(A)}(A) = \{\phi \in \Delta(M(A)) : \phi \mid A \equiv 0 \text{ for each } x \in A\}.$$

Clearly we have

$$\Delta(M(A)) = \Omega \cup h_{M(A)}(A). \tag{119}$$

The equality above is just a set-theoretic decomposition of $\Delta(M(A))$. The next theorem shows how these sets are also topologically linked in the Gelfand topology of $\Delta(M(A))$.

Theorem 4.18. *Let A be a commutative faithful Banach algebra. Then Ω is an open subset of $\Delta(M(A))$ homeomorphic to $\Delta(A)$, and $h_{M(A)}(A)$ is compact in $\Delta(M(A))$ with respect to the Gelfand topology.*

Proof The hull $h_{M(A)}(A)$ is a closed subset of $\Delta(M(A))$, so the equality (119) implies that Ω is open. The bijective mapping $\phi : m \to \phi(m)$, described after Theorem 4.16 is a continuous mapping of $\Delta(A)$ onto Ω, since $\phi(T) = m(Tx)/m(x)$. The inverse mapping is also continuous, since $\phi(L_x) = m(x)$ and $\{L_x : x \in A\}$ is a subset of $M(A)$. Hence the mapping ϕ is an homeomorphism of $\Delta(A)$ onto Ω. This last fact implies, because $\Delta(M(A))$ is compact, that $h_{M(A)}(A)$ is a compact subset of $\Delta(M(A))$. ■

By the last theorem we can identify $\Delta(A)$ with Ω. For this reason we shall always write in the sequel

$$\Delta(M(A)) = \Delta(A) \cup h_{M(A)}(A). \tag{120}$$

It is easy to verify that when $\Delta(A)$ is regarded as a subset of $\Delta(M(A))$ the Gelfand topology of $\Delta(A)$ coincides with the relative Gelfand topology induced by $\Delta(M(A))$. Moreover, if $\widehat{T} : \Delta(M(A)) \to \mathbb{C}$ denotes the Gelfand transform of $T \in M(A)$ we have

$$\widehat{T} \mid \Delta(A) = \varphi_T \quad \text{for each } T \in M(A); \tag{121}$$

and for this reason we shall freely use both notations $\widehat{T}$ and φ_T for the Helgason–Wang function defined on $\Delta(A)$.

As we observed before, the algebra A considered as a subalgebra of $M(A)$ is not generally closed in the $M(A)$ norm. The following result provides an information on the $M(A)$ closure of A in $M(A)$.

Theorem 4.19. *Let A be a commutative faithful Banach algebra. Then the $M(A)$ closure of A is contained in $k_{M(A)}(h_{M(A)}(A))$.*

Proof Let $T \in M(A)$ be an element of the $M(A)$-norm closure of the ideal $A = \{L_x : x \in A\}$. For each $\varepsilon > 0$ then there exists $x \in A$ such that $\|T - L_x\| < \varepsilon$. From the property that the Gelfand transform $\widehat{L}_x = \widehat{x}$ vanishing identically on $h_{M(A)}(A))$ we infer that

$$|\widehat{T}(\psi)| = |\widehat{T}(\psi) - \widehat{L}_x(\psi)| \le \|T - Lx\| < \varepsilon \quad \text{for every } \psi \in h_{M(A)}(A)).$$

Therefore $\widehat{T} \mid h_{M(A)}(A)) \equiv 0$, so $T \in k_{M(A)} h_{M(A)}(A))$. ■

Let us consider the following two sets :

$$M_0(A) := \{T \in M(A) : \widehat{T} \mid \Delta(A) = \varphi_T \text{ vanishes at infinity in } \Delta(A)\}$$

and

$$M_{00}(A) := \{T \in M(A) : \widehat{T} \equiv 0 \text{ on } h_{M(A)}(A)\}.$$

Clearly $M_0(A)$ and $M_{00}(A)$ are proper closed ideals of the Banach algebra $M(A)$, because these ideals do not contain the identity operator I. Because the Gelfand transform $\widehat{a}$ of an element $a \in A$ vanishes at infinity on $\Delta(A)$ we also have

$$A = \{L_a : a \in A\} \subseteq M_{00}(A) \subseteq M_0(A).$$

Later we shall see that both of these inclusions may be strict, for instance when $A = L_1(G)$, G a non-discrete locally compact Abelian group.

Let us consider the set $h_{M_0(A)}(A)$, the hull of the ideal $A = \{L_a : a \in A\}$ relative to the algebra $M_0(A)$. The next theorem exhibits the connection between the two spaces $\Delta(M_0(A))$ and $\Delta(A)$.

Theorem 4.20. *Let A be a commutative semi-simple Banach algebra. Then $\Delta(A)$ is homeomorphic to an open and closed subset of $\Delta(M_0(A))$. Moreover, $\Delta(M_0(A)) = \Delta(A) \cup h_{M_0(A)}(A)$.*

Proof Clearly, as in Theorem 4.18 $\Delta(A)$ is homeomorphic to an open subset Ω of $\Delta(M_0(A))$, and therefore by identifying $\Delta(M_0(A))$ with this set Ω we can write $\Delta(M_0(A)) = \Delta(A) \cup h_{M_0(A)}(A)$.

Now suppose that h_0 is a boundary point of $\Delta(A)$ in $\Delta(M_0(A))$. If $T \in M_0(A)$, for each $\varepsilon > 0$ there exists a compact $K \subset \Delta(A)$ such that

$$|\widehat{T}(m)| < \varepsilon \quad \text{for each } m \in \Delta(A) \setminus K.$$

Since $\Delta(A)$ is open the point h_0 cannot belong to any compact K of $\Delta(A)$, so that $\widehat{T}$ must vanish at the point h_0. But this contradicts that $h_0 \neq 0$. Therefore the point at infinity is the unique boundary point of $\Delta(A)$ and $\Delta(M_0(A)) \setminus \Delta(A)$ in the compactification of $\Delta(M_0(A))$. This shows that $\Delta(A)$ is also a closed subset of $\Delta(M_0(A))$. ■

The maximal ideal space $\Delta(A)$ of a commutative Banach algebra A may also be topologized by means of the so-called *hull-kernel topology* or *hk-topology*. This is determined by the Kuratowski closure operation: given a subset $E \subset \Delta(A)$ the hk-closure of E is the set

$$h_A(k_A(E)) = \{m \in \Delta(A) : m(x) = 0 \text{ for all } x \in A \text{ with } \widehat{x} \mid E \equiv 0\}.$$

Hence the closed sets of the hk-topology are all the sets $h_A(k_A(E))$, where E ranges over the subsets of $\Delta(A)$.

The hk-topology is generally coarser than the Gelfand topology on $\Delta(A)$. To show this let us consider for an arbitrary subset E of $\Delta(A)$ the complement $\Delta(A) \setminus h_A(k_A(E))$. Clearly, from the definition of the hk-closure it follows that the $\Delta(A) \setminus h_A(k_A(E))$ is the set of all $m \in \Delta(A)$ for which there is an element $a \in A$ with $\widehat{a} \equiv 0$ on E and $\widehat{a}(m) = m(a) \neq 0$. From this we infer that $\Delta(A) \setminus hk(E)$ is always open in the Gelfand topology, hence every kh-closed subset of $\Delta(A)$ is closed in the Gelfand topology.

In the sequel we shall need the following two classical results on the maximal ideal spaces of ideals and quotients. A proof of these results may be found in Rickart [**279**], or also in Proposition 4.3.3 and Proposition 4.3.4 of Laursen and Neumann [**214**].

Theorem 4.21. *Let A be commutative complex Banach algebra and J an ideal in A. Then we have*

(i) *The set* $\Delta(A) \setminus h_A(J)$ *is hk-open, and hence Gelfand open, in* $\Delta(A)$. *Moreover, the mapping*

$$m \in \Delta(A) \setminus h_A(J) \to m \mid J$$

is a homeomorphism of $\Delta(A) \setminus h_A(J)$ *to* $\Delta(J)$ *with respect the Gelfand topology, as well as with respect to the hk-topology.*

(ii) *There exists a bijective mapping* $\Psi : \Delta(A/J) \to h_A(J)$ *and this mapping is a homeomorphism with respect the Gelfand and hk-topologies on these two sets.* ∎

Because Theorem 4.21 we shall write

$$\Delta(J) = \Delta(A) \setminus h_A(J) \quad \text{and} \quad \Delta(A/J) = h_A(J),$$

for every ideal J of A. According to Theorem 4.21 these identifications hold both for the Gelfand and the hk-topologies and

$$\Delta(A) = \Delta(J) \cup \Delta(A/J),$$

where the two sets $\Delta(J)$ and $\Delta(A/J)$ are disjoint.

Recall that the *Shilov idempotent theorem* establishes that if $\Delta(A) = \Delta_1 \cap \Delta_2$, where Δ_1, Δ_2 are disjoint non-void compact sets, then there exists an idempotent $e \in A$ with $m(e) = 1$ for all $m \in \Delta_1$ and $m(e) = 0$ for all $m \in \Delta_2$, see Shilov [**298**], or Theorem 21.5 of Bonsall and Duncan [**72**].

In the following theorem we collect some basic results on commutative Banach algebras which will be used in the investigation of multipliers.

Theorem 4.22. *For every commutative Banach algebra* A *the following assertions holds:*

(i) *If* $E \subseteq \Delta(A)$ *is a hull then* E *is compact with respect to the Gelfand topology if and only if* $k_A(E)$ *is a regular ideal;*

(ii) *If* $E \subseteq \Delta(A)$ *is a hull such that there exist an element* $a \in A$ *and a* $\delta > 0$ *for which* $|\widehat{a}| \geq \delta$ *on* E, *then there is an element* $b \in A$ *for which* $\widehat{ab} \equiv 1$ *on* E;

(iii) *If* E_1 *and* E_2 *are two disjoint hulls in* $\Delta(A)$ *and if* E_1 *is compact with respect to the Gelfand topology then there exists* $u \in A$ *such that* $\widehat{u} \mid E_1 \equiv 1$ *and* $\widehat{u} \mid E_2 \equiv 0$.

Proof See Proposition 4.3.12 and Proposition 4.3.13 of Laursen and Neumann [**214**]. ∎

Definition 4.23. *A commutative Banach algebra* A *is said to be* regular *if for every closed subset* E *of* $\Delta(A)$ *in the Gelfand topology and every* $m_0 \in \Delta(A) \setminus E$ *there exists an* $x \in A$ *such that* $\widehat{x}(m_0) = 1$ *and* $\widehat{x} \equiv 0$ *on* E.

Examples of regular commutative Banach algebras are $C_0(\Omega)$, the algebra of all continuous complex-valued functions that vanish at infinity on a locally compact Hausdorff topological space Ω, see [**198**, p. 167], and $L^1(G)$, G being a locally compact Abelian group, see [**198**, Corollary 7.2.3].

Note that if $\Delta(A)$ is totally disconnected in the Gelfand topology then A is regular, for a proof see [**214**, Lemma 4.8.4]. An important example of non-regular commutative Banach algebra is provided by the *disc algebra* $\mathcal{A}(\mathbb{D})$, the algebra of all complex-valued functions continuous on the closed unit disc of $\mathbf{C}$ and analytic on the open unit disc $\mathbb{D}$ provided with the uniform norm:

$$\|f(t)\| := \sup_{t\in\mathbb{D}} |f(t)|.$$

The maximal regular ideal space of $\mathcal{A}(\mathbb{D})$ may be canonically identified with the closed unit disc $\mathbf{D}$ and from the identity theorem for analytic functions it immediately follows that $\mathcal{A}(\mathbb{D})$ is not regular.

It is not complicated to see that the Gelfand topology and the hk-topology on $\Delta(A)$ coincide exactly when A is regular. In fact, if the two topologies coincide then every Gelfand closed set $E \subseteq \Delta(A)$ is hk-closed. Hence $E = h_A(k_A(E))$ so there exists for each $m \in \Delta(A) \setminus E$ an element $a \in A$ with $\widehat{a} \equiv 0$ on E and $m(a) \neq 0$.

Conversely, if A is regular and E is a Gelfand closed subset of $\Delta(A)$ then for every $m \in \Delta(A) \setminus E$ there is an element $x \in k_A(E)$ for which $m(x) \neq 0$. From this it follows that $m \in \Delta(A) \setminus h_A(k_A(E))$ and hence $E = h_A(k_A(E))$, so E is hk-closed and consequently the two topologies coincide.

It should be noted that whilst the Gelfand topology is always Hausdorff the hk-topology need not be Hausdorff. In fact, a classical result establishes that in any unital commutative Banach algebra the hk-topology is Hausdorff if and only if it coincides with the Gelfand topology, or equivalently A is regular, see Theorem 28.3 of Bonsall and Duncan [**72**].

In the sequel, in order to avoid confusion, all topological references on $\Delta(A)$ will be with respect to the Gelfand topology unless explicitly stated otherwise.

If A is a faithful commutative Banach algebra we can consider the hk-topology on $\Delta(A)$ and $\Delta(M(A))$. The next result show that if $\Delta(A)$ and $\Delta(M(A))$ are endowed with the hull-kernel topologies then $\Delta(A)$ is dense in $\Delta(M(A))$.

Theorem 4.24. *Let A be a semi-simple commutative Banach algebra . Suppose that $\Delta(A)$ and $\Delta(M(A))$ are endowed with the hk-topology. Then $\Delta(M(A)) = \Delta(A) \cup h_{M(A)}(A)$, $\Delta(A)$ is open and dense in $\Delta(M(A))$, whilst $h_{M(A)}(A)$ is compact.*

Proof Let us consider the following set

$$\Omega := \{\varphi \in \Delta(M(A)) : \varphi(L_x) \neq 0 \text{ for some } x \in A\}.$$

As we have already observed the set-theoretic decomposition $\Delta(M(A)) = \Omega \cup h_{M(A)}(A)$ holds and $\Delta(A)$ may be identified with Ω. It is evident that the set Ω is open and $h_{M(A)}(A)$ is compact in the hk-topology on $\Delta(M(A))$. We show now that the bijective mapping $\phi : m \in \Delta(A) \to m^* := \phi(m) \in \Omega$ described after Theorem 4.16 is also a homeomorphism if both $\Delta(A)$ and $\Delta(M(A))$ are provided with the hull-kernel topology.

Let W be any subset of $\Delta(A)$ and consider the set $W^* := \phi(W)$. It is easily seen that the kernel in A of W is the set:

$$k_A(W) = \{x \in A : L_x \in k_{M(A)}(W^*)\}. \tag{122}$$

Moreover, there is the following equivalence:

$$\ker m \supseteq k_A(W) \Leftrightarrow \ker m^* \supseteq k_{M(A)}(W^*). \tag{123}$$

From (122) and (123) it then follows that

$$\phi(h_A(k_A(W))) = \Omega \cap h_{M(A)}(k_{M(A)}(W^*))$$

and this clearly implies that the mapping ϕ is an homeomorphism of $\Delta(A)$ onto the set Ω with respect to the hk-topology. Finally, if we take $L_x \in k_{M(A)}(\Omega)$ we have

$$m^*(L_x) = m(x) = 0 \quad \text{for all } m \in \Delta(A)$$

and therefore $x = 0$. This implies that $k_{M(A)}(\Omega) = \{0\}$ and hence $\Delta(M(A)) = h_{M(A)}(k_{M(A)}(\Omega))$. Thus Ω is hk-dense in $\Delta(M(A))$. ∎

Theorem 4.25. *For a commutative faithful Banach algebra A the following statements are equivalent:*

(i) *A is semi-simple;*

(ii) *$M_{00}(A)$ is semi-simple;*

(iii) *$M_0(A)$ is semi-simple;*

(iv) *$M(A)$ is semi-simple.*

Moreover, if $M(A)$ or $M_0(A)$ is regular then A is regular.

Proof Since A is an ideal in $M_{00}(A)$, $M_{00}(A)$ an ideal in $M_0(A)$ and $M_0(A)$ an ideal in $M(A)$, from Bonsall and Duncan [**72**, Corollary 20, p.126] it follows that the implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold.

To show the implication (i) $\Rightarrow$(iv) let suppose that A is semi-simple and let $T \in M(A)$ such that the Gelfand transform $\widehat{T} = \widehat{0}$. Then $\widehat{Tx}(m) = \widehat{T}(m)\widehat{x}(m) = 0$ for every $m \in \Delta(A)$ and $x \in A$ which implies that $\widehat{Tx} = 0$ for all $x \in A$. Since A is semi-simple this implies $T = 0$, thus the Gelfand representation $T \to \widehat{T}$ of $M(A)$ into $C(\Delta(M(A))$ is injective and therefore the algebra $M(A)$ is semi-simple. Hence the statements (i), (ii), and (iii) are equivalent.

Now, to show the last statement suppose that $M_0(A)$ is regular. Let F be closed in $\Delta(A)$ and $m_0 \in \Delta(A) \setminus F$. Then by Theorem 4.20 F is closed in $\Delta(M_0(A))$, so there exists a multiplier $T \in M_0(A)$ such that

$$\widehat{T} \mid F \equiv 0 \quad \text{and} \quad \widehat{T}(m_0) = 1.$$

Let $x \in A$ such that $\widehat{x}(m_0) = 1$. Then we have $\widehat{Tx}(m) = 0$ and $(\widehat{Tx})(m_0) = 1$. This shows that also A is regular. The proof in the case that $M(A)$ is regular is similar. ∎

3. First spectral properties of multipliers

In this section we establish some basic properties of the spectrum of a multiplier. First we need to fix some preliminary notations on Banach algebras.

Let us denote by $\mathcal{A}$ any Banach algebra with unit u. The spectrum of an element $x \in \mathcal{A}$ with respect to $\mathcal{A}$ is canonically defined by

$$\sigma_{\mathcal{A}}(x) := \{\lambda \in \mathbb{C} : \lambda u - x \text{ is invertible in } \mathcal{A}\}.$$

We recall that a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be *inverse closed* if for every $x \in \mathcal{B}$ which has an inverse $x^{-1} \in \mathcal{A}$ then $x^{-1} \in \mathcal{B}$.

As usual, the spectrum of a bounded operator T, the spectrum of T with respect to the Banach algebra $L(A)$, will be denoted, as usual, by $\sigma(T)$.

Theorem 4.26. *If A is any faithful Banach algebra, the following statements holds:*

(i) *$M(A)$ is an inverse closed subalgebra of $L(A)$;*

(ii) *$\sigma(T) = \sigma_{M(A)}(T)$ for each $T \in M(A)$.*

Proof (i) Suppose that $T \in M(A)$ admits an inverse T^{-1} in $L(A)$. For each x, $y \in A$ we have then

$$\begin{aligned}(T^{-1}x)y &= T^{-1}T[(T^{-1}x)y] = T^{-1}[(TT^{-1}x)y] = T^{-1}(xy) \\ &= T^{-1}[x(TT^{-1}y)] = T^{-1}T[x(T^{-1}y)] = x(T^{-1}y),\end{aligned}$$

hence $T^{-1} \in M(A)$ and therefore the multiplier algebra $M(A)$ is an inverse closed subalgebra of $L(A)$.

(ii) This is immediate by part (i). ∎

Recall that for every $T \in L(X)$, where X is a Banach space, the *residual spectrum* of T is defined to be the set

$$\sigma_r(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is injective, } (\lambda I - T)(X) \text{ is not dense in } X\}.$$

The *continuous spectrum* $\sigma_{\mathrm{c}}(T)$ of a bounded operator T on a Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is injective and $(\lambda I - T)(X)$ is a proper non-dense subspace of X. Evidently the three spectra $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{ap}}(T)$ are disjoint.

In the following theorem we collect some basic relationships between these spectra.

Theorem 4.27. *Let $T \in L(X)$, where X is a Banach space. Then:,*

(i) *$\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$;*

(ii) *$\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T)$;*

(iii) *$\sigma_r(T) \subseteq \sigma_{\mathrm{p}}(T^*) \subseteq \sigma_{\mathrm{p}}(T) \cup \sigma_r(T)$.*

Proof The statements (i) and (ii) are immediate from definitions. To show (iii) assume that $\lambda \in \sigma_r(T)$. Then $\lambda I - T$ is one to one but $\overline{\lambda I - T)(X)} \neq X$. Hence there exists a non-zero $f \in X^*$ such that

$$f[(\lambda I - T)x] = [(\lambda I^* - T^*)f](x) = 0 \quad \text{for all } x \in X.$$

Therefore $T^* f = \lambda f$, and consequently $\lambda \in \sigma_{\mathrm{p}}(T^*)$. This shows the first inclusion of (iii).

To show the second inclusion let $\lambda \in \sigma_{\mathrm{p}}(T^*)$. If $\lambda I - T$ is not injective then $\lambda \in \sigma_{\mathrm{p}}(T)$. Suppose then that $\lambda I - T$ is injective. Since $\lambda \in \sigma_{\mathrm{p}}(T^*)$, there exists a non-zero $f \in X^*$ such that

$$[(\lambda I^* - T^*)f](x) = f[(\lambda I - T)x] = 0 \quad \text{for all } x \in X.$$

The last equality shows that $f(y) = 0$ for all $y \in \overline{(\lambda I - T)(X)}$ and hence $\overline{(\lambda I - T)(X)} \neq X$. In this case $\lambda \in \sigma_r(T)$, so the proof is complete ■

In the following theorem we give more information on the fine structure of the spectrum of a multiplier.

Theorem 4.28. *For every semi-simple commutative Banach algebra A the following statements hold:*

(i) $\sigma(T) = \widehat{T}(\Delta(M(A))$ *if* $T \in M(A)$, *and* $\sigma(T) = \widehat{T}(\Delta(M_0(A))$ *if* $T \in M_0(A)$;

(ii) $\sigma_p(T) \subseteq \widehat{T}(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_{\mathrm{r}}(T)$ *for every* $T \in M(A)$;

(iii) *If A has no unit and $T \in M_0(A)$ then*

$$\widehat{T}(\Delta(A)) \cup \{0\} = \overline{\widehat{T}(\Delta(A))} \subseteq \sigma(T);$$

(iv) *If A has no unit and $T \in M_{00}(A)$ then*

$$\sigma(T) = \widehat{T}(\Delta(A)) \cup \{0\} = \overline{\widehat{T}(\Delta(A))};$$

(v) $\sigma(T) = \overline{\widehat{T}(\Delta(A))}$ *if $\widehat{T}$ is hk-continuous on $\Delta(M(A))$.*

Proof (i) By Theorem 4.26 and the Gelfand theory we have $\sigma(T) = \sigma_{M(A)}(T) = \widehat{T}(\Delta(M(A))$.

To prove the second assertion denote by $[M_0(A)]_e$ the Banach algebra obtained by adjoining an unit to $M_0(A)$.

Let $T = \lambda I - S \in [M_0(A)]_e$, where $\lambda \in \mathbb{C}$ and $S \in M_0(A)$, and let us suppose that T is invertible in $M(A)$. Then by Theorem 4.26 $T^{-1} \in M(A)$ and since $M_0(A)$ is a proper ideal of $M(A)$ we also have $\lambda \neq 0$. From this we obtain

$$\begin{aligned} T^{-1} &= (\lambda I - S)^{-1} = \frac{1}{\lambda} I - [\frac{1}{\lambda} I - (\lambda I - S)^{-1}] \\ &= \frac{1}{\lambda} I - \frac{1}{\lambda}(\lambda I - S)^{-1}[(\lambda I - S) - \lambda I] \\ &= \frac{1}{\lambda} I + \frac{1}{\lambda}(\lambda I - S)^{-1} S \in [M_0(A)]_e, \end{aligned}$$

since $S \in M_0(A)$ and $\dfrac{1}{\lambda}(\lambda I - S)^{-1} S \in [M_0(A)]_e$. This shows that $[M_0(A)]_e$ is an inverse closed algebra of $M(A)$, and therefore for every $T \in M_0(A)$, we have

$$\sigma(T) = \sigma_{[M_0(A)]_e}(T) = \sigma_{M_0(A)}(T) = \widehat{T}(\Delta(M_0(A)).$$

(ii) First we prove the inclusion $\sigma_p(T) \subseteq \widehat{T}(\Delta(A))$. If λ is an eigenvalue of T then there exists an element $x \neq 0$ of A such that $(\lambda I - T)x = 0$. By hypothesis A is semi-simple, and hence there is an element $m_0 \in \Delta(A)$ such that $\widehat{x}(m_0) \neq 0$. From the equality $(\widehat{\lambda I - T})x = (\lambda - \widehat{T})\widehat{x} \equiv \widehat{0}$ we obtain now that $(\lambda - \widehat{T})(m_0) = 0$, and consequently $\widehat{T}(m_0) = \lambda$.

To prove the second inclusion let us consider the dual T^* of T. For each multiplicative linear functional $m \in \Delta(A)$ and for each $x \in A$ we have

$$T^*(m)(x) = m(Tx) = \widehat{Tx}(m) = \widehat{T}(m)\widehat{x}(m) = \widehat{T}(m)m(x),$$

so $T^*(m) = \widehat{T}(m)m$, and therefore $\widehat{T}(m)$ is an eigenvalue of T^*. The second inclusion then follows immediately from the second inclusion of part (ii) of Theorem 4.27.

(iii) If $T \in M_0(A)$ we have

$$\begin{aligned} \sigma(T) &= \widehat{T}(\Delta(M(A)) = \overline{[\widehat{T}(\Delta(M(A))]} \\ &\supseteq \overline{[\widehat{T}(\Delta(A)]} \supseteq \widehat{T}(\overline{\Delta(A))} = \widehat{T}(\Delta(A)) \cup \{0\}. \end{aligned}$$

(iv) The equalities follow easily because $\Delta(M(A)) = h_{M(A)}(A) \cup \Delta(A)$, by Theorem 4.18 and $\widehat{T} \mid h_{M(A)}(A) \equiv 0$, by definition.

(v) This is clear from part (i) and Theorem 4.24. ∎

A natural question is whether the two sets $M(A)$ and $[M_0(A)]_e$ coincide. The following simple example shows that, generally, these two algebras may be different.

Example 4.29. Let us consider the Banach algebra $C_0(\Omega)$ of all continuous complex valued functions which vanish at infinity on a locally compact Hausdorff space Ω. We show first that $M(C_0(\Omega))$ is isometrically isomorphic to the Banach algebra $C_b(\Omega)$ of all bounded continuous complex-valued functions defined on Ω, endowed with the supremum-norm $\|\cdot\|_\infty$.

Clearly, for each $g \in C_b(\Omega)$ the operator defined by

$$T_g(f) := gf \quad \text{for all } f \in C_0(\Omega)$$

is a multiplier on $C_0(\Omega)$. Conversely, let $T \in M(C_0(\Omega))$ be arbitrary. The classical Urysohn lemma ensures that for every $\omega_0 \in \Omega$ there exists $f \in C_0(\Omega)$ such that $f(\omega) = 1$ and $0 \le f \le 1$ on Ω. Clearly $f_1 T f_2 = f_2 T f_1$ for all $f_1, f_2 \in C_0(\Omega)$, so if we define

$$g(\omega) := (Tf)(\omega)/f(\omega) \quad \text{whenever } f(\omega) \neq 0$$

then $g(\omega)$ is independent of $f \in C_0(\Omega)$. The function g so defined is continuous on Ω and being T continuous, by Theorem 4.3, we also have $\|g\|_\infty \le T$, and hence $g \in C_b(\Omega)$. It is easily seen that T is the multiplication operator by g, so the mapping $g \in C_b(\Omega) \to T_g$ defines an isometric isomorphism from $C_b(\Omega)$ onto the multiplier algebra $M(C_0(\Omega))$.

A similar argument shows that $M_0(C_0(\Omega))$ can be identified with the same space $C_0(\Omega)$. Let $\beta\Omega$ be the Stone–Čech compactification of Ω. By a standard result of functional analysis we have that $C_b(\Omega) = C(\beta\Omega)$. A classical result of Čech establishes that no point of $\beta\Omega = \Omega$ is a G_δ in $\beta\Omega$, so that the set $\beta\Omega = \Omega$ contains uncountably many points. From that it easily follows that the multiplier algebra $M(C_0(\Omega))$ contains $[M_0(C_0(\Omega))]_e$ properly.

A standard result on Hilbert spaces establishes that every normal operator has empty residual spectrum. The following example shows that a similar property, in general, does not hold for arbitrary multipliers of semi-simple Banach algebras.

Example 4.30. Let $A := C([0,1])$ denote the uniform Banach algebra of all continuous complex-valued functions on the unit interval. Let us consider the operator $T \in M(A)$ of multiplication by the independent variable,

$$(Tf)(\lambda) := \lambda f(\lambda) \quad \text{for all } f \in A,\ \lambda \in [01].$$

Evidently

$$\sigma(T) = \sigma_{\mathrm{r}}(T) = \widehat{T}([0,1]) = [0,1],$$

and it is easy to see that $\sigma_{\mathrm{p}}(T) = \varnothing$. Indeed, if $(\lambda - \omega)f(\lambda) = 0$ for some $\omega \in [0,1]$ then $f(\lambda) = 0$ for all λ.

The example above also shows that the first inclusion of part (ii) of Theorem 7.79 may be strict. In the following result we give further information, on the point spectrum $\sigma_{\mathrm{p}}(T)$ of a multiplier $T \in M(A)$ whenever we assume that the maximal ideal space $\Delta(A)$ is discrete.

Theorem 4.31. *Let A be a commutative semi-simple Banach algebra with discrete maximal ideal space $\Delta(A)$. Then*

$$\sigma_{\mathrm{p}}(T) = \widehat{T}(\Delta(A)) \quad \textit{for every } T \in M(A).$$

Proof By Theorem 7.79 the inclusion $\sigma_{\mathrm{p}}(T) \subseteq \widehat{T}(\Delta(A))$ is true for every multiplier, so we have only to prove the inclusion $\widehat{T}(\Delta(A)) \subseteq \sigma_{\mathrm{p}}(T)$. Since $\Delta(A)$ is a discrete space the Shilov idempotent theorem ensures that A is a regular algebra, and hence if m_0 is a fixed multiplicative functional there exists an element $x \in A$ such that $\widehat{x}(m_0) = 1$ and $\widehat{x}$ vanishes identically in the set $\Delta(A) \setminus \{m_0\}$. From this we obtain

$$\widehat{[(\widehat{T}(m_0)I - T)x]}\ (m) = [\widehat{T}(m_0) - \widehat{T}(m)]\widehat{x}(m) = 0$$

for every $m \in \Delta(A)$. Therefore $[(\widehat{T}(m_0)I - T]x = 0$ and because $x \neq 0$ we conclude that $\widehat{T}(m_0) \subseteq \sigma_{\mathrm{p}}(T)$, so $\sigma_{\mathrm{p}}(T) = \widehat{T}(\Delta(A))$. ∎

In the next chapter we shall prove that the residual spectrum is empty for every multiplier T acting on a semi-simple commutative Banach algebra A which have dense socle.

The next result establishes elementary properties of a multiplier from the point of view of local spectral theory. In particular, it is shown that every multiplier of a semi-prime, not necessarily commutative, Banach algebra has SVEP, so that many results of the previous chapters apply.

Theorem 4.32. *Let A be a semi-prime Banach algebra and $T \in M(A)$. Then:*

(i) $p(T) \leq 1$. *Moreover,* $\ker T \cap T(A) = \{0\}$*;*

(ii) *T has the SVEP;*

(iii) $\sigma(T) = \sigma_{\rm su}(T)$ *and* $\sigma_{\rm se}(T) = \sigma_{\rm ap}(T)$*;*

(iv) $E_T(\Omega) = \bigcap_{\lambda \notin \Omega} (\lambda I - T)^\infty(A)$ *for every* $\Omega \subset \mathbb{C}$*.*

Proof (i) We have to show that $\ker T = \ker T^2$ for every $T \in M(A)$. The inclusion $\ker T \subseteq \ker T^2$ is true for every linear operator on a vector space.

To show the reverse inclusion let $x \in \ker T^2$ and set $z := Tx$. Obviously $z \in \ker T$ and

$$zaz = (Tx)az = xa(Tz) = 0$$

for every $a \in A$. From this it follows that $zAzA = (zA)^2 = \{0\}$, hence since A is semi-prime $zA = \{0\}$, which implies $z = Tx = 0$. This proves the inclusion $\ker T^2 \subseteq \ker T$.

The equality $\ker T \cap T(A) = \{0\}$ is an obvious consequence of Lemma 3.2.

(ii) Since $\lambda I - T \in M(A)$ for every $\lambda \in \mathbb{C}$, from the first part we obtain that $p(\lambda I - T) \leq 1$ for every $\lambda \in \mathbb{C}$. This condition ensures by Theorem 5.4 that T has the SVEP at every $\lambda \in \mathbb{C}$.

(iii) This is a consequence of the the SVEP, by Corollary 2.45.

(iv) The condition $p(\lambda I - T) \leq 1$ implies that the algebraic core $C(\lambda I - T) = (\lambda I - T)^\infty(A)$ for all $\lambda \in \mathbb{C}$, by Theorem 3.5. Hence

$$E_T(\Omega) = \bigcap_{\lambda \notin \Omega} C(\lambda I - T) = \bigcap_{\lambda \notin \Omega} (\lambda I - T)^\infty(A)$$

for every $\Omega \subset \mathbb{C}$. ∎

Note that if a faithful commutative Banach algebra A is not semi-prime and $T \in M(A)$ then the inequality $p(T) \leq 1$ need not to be true. In fact, since A is not semi-prime we know that there exists a nilpotent element $0 \neq a \in A$. Let denote by L_a the corresponding multiplication operator and let $n \in \mathbb{N}$ be the smallest natural number for which $a^n = 0$. Clearly $n > 1$ and $a^{n-1} \neq 0$. Since A is faithful we have

$$\ker (L_a)^{n-1} = \ker L_{a^{n-1}} \neq \{0\} \quad \text{for all } n \in \mathbb{N},$$

whilst $\ker\, {L_a}^k = \{0\}$ for all $k \geq n$. Therefore $p(L_a) = n > 1$. This argument shows that a faithful commutative Banach algebra A is semi-prime precisely when $p(T) \leq 1$ for all $T \in M(A)$.

Part (iii) of Theorem 4.32 shows that for a multiplier $T \in M(A)$, A a semi-prime Banach algebra, we have the following implication:

$$T \text{ surjective } \Rightarrow \text{ semi-regular } \Leftrightarrow T \text{ bounded below.}$$

Note that all these implications may be proved directly, and in a simpler way, by arguing as follows: since $T(A) \cap \ker\, T = \{0\}$ the surjectivity of T trivially implies the closedness of $T(A)$ and the injectivity of T. Moreover, if T is semi-regular T has by definition a closed range and $\ker\,(T) \subseteq T(A)$, thus T is injective. The converse is true for all operators $T \in L(X)$, X a Banach space, so the implications above described are proved.

Later we shall see that also the first implication is an equivalence whenever T is a multiplier of a semi-simple regular Tauberian commutative Banach algebra A.

Theorem 4.33. *Let A be a semi-simple Banach algebra and $T \in M(A)$, Then*

$$X_T(\{0\}) = H_0(T) = \ker\, T.$$

Proof The first equality is clear from Theorem 2.20, since T has the SVEP. We know that $\ker\, T \subseteq H_0(T)$, so it remains to prove the inverse inclusion.

Suppose that $x \in H_0(T)$. By an easy inductive argument we have

$$(Ty)^n = (T^n y)y^{n-1} \quad \text{for every } y \in A \text{ and } n \in \mathbb{N}.$$

From that it follows

$$\begin{aligned} \|(aTx)^n\| &= \|(Tax)^n\| = \|T^n(ax)(ax)^{n-1}\| \\ &\leq \|a\|\|T^n x\|\|(ax)^{n-1}\| \end{aligned}$$

for every $a \in A$, so the spectral radius of the element aTx is

$$r(aTx) = \lim_{n\to\infty} \|(aTx)^n\|^{1/n} = 0$$

for every $a \in A$. This implies that $Tx \in \text{rad}\, A$ (see Bonsall and Duncan [**72**, Proposition 1, page 126]). Since A is semi-simple $Tx = 0$, hence $x \in \ker T$, and consequently $H_0(T) \subseteq \ker\, T$, which concludes the proof. ∎

Corollary 4.34. *Let A be a semi-simple Banach algebra and $T \in M(A)$. Then T is quasi-nilpotent if and only if $T = 0$.*

Proof Suppose $T \in M(A)$ quasi-nilpotent. Combining Theorem 4.33 and Theorem 1.68 we have $A = H_0(T) = \ker\, T$ and hence $T = 0$. ∎

The following example shows that the assumption that A is semi-simple in Theorem 4.33 cannot replaced by the weaker assumption that A is semi-prime.

Example 4.35. Let $\omega := (\omega_n)_{n\in\mathbb{N}}$ be a sequence with the property that

$$0 < \omega_{m+n} \le \omega_m \omega_n \quad \text{for all } m, n \in \mathbb{N}.$$

Let $\ell^1(\omega)$ denote the space of all complex sequences $x := (x_n)_{n\in\mathbb{N}}$ for which $\|x\|_\omega := \sum_{n=0}^\infty \omega_n |x_n| < \infty$. The space $\ell^1(\omega)$ equipped with convolution

$$(x \star y)_n := \sum_{j=0}^n x_{n-j} y_j \quad \text{for all } n \in \mathbb{N} ,$$

is a commutative unital Banach algebra with unit $e_1 = (1, 0, \ldots)$. Therefore the multiplier algebra $M(\ell^1(\omega))$ may be identified with the same algebra $\ell^1(\omega)$. Let A_ω denote the maximal ideal of $\ell^1(\omega)$ given by

$$A_\omega := \{(x_n)_{n\in\mathbb{N}} \in \ell^1(\omega) : x_0 = 0\}.$$

The Banach algebra A_ω is, obviously, an integral domain and hence semi-prime. Consider the standard basis (e_n), where $e_n := (\delta_{nj})_{j\in\mathbb{Z}}$. It is easily seen that ${e_1}^n = e_n$, so e_1 generates A_ω. From the estimate $\|e_1\|_\omega = \omega_n$ for all $n \in \mathbb{N}$, we obtain that the spectral radius of e_1 is given by the limit

$$\rho_\omega := \lim_{n\to\infty} \omega_n^{1/n} = \inf_{n\in\mathbb{N}} \omega_n^{1/n}.$$

From the Gelfand theory we know that the set Q_ω of quasi-nilpotent elements of a commutative A is the kernel of the Gelfand transform, so that Q_ω is a closed subalgebra of A. These facts imply that A_ω is a radical algebra, *i.e.* A_ω coincides with its radical, if and only if $\rho_\omega = 0$.

It can been shown that for A_ω we have the following dichotomy: A_ω is radical or is semi-simple and the last case occurs precisely when $\rho_\omega > 0$, for more information and details see Example 4.1.9 of Laursen and Neumann [**214**]).

Now fix $0 \ne a \in A_\omega$ and let $T_a(x) := a \star x$, $x \in A_\omega$, denote the multiplication operator by the element a. From the estimate

$$\|T^n x\|^{1/n} = \|a^n \star x\|^{1/n} \le \|a^n\|^{\frac{1}{n}} \|x\|^{1/n}$$

we see that T_a is quasi-nilpotent. Thus $H_0(T_a) = A_\omega$, by Theorem 1.68. On the other hand, A_ω is an integral domain so that $\ker T_a = \{0\}$.

The poles of a multiplier of a semi-simple Banach algebra may be characterized in a very simple way. Recall that a spectral point λ_0 of $T \in L(X)$, where X is a Banach space, is a *simple pole* of the resolvent $R(\lambda, T)$ if λ_0 is a pole of order 1. This is equivalent to saying that $p(\lambda_0 I - T) = q(\lambda_0 I - T) = 1$, see Remark 3.7, part (f). By Theorem 4.32 a pole of a multiplier of a semi-simple Banach algebra is then necessarily simple.

The next result shows that the poles of a multiplier of a semi-simple Banach algebra are precisely the isolated points of $\sigma(T)$.

Theorem 4.36. *Let $T \in M(A)$, A a semi-simple Banach algebra. Then λ_0 is a simple pole of $R(\lambda, T)$ if and only if λ_0 is an isolated point of the spectrum. In particular, every isolated point of $\sigma(T)$ is an eigenvalue of T.*

Proof Clearly we have only to prove that if λ_0 is an isolated point of the spectrum then $p(\lambda_0 I - T) = q(\lambda_0 I - T) = 1$, or equivalently, see Theorem 3.6, that

$$A = (\lambda_0 I - T)(A) \oplus \ker\,(\lambda_0 I - T).$$

Let P_0 be the spectral projection associated with $\{\lambda_0\}$. The spectral projection P_0 generates the decomposition

$$A = M_0 \oplus N_0 \text{ where } M_0 := P_0(A) \text{ and } N_0 := \ker\, P_0.$$

The subspaces M_0 and N_0 are invariant under T, the spectrum of $T\,|M_0$ is $\{\lambda_0\}$ and the spectrum of $T\,|N_0$ is $\sigma(T) \setminus \{\lambda_0\}$. By Theorem 4.33 and Theorem 3.74 we have

$$M_0 = H_0(\lambda_0 I - T) = \ker\,(\lambda_0 I - T).$$

Since the restriction $\lambda_0 I - T\,|N_0$ is invertible we also have

$$N_0 = (\lambda_0 I - T)(N_0) \subseteq (\lambda_0 I - T)(A).$$

Thus

$$A = (\lambda_0 I - T)(A) + \ker\,(\lambda_0 I - T),$$

and this sum is direct, from part (i) of Theorem 4.32.

Note that, the fact that every isolated point of the spectrum is a pole of the resolvent also follows from Theorem 3.96 and Theorem 4.33. ■

The SVEP of a multiplier implies by part (i) of Corollary 3.53 that the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ coincide. Generally these two spectra differ from the Fredholm spectrum $\sigma_f(T)$, as the following example shows.

Example 4.37. Let L_z be the multiplication operator defined on the disc algebra $A := \mathcal{A}(\mathbb{D})$ by pointwise multiplication by the independent variable z. It is easily seen that $\sigma_w(L_z) = \sigma_b(L_z)$ is the closed unit disc $\mathbf{D}$, whilst the Fredholm spectrum $\sigma_f(L_z)$ is the boundary of $\mathbf{D}$. This example also shows by part (iii) of Corollary 3.53 that in general the dual $T^\star$ of a multiplier need not have the SVEP.

The following result gives a precise description of the isolated points λ of $\sigma(T)$ which do not belong to the Fredholm spectrum $\sigma_f(T)$.

Theorem 4.38. *Let $T \in M(A)$, where A is a semi-simple commutative Banach algebra, and suppose that $\lambda_0 \in \mathbb{C}$ is isolated in $\sigma(T)$. Then the following statements are equivalent:*

(i) *$\lambda_0 I - T$ is Fredholm;*

(ii) *$\alpha(\lambda_0 I - T) < \infty$;*

(iii) *$\beta(\lambda_0 I - T) < \infty$;*

(iv) *The set $\{m \in \Delta(A) : \widehat{T}(m) = \lambda\}$ is finite.*

Proof The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose that $\alpha(\lambda_0 I - T) < \infty$ and that λ_0 is isolated in $\sigma(T)$. From Theorem 4.36 we know that $p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$ and this implies by Theorem 3.4 that $\alpha(\lambda_0 I - T) = \beta(\lambda_0 I - T)$, so $(\lambda_0 I - T)(A)$ is finite-codimensional.

(iii) $\Rightarrow$ (iv) If the range $(\lambda_0 I - T)(A)$ is finite-codimensional the hull in A of $(\lambda_0 I - T)(A)$,

$$h_A((\lambda_0 I - T)(A)) = \{m \in \Delta(A) : m(y) = 0 \text{ for every } y \in (\lambda_0 I - T)(A)\},$$

is a finite set and evidently coincides with the set

$$\Gamma := \{m \in \Delta(A) : \widehat{T}(m) = \lambda_0\}.$$

(iv) $\Rightarrow$ (i) Assume that $\Gamma := \{m \in \Delta(A) : \widehat{T}(m) = \lambda_0\}$ is a finite set. Then the points of Γ are clopen in $\Delta(A)$, and obviously Γ supports the set $\ker(\lambda_0 I - T)$, so $\ker(\lambda_0 I - T)$ is finite-dimensional. The point λ_0 being isolated in $\sigma(T)$, the condition $p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$ then implies, again by Theorem 3.4, that the range $(\lambda_0 I - T)(A)$ is finite-codimensional, so $\lambda_0 I - T$ is Fredholm. ∎

Corollary 4.39. *Let A be a semi-simple commutative Banach algebra. Then $T \in M(A)$ is a Riesz operator if and only if each $\lambda \in \sigma(T) \setminus \{0\}$ is isolated and the set $\{m \in \Delta(A) : \widehat{T}(m) = \lambda\}$ is a finite set.*

Proof Each $\lambda \neq 0$ of the spectrum of a Riesz operator is an isolated point of $\sigma(T)$, so the assertion is immediate by Theorem 4.38. ∎

4. Multipliers of group algebras

In this section we shall consider examples of multipliers of some important classes of commutative semi-simple Banach algebras. For some of these algebras we shall see that it is possible to construct a model for the algebra $M(A)$. Moreover, for these algebras we shall characterize the ideal $K_M(A)$ in $M(A)$ of all multipliers which are compact operators. These characterizations will be useful in the subsequent chapter in order to describe the multipliers which are Fredholm operators or Riesz operators.

The first Banach algebra which we shall consider is the so called *group algebra* $L_1(G)$, which, in a sense, is the paradigm of many of the results on multiplier theory. We shall review briefly, without proofs, some of the basic results about this important algebra and shall refer for a full treatment to the classical monographs by Hewitt and Ross [**161**], [**162**], Rudin [**282**], Loomis [**223**] and Gaudry [**123**].

Let G be a locally compact Abelian group and suppose that λ is a Haar measure on G. Let $L_1(G)$ denote the so called *group algebra* of all equivalence classes of complex valued functions which are absolutely integrable with respect to λ. As usual, we shall not distinguish between an integrable function f on G and the equivalence class in $L_1(G)$ to which f belongs. We recall that $L_1(G)$ under the norm

$$\|f\| := \int_G |f(t)| \, \mathrm{d}\lambda(t) \quad \text{for every } f \in L_1(G)$$

and with the pointwise operations is a Banach space. Moreover, $L_1(G)$ with the usual convolution defined by

$$(f * g)(s) := \int_G f(st)g(t^{-1}) \, \mathrm{d}\lambda(t) \quad \text{for } f, g \in L_1(G)$$

as multiplication is a semi-simple regular commutative Banach algebra. Moreover, $L_1(G)$ possesses a unit if and only if G is discrete, see Rudin [**282**, 1.1.8], or Loomis [**223**].

Let $L_p(G)$, $1 < p < \infty$, denote the space of all equivalence classes of measurable complex valued functions on G whose p-th powers are absolutely integrable with respect to λ. Also here we do not distinguish between an integrable function and the corresponding equivalence class. The space $L_p(G)$ under the norm

$$\|f\|_p := \left(\int_G |f(t)|^p d\lambda(t) \right)^{\frac{1}{p}} \quad \text{for every } f \in L_p(G)$$

and with the pointwise operations is a Banach space. In general, if G is not compact and $1 < p < \infty$, the convolution product of two elements f, $g \in L_p(G)$ need not belong to $L_p(G)$, thus $L_p(G)$ is not an algebra. This is not the case of a compact Abelian group. Indeed, for a compact Abelian group G it is not hard to check that $L_p(G)$ is a semi-simple regular commutative Banach algebra for every $1 \leq p < \infty$. Moreover, $L_p(G)$, for $1 < p < \infty$, is a closed ideal of $L_1(G)$.

Finally, let $L_\infty(G)$ denote, for an arbitrary locally compact Abelian group G, the space of equivalence classes of essentially bounded measurable complex functions on G. It is easily seen that $L_\infty(G)$ is a semi-simple commutative Banach algebra with respect to pointwise operations. Moreover, if G is a compact Abelian group, $L_\infty(G)$ is also a semi-simple regular commutative Banach algebra under convolution as multiplication, and precisely $L_\infty(G)$ is a closed ideal in $L_1(G)$.

Although the group algebra $L_1(G)$, G a locally compact Abelian group, does not have a unit unless G is discrete, it always admits a minimal approximate identity. If G is compact, also the algebra $L_p(G)$, where $1 < p < \infty$, does always possess an approximate identity, but this never is minimal unless G is finite, see Hewitt and Ross [**161**].

Note that if G is compact and $1 \leq p < \infty$ then there exist approximate identities consisting of *trigonometric polynomials*. These are defined to be the finite linear combinations of continuous characters and denoting by $P(G)$ this set of polynomials a remarkable property is that $P(G)$ is norm dense in $L_p(G)$, [**282**].

In order to describe the multiplier algebra for group algebras we recall some well known results of Gelfand theory about these algebras. The regular maximal ideal space $\Delta(A)$ of $A = L_1(G)$, G a locally compact Abelian group, or of $A = L_p(G)$, $1 \leq p \leq \infty$, G a compact Abelian group, is easy to describe. In fact, in both cases $\Delta(A)$ corresponds precisely to the so called *continuous characters* on G, where a continuous character γ is a continuous group homomorphism of G onto the circle group $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. This is a consequence of the result that for every multiplicative functional $m \in \Delta(A)$, where A is one of these algebras, there exists a continuous character γ on G such that

$$m(f) = \int_G (t^{-1}, \gamma) f(t) \, \mathrm{d}\lambda(t) \quad \text{for every } f \in A,$$

where (t^{-1}, γ) denotes the values of γ at $t^{-1} \in G$.

Let $\widehat{G}$ be the collection of all such homomorphisms. For each real $\varepsilon > 0$, let $\mathcal{U}_\varepsilon : \{\lambda \in \mathbb{T} : |\lambda - 1| < \varepsilon\}$, and consider for every compact subset K of G the set

$$N(K, \varepsilon) := \{\gamma \in \widehat{G} : (t, \gamma) \in \mathcal{U}_\varepsilon \text{ for all } t \in K\}.$$

The family $\{N(K, \varepsilon)\}$ and their translates form a basis for a topology on $\widehat{G}$. With respect to this topology $\widehat{G}$ becomes a locally compact Abelian group, called the *dual group* of G. Hence we have the following two important relations :

$$\Delta(L_1(G)) = \widehat{G} \quad \text{for every locally compact abelian group } G$$

and

$$\Delta(L_p(G)) = \widehat{G}, \quad 1 \leq p \leq \infty, \quad \text{for every compact abelian group } G.$$

Moreover, for these algebras the Gelfand topology on the regular maximal ideal space $\Delta(A) = \widehat{G}$ coincides with the topology mentioned above on the dual group $\widehat{G}$. Observe that G is compact (respectively, discrete) if and only if $\widehat{G}$ is discrete (respectively, compact), see Hewitt and Ross [**161**, Theorem 23.17].

The Gelfand transform $\widehat{f}$ of an element in anyone of these algebras is the so called *Fourier transform* of f defined by

$$\widehat{f}(\gamma) := \int_G (t^{-1}, \gamma) f(t) \, \mathrm{d}\ \lambda(t), \quad \gamma \in \widehat{G}.$$

In the cases of $G = \mathbb{R}$, $G = \mathbb{T}$, $\mathbb{T}$ the circle group, and $G = \mathbb{Z}$ we obtain some standard concepts of classical Fourier analysis. Precisely,

(1) If $G = \mathbb{R}$ then $\widehat{G} = \mathbb{R}$ and

$$\widehat{f}(x) = \int_{\mathbb{R}} f(t) e^{-ixt} \, \mathrm{d}t \quad \text{for every } x \in \mathbb{R}.$$

(2) If $G = \mathbb{T}$ then $\widehat{G} = \mathbb{Z}$ and

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \, \mathrm{d}\theta \quad \text{for every} \quad n \in \mathbb{Z}.$$

(3) If $G = \mathbb{Z}$ then $\widehat{G} = \mathbb{T}$ and

$$\widehat{f}(z) = \sum_{-\infty}^{+\infty} f(n) z^{-n} \quad \text{for every } z \in \mathbb{T}.$$

Let $\mathcal{M}(G)$ denote the space of all regular complex valued Borel measures on G with finite total mass. The set $\mathcal{M}(G)$ with respect to the pointwise operation and endowed with the total variation norm given by $\|\mu\| := |\mu|(G)$ is a Banach space. If one defines, for μ, $\nu \in \mathcal{M}(G)$, the convolution product $\mu * \nu$ by

$$(\mu * \nu)(E) := \int_G \nu(Es^{-1}) \, \mathrm{d}\mu(s) \quad \text{for all Borel sets } E \subseteq G,$$

then $\mathcal{M}(G)$ is a commutative Banach algebra called the *measure algebra* of G. For fixed $g \in G$ define $u_g(E) := 1$ or 0 according as $g \in E$ or $g \notin E$. It is easy to see that if 1 denotes the identity of G then $u_1 * \mu = \mu$ for all $\mu \in \mathcal{M}(G)$. Hence u_1 is a unit for $\mathcal{M}(G)$, the so called *Dirac measure* concentrated at the identity of G.

For every measure $\mu \in \mathcal{M}(G)$ the *Fourier–Stieltjes transform* is the function $\widehat{\mu} : \widehat{G} :\to \mathbb{C}$ defined by

$$\widehat{\mu}(\gamma) := \int_G (t^{-1}, \gamma) \, \mathrm{d}\mu(t) \quad \text{for all } \gamma \in \widehat{G}.$$

The Fourier–Stieltjes transform is bounded and uniformly continuous on $\widehat{G}$ and satisfies the important equality

$$\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu} \quad \text{for all } \mu, \nu \in \mathcal{M}(G).$$

If we define $m_\gamma(\mu) := \widehat{\mu}(\gamma)$ for each $\mu \in \mathcal{M}(G)$ then m_γ is a non-trivial multiplicative functional on $\mathcal{M}(G)$, so the dual group $\widehat{G}$ may be embedded into the maximal ideal space $\Delta(\mathcal{M}(G))$. From the uniqueness theorem for the Fourier–Stieltjes transform it then follows that $\mu = 0$ whenever $\widehat{\mu} \equiv 0$ on $\widehat{G}$. This of course implies that $\mathcal{M}(G)$ is semi-simple.

The regular maximal deal space $\Delta(\mathcal{M}(G))$ is considerable larger than $\Delta(L_1(G)) = \widehat{G}$, whenever G is non-discrete. This is a consequence of an important classical result of harmonic analysis, known as the *Wiener Pitt phenomenon*, which ensures that on a non-discrete locally Abelian group there always exists a non-invertible measure $\mu \in \mathcal{M}(G))$ such that $|\widehat{\mu}| \geq 1$, see Rudin [**282**] or Graham and McGehee [**140**].

If for $f \in L_1(G)$ we define

$$\mu_f(E) := \int_E f(t) d\lambda(t) \quad \text{for every Borel set } E \subset G$$

then the mapping $f \to \mu_f$ is an isometric isomorphism of the algebra $L_1(G)$ into the algebra $\mathcal{M}(G)$. For this reason $L_1(G)$ may be regarded as a subalgebra of $\mathcal{M}(G)$. Precisely, the well known Radon-Nikodym theorem establishes that $L_1(G)$ may be identified with the closed ideal of $\mathcal{M}(G)$ of all measures which are absolutely continuous with respect to λ, see Rudin [**282**].

The algebra of multipliers of the group algebra $L_1(G)$, G a locally compact Abelian group, may be characterized in a precise way. First we recall that to each $\mu \in \mathcal{M}(G)$, G a locally compact Abelian group, there corresponds a *convolution operator* $T_\mu : L_1(G) \to L_1(G)$ defined by

$$T_\mu(f) := \mu * f \quad \text{for every } f \in L_1(G).$$

Trivially

$$T_\mu(f * g) = (T_\mu f) * g = f * (T_\mu g) \quad \text{for every } f, g \in L_1(G),$$

so every convolution operator is a multiplier of $L_1(G)$ and therefore a bounded linear operator by Theorem 4.3.

The following crucial result has been established independently by Wendel [**319**] and Helson [**157**].

Theorem 4.40. *Let G be a locally compact Abelian group. For a bounded operator $T : L_1(G) \to L_1(G))$ the following statements are equivalent:*

(i) *$T \in M(L_1(G))$;*

(ii) *There exists a unique measure $\mu \in \mathcal{M}(G)$ such that $T = T_\mu$;*

Moreover, $M(L_1(G))$ is isometrically isomorphic to $\mathcal{M}(G)$.

Proof The implication (ii) $\Rightarrow$ (i) is clear, so we need only to prove the converse implication (i) $\Rightarrow$ (ii). Suppose that $T \in M(L_1(G))$. By Theorem 4.14 there is a unique function φ defined on $\Delta(L_1(G)) = \widehat{G}$ such that $(Tf)^{=}\varphi\widehat{f}$ for every $f \in L_1(G)$, and hence $\varphi\widehat{f} \in \widehat{L_1(G)}$ whenever $\widehat{f} \in \widehat{L_1(G)}$. Define $\|f\|_1 := \|\widehat{f}\|$. Then $\widehat{L_1(G)}$ provided with the norm $\|\cdot\|_1$ becomes a commutative Banach algebra under pointwise multiplication.

Define the linear mapping $S : \widehat{L_1(G)} \to \widehat{L_1(G)}$ by $S\widehat{f} := \varphi\widehat{f}$. Choose $f_n, f, g \in L_1(G)$ such that

$$\lim_{n\to\infty} \|\widehat{f_n} - \widehat{f}\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi\widehat{f_n} - \widehat{g}\|_1 = 0.$$

From $\|\widehat{f}\|_\infty \le \|f\|_1$ we obtain, for every character $\gamma \in \widehat{G}$, that

$$\widehat{g}(\gamma) = \lim_{n\to\infty} \varphi(\gamma)\widehat{f_n}(\gamma) = \varphi(\gamma)\widehat{f}(\gamma),$$

so that S is a closed mapping. The closed graph theorem then implies that S is bounded, and hence there is $K > 0$ such that $\|S\widehat{f}\| = \|\varphi\widehat{f}\| \le K\widehat{f}\|$ for all $f \in L_1(G)$.

Let $\gamma_1, \gamma_2, \cdots, \gamma_n \in \widehat{G}$, $\varepsilon > 0$ and choose $f \in L_1(G)$ such that $\|\widehat{f}\| = \|f\|_1 < 1 + \varepsilon$, and $\widehat{f}(\gamma_k) = 1$ for $k = 1, 2, \ldots, n$. A such choice is always possible by Theorem 2.6.1 of Rudin [**282**]. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be arbitrary chosen in $\mathbb{C}$ and put $\widehat{g} := \varphi\widehat{f}$. Then

$$\begin{aligned}\left|\sum_{k=1}^{n} \lambda_k \varphi(\gamma_k)\right| &= \left|\sum_{k=1}^{n} \lambda_k \widehat{g}(\gamma_k)\right| = \left|\int_G \left[\sum_{k=1}^{n} (\lambda_k (t^{-1}, \gamma_k)\right] g(t)\, \mathrm{d}\lambda(t)\right| \\ &\leq \|g\|_1 \left\|\sum_{k=1}^{n} \lambda_k (\cdot, {\gamma_k}^{-1})\right\|_\infty < K(1+\varepsilon) \left\|\sum_{k=1}^{n} \lambda_k (\cdot, {\gamma_k}^{-1})\right\|_\infty .\end{aligned}$$

By a well known characterization of Fourier–Stieltjes transforms, see Rudin [**282**, Theorem 1.9.11], from these estimates it follows that there exists a unique $\mu \in \mathcal{M}(G)$ such that $\varphi = \mu$. Since $L_1(G)$ in $\mathcal{M}(G)$ we then conclude that $T = T_\mu$.

Evidently the equation $Tf = \mu * f$ defines a bijective isomorphism between $M(L_1(G)$ and $\mathcal{M}(G)$. Moreover, $\|T_\mu\| \leq \|\mu\|$, so to prove the last assertion we need only to prove that $\|\mu\| \leq \|T_\mu\|$.

If $\gamma_1, \ldots, \gamma_n$ in $\widehat{G}$, $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, and $\varepsilon > 0$ are arbitrarily given, choose $f \in L_1(G)$ as above. Then

$$\begin{aligned}\left|\sum_{k=1}^{n} \lambda_k \widehat{\mu}(\gamma_k)\right| &= \left|\sum_{k=1}^{n} \lambda_k \widehat{f * \mu}(\gamma_k)\right| = \left|\sum_{k=1}^{n} \lambda_k \widehat{T_\mu f}(\gamma_k)\right| \\ &\leq \|T\|(1+\varepsilon) \left\|\sum_{k=1}^{n} \lambda_k (\cdot, {\gamma_k}^{-1})\right\|_\infty ,\end{aligned}$$

and therefore because ε is arbitrary

$$\left|\sum_{k=1}^{n} \lambda_k \widehat{\mu}(\gamma_k)\right| \leq \|T\| \left\|\sum_{k=1}^{n} \lambda_k (\cdot, {\gamma_k}^{-1})\right\|_\infty . \tag{124}$$

Again from Theorem 1.9.1 of Rudin [**282**] it follows that $\|\mu\| \leq \|T_\mu\|$, and this concludes the proof. ■

The richness of the structure of group algebras is the main reason for the possibility of characterizing the multipliers of $L_1(G)$ in several equivalent ways. For instance, if τ_s denotes the *translation operator* on $L_1(G)$ defined by

$$\tau_s(f)(t) := f(ts^{-1}) \quad \text{for each } f \in L_1(G)$$

then a bounded operator T on $L_1(G)$ is a multiplier if and only if T commutes with the translation operators τ_s for each $s \in G$. The proof of this result, as well as of some other characterizations of multipliers of $L_1(G)$, may be found in Chapter 0 of Larsen [**199**].

The multiplier theory of $L_p(G)$, G a locally compact Abelian group, $1 < p < \infty$, presents some complications compared to the case $p = 1$. We briefly describe this situation and refer the interested reader to the monograph by Larsen [**199**]. As observed above, $L_p(G)$ in the non-compact case is not a Banach algebra, so the definition (110) is not meaningful.

Nevertheless, to cover this case a multiplier may be defined as a bounded operator on $L_p(G)$ which commutes with all translation operators τ_s as s ranges in G. For compact groups this definition is equivalent to the usual one, see Larsen [**199**].

Evidently, given a measure $\mu \in \mathcal{M}(G)$ on a locally compact group G the convolution operator T_μ is a multiplier of $L_p(G)$. The converse is not true, since not all multipliers on $L_p(G)$ have this form. In fact, if $p > 1$ and G is an infinite compact group then there always exists a multiplier $T \in M(L_p(G))$ which cannot be written as a convolution operator, see Chapters 3 and 4 of Larsen [**199**].

Also the case $A = L_\infty(G)$, where G any locally compact Abelian group, presents some difficulties. However, if G is compact the definition (110) is meaningful, since A is a commutative Banach algebra, and $T \in M(L_\infty(G))$ precisely when there exists $\mu \in \mathcal{M}(G)$ such that $T = T_\mu$, see Theorem 4.4.2 of Larsen [**199**]. Note that for an arbitrary locally compact Abelian group G there exists a continuous homomorphism of $M(L_\infty(G)$ onto $\mathcal{M}(G)$, see Theorem 4.4.3 of Larsen [**199**].

The case $A = C_0(G)$, G a locally compact Abelian group, is very simple. In fact we have that $M(C_0(G))$ is isometrically isomorphic to the measure algebra $\mathcal{M}(G)$, see Larsen [**199**, Theorem 3.3.2].

We conclude this section by giving some information about convolution operators which are compact operators. The characterization of compact convolution operators of $L_1(G)$, G a non-compact Abelian group, is an immediate consequence of following result owed to Kamowitz [**181**].

Theorem 4.41. *Let A a commutative semi-simple Banach algebra. Suppose that $\Delta(A)$ has no isolated points. Then $T \in M(A)$ is compact if and only if $T = 0$.*

Proof Assume that $T \in M(A)$ is compact and that $\Delta(A)$ has no isolated points. Looking at the proof of part (ii) of Theorem 7.79 we have $T^*m = \widehat{T}(m)m$ for every $m \in \Delta(A)$, so $\widehat{T}(m)$ is an eigenvalue of T^*. Now, since T is compact T^* also is compact, and hence the spectrum $\sigma(T^*)$ is a finite set or a denumerable set with 0 as its only possible cluster point. By the spectral theory of compact operators we also know that every nonzero element in $\sigma(T^*)$ is an eigenvalue of finite multiplicity.

Now, let us denote by m_0 an arbitrary point of $\Delta(A)$. We claim that $\widehat{T}(m_0) = 0$. Suppose, an absurdity, that $\widehat{T}m_0 \neq 0$. Let $(m_k) \subset \Delta(A)$ be a sequence which converges to m_0 and suppose that $m_k \neq m_0$ for every $k \in \mathbb{N}$. Since $\widehat{T}$ is continuous on $\Delta(A)$ the sequence $(\widehat{T}(m_k))$ converges to $\widehat{T}(m_0)$. However, since each nonzero eigenvalue of T' has finite multiplicity we have $\widehat{T}(m_k) = \widehat{T}(m_0)$ for only finitely many $n \in \mathbb{N}$. Therefore $\widehat{T}(m_0)$ is a cluster point of the set $\{\widehat{T}(m_k) : k \in \mathbb{N}\}$.

Since 0 is the only possible cluster point of the spectrum $\sigma(T)$ we have

$\widehat{T}(m_0) = 0$, contradicting the assumption $\widehat{T}(m_0) \neq 0$. Since m_0 is an arbitrary element of $\Delta(A)$ we then conclude that $\widehat{T}(m)) = 0$ for all $m \in \Delta(A)$ and from the semi-simplicity of A it follows that $T = 0$. ∎

In the sequel, by $K_M(L^1(G))$ we shall denote the ideal of all compact convolution operators on $L_1(G)$.

Corollary 4.42. *Let G be a non-compact locally compact Abelian group for which the dual group $\widehat{G}$ contains no isolated point. Then $K_M(L^1(G)) = \{0\}$.*

The finite-dimensional convolution operators on $L^1(G)$ in the case of compact groups may be characterized as follows:

$$\begin{aligned} \mu \in P(G) \quad &\Leftrightarrow \quad T_\mu : \mathcal{M}(G) \to \mathcal{M}(G) \text{ is finite-dimensional} \\ &\Leftrightarrow \quad T_\mu : L_1(G) \to L_1(G) \text{ is finite-dimensional,} \end{aligned}$$

whilst

$$\begin{aligned} \mu \in L_1(G) \quad &\Leftrightarrow \quad T_\mu : \mathcal{M}(G) \to \mathcal{M}(G) \text{ is compact} \\ &\Leftrightarrow \quad T_\mu : L_1(G) \to L_1(G) \text{ is compact.} \end{aligned}$$

Moreover, $K_M(L_1(G))$ is isometrically isomorphic to $L_1(G)$. These equivalences will be proved in a more abstract setting in the next chapter.

We end this section by mentioning the multiplier theory of Beurling algebras, a variant of group algebras. Let ω be a continuous positive submultiplicative weight function on a locally compact Abelian group G and denote by $L_1(\omega)$ the space of all equivalence classes of Borel measurable functions f on G for which $f\omega \in L_1(G)$. If we define the weighted L_1 norm by

$$\|f\|_\omega := \int_g |f(t)|\omega(t)\, \mathrm{d}m(t) \quad \text{for all } f \in L_1(\omega),$$

then the space $L_1(\omega)$, endowed with this norm and with convolution as multiplication, becomes a commutative Banach algebra, called the *Beurling algebras* for the weight ω. All Beurling algebras on a compact Abelian group, or on the additive groups $G = \mathbb{R}$ and $G = \mathbb{Z}$, are semi-simple and possess a bounded approximate identity, see Example 4.1.7 of Laursen and Neumann [**214**]. The multiplier algebra of $L_1(G)$ may be identified with an appropriate weighted measure algebra $M(\omega)$ and, as in Theorem 4.40, every multiplier T on $L_1(G)$ is a convolution operator for a measure $\mu \in M(\omega)$, see also Dales [**87**] and Ghahramani [**121**].

5. Multipliers of Banach algebras with orthogonal basis

The Banach algebras with an orthogonal basis are a natural framework for unifying the study of the properties of multipliers for several commutative semi-simple Banach algebras. We first recall some standard notions from the theory of Banach algebras with orthogonal basis and refer for a full treatment of this theory to the monographs of Singer [**299**] or Husain [**168**].

Definition 4.43. *Given a complex Banach algebra A, a countable subset (e_k) of A is said to be an* orthogonal basis *of A if the following two conditions are satisfied:*

(a) $e_k e_j = \delta_{kj} e_k$ *for all integers* $k, j \geq 1$*;*

(b) (e_k) *is a basis of A, i.e., for each $x \in A$ there exists a unique sequence $(\lambda_k(x))$ of scalars such that*

$$x = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k(x) e_k = \sum_{k=1}^{\infty} \lambda_k(x) e_k.$$

The basis (e_k) is said to be unconditional *if for each $x \in A$ the series $\sum_{k=1}^{\infty} \lambda_k(x) e_k$ converges unconditionally, i.e., $\sum_{k=1}^{\infty} \lambda_{\pi(k)}(x) e_{\pi(k)}$ converges for every permutation π of the positive integers.*

It is easy to verify that the mappings $\lambda_k : x \to \lambda_k(x)$ are multiplicative linear functionals on A, and a well known consequence of the open mapping theorem asserts that all these functionals are continuous, so the set (e_k) is a *Schauder basis* of A. From this it also follows that a Banach algebra with an orthogonal basis is necessarily infinite-dimensional and separable.

The product of two elements x and y of A may be written in the form

$$xy = \sum_{k=1}^{\infty} \lambda_k(x) \lambda_k(y) e_k,$$

so every Banach algebra A with an orthogonal basis is commutative. Any Banach algebra A with an orthogonal basis is also non-unital. To see this assume that A has unit u. Then

$$x = xu = \sum_{k=1}^{\infty} \lambda_k(x) \lambda_k(u) \lambda_k(y) e_k,$$

from which it follows that $\lambda_k(u) = 1$ for all $k \in \mathbb{N}$. Therefore $u = \sum_{k=1}^{\infty} e_k$ and hence the sequence (e_k) converges to 0 as $k \to \infty$. On the other hand, the inequality $\|e_k\| = \|{e_k}^2\| \geq \|e_k\|^2$ shows that $\|e_k\| \geq 1$, so (e_k) cannot converge to 0, a contradiction.

Any Banach algebra A with an orthogonal basis is faithful . Indeed, if $Ax = \{0\}$, from the equality

$$0 = e_j x = e_j (\sum_{k=1}^{\infty} \lambda_k(x)) e_k) = \lambda_j x$$

we obtain that $\lambda_j(x) = 0$ for all integers $j \geq 1$, which obviously implies that $x = 0$. A consequence of (e_k) being a Schauder basis is that A is semi-simple [**167**]. Since the associated coordinate functionals λ_k are multiplicative, the sets ker λ_k are regular closed maximal ideals of A. We show now that the converse holds, namely that every regular maximal ideal of A is equal to ker λ_k for some $k \in \mathbb{N}$.

To see this we prove that given an arbitrary proper closed ideal J in A then J is contained in ker λ_k for some $k \in \mathbb{N}$. Suppose that there does not exist a linear multiplicative functional λ_k such that J is contained in ker λ_k. Then for every $k \in \mathbb{N}$ we can find an element x_k such that $\lambda_k(x_k) = 1$. Then

$$x_k e_k = \sum_{k=1}^{\infty} \lambda_j(x_k) e_j e_k = e_k \quad \text{for all } k \in \mathbb{N},$$

and since J is a closed ideal this implies that $e_k \in J$ for all $k \in \mathbb{N}$. Hence $J = A$, a contradiction. From this argument it follows that we can identify the regular maximal ideal space $\Delta(A)$ with the discrete set $\{\lambda_k : k \in \mathbb{N}\}$, or, which is the same, we can identify $\Delta(A)$ with $\mathbb{N}$. From the equality

$$\{\lambda_k\} = \{m \in \Delta(A) : |m(e_k) - \lambda_k(e_k)| < 1\} \quad \text{for all } k \in \mathbb{N},$$

we then conclude that $\Delta(A)$ is discrete. This also implies that every Banach algebra with an orthogonal basis is regular.

In the following we list some important examples of Banach algebras with an orthogonal basis:

(i) The algebras ℓ^p for any $1 \leq p < \infty$ and c_0 (all with respect to pointwise operations) are immediate examples of Banach algebras with an orthogonal basis. An orthogonal basis is given by the standard basis (e_k), where e_k: $= (\lambda_{kj})_{j=0,1,\ldots}$. This basis is obviously unconditional. Note that ℓ^∞ has no orthogonal basis since it is not separable.

(ii) The algebras $L_p(\mathbb{T})$, $1 < p < \infty$, $\mathbb{T}$ the circle group, with convolution as multiplication. The sequence (u_k) where $u_k(z) := z^k$, $z \in \mathbb{T}$, for all $z \in \mathbb{T}$ and $k \in \mathbb{Z}$, is an orthogonal basis for $L_p(\mathbb{T})$. Except for the case $p = 2$, this basis is not unconditional, see Singer [**299**, Section 2.14] or also Husain and Watson [**169**].

(iii) The *Hardy algebra* $H^p(\mathbb{D})$, $1 < p < \infty$, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$. This algebra is the space of all complex-valued analytic functions defined on $\mathbb{D}$ for which the integrals $\int_{\mathbb{T}} |f(rt)|^p \mathrm{d}m(t)$ are bounded for every $0 < r < 1$, where m is the usual normalized Lebesgue measure on $\mathbb{T}$. The multiplication on $H^p(\mathbb{D})$ is defined by the *Hadamard product*

$$(f * g)(x) := \frac{1}{2\pi i} \int_{|z|=r} f(z) g(xz^{-1}) z^{-1} dz,$$

where f, $g \in H^p(\mathbb{D})$ and $|x| < r < 1$. If we let $e_k(z) := z^k$, $z \in \mathbb{D}$, $k \in \mathbb{N}$, then the sequence (e_k) is an orthogonal basis for $H^p(\mathbb{D})$, see Husain [**168**] and Porcelli [**265**]. This basis is unconditional. This follows from the product $f * g$ being able to be represented by the power series $\sum_{k=0}^{\infty} a_k b_k x^k$, where $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ are the power series relative to f and g, respectively.

Next, we show that each multiplier T of a Banach algebra A with an orthogonal basis can be represented in a very simple way. Clearly, for each $x \in A$ and for all $k \in \mathbb{N} \cong \Delta(A)$ we have $\widehat{x}(k) = \lambda_k(x)$. Thus the Helgason–Wang equation of Theorem 4.14 assumes, for every $T \in M(A)$, the form

$$\widehat{Tx}(k) = \widehat{T}(k)\widehat{x}(k) = \widehat{T}(k)\lambda_k(x) \quad \text{for all } k \in \mathbb{N}.$$

Choosing $x = e_k$ yields

$$\widehat{T}(k) = \lambda_k(Te_k) \quad \text{for each } k \in \mathbb{N}, \tag{125}$$

and therefore

$$Tx = \sum_{k=1}^{\infty} \lambda_k(Tx)e_k = \sum_{k=1}^{\infty} \widehat{T}(k)\lambda_k(x)e_k = \sum_{k=1}^{\infty} \lambda_k(Te_k)\lambda_k(x)e_k.$$

Therefore to every $T \in M(A)$ we can associate a sequence $\lambda_k(Te_k)$. This sequence is bounded. In fact, putting $x = e_j$ in the equality above we obtain

$$Te_j = \sum_{k=1}^{\infty} \lambda_k(Te_k)\lambda_k(e_j)e_k \quad \text{for each } j \in \mathbb{N}.$$

Since $\lambda_k(e_j) = \delta_{kj}$ for all integers $k, j \in \mathbb{N}$ we then have

$$Te_j = \lambda_j(Te_j)e_j \quad \text{for each } j \in \mathbb{N}, \tag{126}$$

thus each $\lambda_j(Te_j)$ is an eigenvalue of T and therefore $\|\lambda_j(Te_j)| \leq \|T\|$.
It is easily seen that if A has an unconditional orthogonal basis (e_k) then the mapping $x \to x^*$, where

$$x^* := \sum_{k=1}^{\infty} \overline{\lambda_k(x)} e_k \quad \text{for each } x \in A,$$

defines a natural involution on A.

Now suppose that (e_k) is an unconditional orthogonal basis of A and $T \in M(A)$. Since the sequence $(\lambda_k(Te_k))$ is bounded the series defined by

$$T^*x := \sum_{k=1}^{\infty} \lambda_k(x)\overline{\lambda_k(Te_k)}e_k \tag{127}$$

converges in A. Consequently the mapping $T \to T^*$ is an involution on the multiplier algebra $M(A)$. Combining the two equalities (125) and taking into account that $T^*e_k = \overline{\lambda_k(Te_k)}e_k$ we then obtain

$$\widehat{T^*}(k) = \lambda_k(T^*e_k) = \overline{\lambda_k(Te_k)}\lambda_k(e_k) = \overline{\lambda_k(Te_k)} = \overline{\widehat{T}(k)},$$

which shows that Helgason–Wang function of T^* is just $\widehat{T^*} \equiv \overline{\widehat{T}}$.

Theorem 4.44. *Let A be a Banach algebra with an orthogonal basis (e_k). Then there exists a continuous isomorphism of $M(A)$ onto a subalgebra of ℓ^∞. If the basis (e_k) is unconditional and $M(A)$ is provided with the involution defined above, then is even a $*$-isomorphism of $M(A)$ onto ℓ^∞.*

Proof Let $\Psi : M(A) \to \ell^\infty$ be the mapping defined by $\Psi(T) := (\lambda_k(Te_k))$ for each $T \in M(A)$. It is easy to verify that Ψ is linear. Moreover, we have

$$\begin{aligned}\Psi(ST) &= (\lambda_k(STe_k)) = (\lambda_k(S(Te_k))) = (\lambda_k(S(\lambda_k(Te_k)e_k))) \\ &= (\lambda_k(\lambda_k(Te_k)Se_k)) = (\lambda_k(Te_k))(\lambda_k(Se_k)) = \Psi(S)\Psi(T).\end{aligned}$$

Clearly the map Ψ is injective and from the estimate

$$\|\Psi(T)\|_\infty = \sup_{k\in\mathbb{N}} |\lambda_k(Te_k)| \le \|T\|,$$

we see that Ψ is also continuous.

Let us now suppose that the basis (e_k) is unconditional and let (t_k) be any bounded sequence of complex numbers. It is easily seen that the series

$$Tx := \sum_{k=1}^{\infty} \lambda_k(x)t_k e_k$$

converges for each $x \in A$, and that the mapping T so defined is a multiplier of A. Moreover, $\Psi(T) = (t_k)$, thus Ψ is surjective. Then by definition and from the surjectivity of T it easily follows that Ψ is a $*$-homomorphism of $M(A)$ onto ℓ^∞. ■

Let us consider the sets

$$K_M(A) := K(A) \cap M(A)$$

and

$$K_{M_0}(A) := \{T \in K_M(A) : (\lambda_k(Te_k)) \in c_0\}.$$

Clearly, $K_{M_0}(A)$ is a closed ideal of $M(A)$. Moreover, if we denote by $F_M(A)$ the ideal of all finite-dimensional multipliers of A, we have

$$F_M(A) \subset K_{M_0} \subseteq K_M(A).$$

Indeed, if $S \in M(A)$ is finite-dimensional, since $Se_k = \lambda_k(Se_k)e_k$ for all integers $k \in \mathbb{N}$, we infer that $\lambda_k(Se_k) = 0$ for all but finitely many of the indices k (otherwise the vectors e_k being linearly independent the image $S(A)$ would be infinite-dimensional). This shows that under the map constructed in Theorem 4.44 the finite-dimensional multipliers correspond to the sequences in c_0 having only a finite number of non-zero terms.

Theorem 4.45. *Let A be a Banach algebra with an orthogonal basis (e_k). Then there exists a continuous isomorphism Ψ_0 of $K_{M_0}(A)$ onto a subalgebra of c_0. Furthermore, if the basis (e_k) is unconditional, in the involution on $K_{M_0}(A)$ defined by the series (127) Ψ_0 is a $*$-isomorphism of $K_{M_0}(A)$ onto c_0.*

Proof Let us consider the mapping Ψ_0 defined by $\Psi_0(T) := (\lambda_k(Te_k))$ for each $T \in K_{M_0}(A)$. Trivially Ψ_0 maps $K_{M_0}(A)$ onto a subalgebra of c_0. Suppose that the basis (e_k) is unconditional and let c_{00} denote the algebra of all complex sequences (α_k) such that $\alpha_k = 0$ for all k but a finite number of them. For any natural $p \geq 1$ let (τ_k) be a sequence of c_{00} such that $\tau_k = 0$ for all $k \geq p$. If we define

$$Tx := \sum_{k=1}^{\infty} \tau_k \lambda_k e_k = \sum_{k=1}^{p} \tau_k \lambda_k e_k$$

we have $T \in M(A)$. Clearly T is a finite-dimensional operator and therefore belongs to $K_{M_0}(A)$. We also have $\Psi_0(K_{M_0}(A)) \supseteq c_{00}$, and since Ψ_0 is the restriction of the isomorphism Ψ defined in the proof of Theorem 4.44 on the closed subalgebra $K_{M_0}(A)$ we also have that $\Psi_0(K_{M_0}(A))$ is a closed subalgebra of c_0. From this we obtain that

$$\Psi_0(K_{M_0}(A)) \supseteq \overline{c_{00}} = c_0,$$

so Ψ_0 is surjective. Clearly Ψ_0 is a isomorphism and for every $T \in K_{M_0}(A)$ the sequence $(\overline{\lambda_k(Te_k)}) \in c_0$, from which we conclude that the mapping T^* defined by (127) is compact. Hence $K_{M_0}(A)$ is an involution algebra and Ψ_0 is a $*$-isomorphism of $K_{M_0}(A)$ onto c_0. ■

Note that compact multipliers of Banach algebras with an orthogonal basis do not always correspond to sequences which belong to c_0, i.e. in the notation of Theorem 4.44 generally $K_{M_0}(A)$ is a proper subalgebra of c_0. In fact, let consider the case $A = L_p(\mathbb{T})$, $p < 1 < \infty$. Then $\Delta(A) = \mathbb{Z}$, and if $f \in L_p(\mathbb{T})$

$$\widehat{Tf}(k) = \widehat{T}k)\widehat{f}(k) = \lambda_k(Te_k)\widehat{f}(k) \quad \text{for each } k \in \mathbb{Z},$$

where $\widehat{f}$ denotes the Fourier transform of f. If $p \neq 2$, Figá-Talamanca and Gaudry [**114**, Theorem B and Remark b] have shown that there exists a non-compact multiplier T such that the corresponding sequences $(\lambda_k(Te_k))$ belongs to c_0.

If A has an orthogonal basis (e_k) let L_k denote the multiplication operator defined by

$$L_k(x) := xe_k \quad \text{for every integer } k \geq 1.$$

Trivially $L_k^2 = L_k$ and, since $xe_k = \lambda_k(x)e_k$, L_k is a bounded projection of A onto the one-dimensional ideal e_kA.

In the next result we describe some distinguished parts of the spectrum of a multiplier of a Banach algebra with orthogonal basis.

Theorem 4.46. *Let A be a Banach algebra with an orthogonal basis e_k and let (λ_k) be the sequence of the corresponding coefficients functionals. Then, for every $T \in M(A)$ we have:*

(i) $\sigma_{\mathrm{p}}(T) = \{\lambda_k(Te_k) : k \in \mathbb{N}\}$;

(ii) $\sigma_r(T) = \varnothing$;

(iii) $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

Proof (i) This follows from Theorem 4.31 since the maximal ideal space $\Delta(A)$ is discrete and the set $\{\lambda_k(Te_k) : k \in \mathbb{N}\}$ is the range of the Wang–Helgason representation of T.

(ii) Let us suppose that $\sigma_r(T) \neq \varnothing$. If $\lambda \in \sigma_r(T)$ then λ does not belong to $\sigma_{\mathrm{p}}(T)$. From part (i) we then obtain that $\lambda \neq \lambda_k(Te_k)$ for every $k \in \mathbb{N}$. Let us consider for every $k \in \mathbb{N}$ the element $z_k := [\lambda - \lambda_k]^{-1} e_k$. Then

$$(\lambda I - T)z_k = [\lambda \lambda_k(Te_k)]^{-1}(\lambda I - T)e_k \quad \text{for each } k \in \mathbb{N},$$

and since $Te_k = \lambda_k(Te_k)e_k$ it easily follows that $(\lambda I - T)z_k = e_k$ for each$k \in \mathbb{N}$. Therefore, for each $k \in \mathbb{N}$ we have $e_k \in (\lambda I - T)(A)$.

Let us denote by Z the linear space spanned by the sequence (e_k). Clearly, Z is norm dense in the algebra A so that $\overline{Z} = \overline{(\lambda I - T)(A)} = A$. This of course implies that $\lambda \notin \sigma_r(T)$, a contradiction.

(iii) By part (ii) and the relationships (i) and (ii) of Theorem 4.27 we have

$$\sigma_{\mathrm{ap}}(T) \subseteq \sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \subseteq \sigma_{\mathrm{ap}}(T),$$

so $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. ∎

The preceding result leads to the following spectral decomposition theorem for multipliers on a Banach algebra with an unconditional . Not surprisingly, this result is remarkably similar to the classical spectral theorem for compact, or meromorphic, normal operators on Hilbert spaces, see Heuser [**159**, Proposition 71.1].

We first need the following lemma

Lemma 4.47. *Let A be a Banach algebra with an unconditional orthogonal basis (e_k). Then the sequence (L_k) is an unconditional basis of $K_{M_0}(A)$.*

Proof By Theorem 4.45 the mapping $\Psi_0 : K_{M_0}(A) \to c_0$ defined by $\Psi_0(T) := (\lambda_k(Te_k))$ is a homeomorphism of $K_{M_0}(A)$ onto c_0. Obviously Ψ_0 corresponds to (L_k), the standard basis (u_k) of c_0, where $(u_k) := (\delta_{kj})_{j=0,1,\dots}$. Moreover, since $T = \sum_k^\infty \lambda_k(Te_k)L_k$ the sequence (L_k) is an unconditional basis of $K_{M_0}(A)$. ∎

Theorem 4.48. *Let A be a Banach algebra with an unconditional orthogonal basis (e_k). Then every $T \in M_0(A)$ is a compact operator and*

$$\sigma(T) = \{\lambda_k(Te_k) : k \in \mathbb{N}\} \cup \{0\}. \tag{128}$$

Moreover, if $\mu_k := \lambda_k(Te_k)$ then T admits the unconditionally convergent expansion

$$T = \sum_{k=1}^{\infty} \mu_k P_k,$$

where for each $k \in \mathbb{N}$ P_k is the spectral projection associated with $\{\mu_k\}$. The projections P_k are multipliers on A and project the finite-dimensional ideal $M_k = \ker(\mu_k I - T)$ along its orthogonal ideal ${M_k}^\top = (\mu_k I - T)(A)$. The dimension of M_k coincides exactly with the number of the coefficients $\lambda_j(Te_j)$ winch are equal to μ_k.

Proof By Lemma 4.47, if $T \in M_0(A)$ then $T = \sum_k^\infty \lambda_k(Te_k)L_k$. From

$$Le_j(x) = e_j x = e_j \sum_{k=1}^\infty \lambda_k(x)e_k = \lambda_j(x)e_j$$

we then conclude that the operator L_{e_j} has a range of dimension 1, so T is a compact operator, since it is the norm limit of finite-dimensional operators. A compact operator on an infinite-dimensional Banach space has as its spectrum $\sigma(T) = \sigma_{\rm p}(T) \cup \{0\}$, so by Theorem 4.46 the equality (128) holds for every $T \in M_0(A)$.

Let (μ_k) be the sequence of all distinct eigenvalues obtained from the sequence $(\lambda_k(Te_k))$ by removing eventual repetitions of them. Since by Theorem 4.45, $(\lambda_k(Te_k)) \in c_0$ each $\lambda_k(Te_k) \neq 0$ may appear in the sequence $(\lambda_k(Te_k))$ for only a finite number of indices k. The convergence of the series $\sum_{k=1}^\infty \lambda_k(e_k)L_k$ is unconditional, so

$$T = \sum_{k=1}^\infty \mu_k \sum_{\lambda_j = \mu_k} L_j,$$

and hence if P_k denotes the finite sum

$$P_k := \sum_{\lambda_j = \mu_k} L_j,$$

we obtain

$$T = \sum_{k=1}^\infty \mu_k P_k. \tag{129}$$

Trivially, every P_k is a projection and belongs to $M(A)$. Precisely by Theorem 4.10 P_k projects the algebra A onto the finite-dimensional ideal $M_k := \sum_{\lambda_j=\mu_k} e_j A$ along its orthogonal ideal ${M_k}^\top$.

We claim that $M_k = \ker(\mu_k I - T)$. The inclusion $M_k \subseteq \ker(\mu_k I - T)$ follows from the equalities (126). Conversely, let us suppose $x \in \ker(\mu_k I - T)$. Then

$$\sum_{j=1}^\infty \lambda_j(x)\lambda_j(Te_j)e_j = Tx = \mu_k x = \sum_{j=1}^\infty \mu_k \lambda_j(x)e_j.$$

The uniqueness of coefficients implies that $\mu_k\lambda_j(x) = \lambda_j(Te_j)\lambda_j(x)$ for each $j \in \mathbb{N}$, so that

$$\lambda_j(Te_j) = \mu_k \quad \text{for all } j \text{ with } \lambda_j(x) \neq 0,$$

and therefore $x \in M_k$. Hence $M_k = \ker(\mu_k I - T)$.

We show now that ${M_k}^\top = (\mu_k I - T)(A)$. Clearly, if $\mu_k = 0$ for all except a finite number of the k then the set of all eigenvalues is finite, so the series (129) is to be interpreted as a finite sum. For every k let Q_k be the spectral projection associated to the isolated point μ_k:

$$Q_k := \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} \, \mathrm{d}\lambda.$$

Clearly, we have $Q_k \in M(A)$ since $(\lambda I - T)^{-1} \in M(A)$. By Theorem 4.36 we know that μ_k is a simple pole of $R(\lambda, T)$, hence Q_k projects A onto $\ker(\mu_k I - T)$ along $(\mu_k I - T)(A)$, see part (b) of Remark 3.7. Therefore $Q_k = P_k$ and ${M_k}^\top = (\mu_k I - T)(A)$. Finally, the dimension of M_k is exactly given by the number of coefficients $\lambda_j(Te_j)$ which coincides with μ_k, so the proof is complete.

■

6. Multipliers of commutative H^* algebras

A Banach algebra A with an involution $*$ is said to be a H^* algebra if its underlying Banach space is a Hilbert space whose scalar product $\langle \cdot, \cdot \rangle$ verifies the following properties:

(a) The Hilbert space norm agrees with the Banach space norm $\| \cdot \|$, $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$;

(b)$\|x^\star\| = \|x\|$ for each $x \in A$;

(c) If $x \neq 0$ then $x^* x \neq 0$;

(d) $\langle xy, z \rangle = \langle y, x^* z \rangle$ for all x, y, $z \in A$.

The standard example of a commutative H^* algebra is the group algebra $L^2(G)$, where G is a compact Abelian group. For a systematic study of H^*-algebras we refer to Naimark [**242**]. In the sequel we sketch the Gelfand theory of commutative H^* algebras.

The regular maximal ideal space $\Delta(A)$ of a commutative H^* algebra A is easy to describe. We recall that an idempotent $e \in A$ is said to be *irreducible* if e is not the sum of two nonzero orthogonal (in the Banach algebra sense) self-adjoint idempotents, i.e., there do not exist elements e_1, $e_2 \in A$ such that $e = e_1 + e_2$ and $e_1 e_2 = 0$. Let E be the set defined by

$$E := \left\{ \frac{e_\beta}{\|e_\beta\|} : e_\beta \ \text{ is an irreducible self-adjoint idempotents of } A \right\}. \tag{130}$$

Evidently the set E forms a complete orthogonal system in the Hilbert space A. Each minimal ideal J_β of A is the one-dimensional ideal generated by an element $e_\beta \in E$. The orthogonal (in the Hilbert sense) of each J_β is a regular maximal ideal of A and all regular maximal ideals of A may be obtained in this way. Hence there exists a one to one correspondence between points of $\Delta(A)$ and elements of the set E. If we equip E with the discrete topology then the maximal regular ideal space $\Delta(A)$ and E are also topologically equivalent. By identifying $\Delta(A)$ and E the Gelfand transform assumes the form:

$$\widehat{x}(e) = \frac{\langle x, e\rangle}{\|e_\beta\|} \quad \text{for all} \quad e := \frac{e_\beta}{\|e_\beta\|} \in E.$$

Note that a commutative H^* algebra A is also semi-simple, [**242**]. Furthermore, A is regular since $\Delta(A)$ is discrete.

If $T \in M(A)$ and T^* denotes the Hilbert adjoint of T, for each x, y, $z \in A$ we have

$$\begin{aligned} \langle (T^*x)y, z\rangle &= \langle T^*x, zy^*\rangle = \langle x, T(zy^*)\rangle \\ &= \langle x, (Tz)y^*\rangle = \langle xy, Tz\rangle = \langle y, x^*(Tz)\rangle \\ &= \langle y, T(x^*z)\rangle = \langle T^*y, x^*z\rangle = \langle x(T^*y), z)\rangle \end{aligned}$$

Therefore $(T^*x)y = x(T^*y)$ and hence $T^* \in M(A)$. Consequently the mapping $T \to T^*$ is an involution on $M(A)$.

Theorem 4.49. *Suppose that A is a commutative H^* algebra. Then $M(A)$ is isometrically *-isomorphic to $C(\Delta(A))$.*

Proof Clearly, the mapping $T \to \widehat{T}$ is an algebra isomorphism of $M(A)$ onto $\mathcal{M}(A) := \{\widehat{T} : T \in M(A)\} \subseteq C(\Delta(A))$. Let us identify $\Delta(A)$ with the set E defined in (130) and suppose $\varphi \in C(E)$. If $x \in A$ then the series

$$z := \sum_{e\in E} \langle x, e\rangle \varphi(e) e$$

defines an element of A. It is easy to see that

$$\widehat{z}(e) = \varphi(e)\frac{\langle x, e\rangle}{\|e_\beta\|} = \varphi(e)\widehat{x}(e).$$

From the equality $\widehat{z}(e) = \varphi(e)\widehat{x}(e)$ it then follows that $\varphi\widehat{A} \subseteq \widehat{A}$ thus every $\varphi \in C(E)$ defines a multiplier of A, and $\mathcal{M}(A) = C(\Delta(A))$. The mapping $T \in M(A) \to \widehat{T} \in \mathcal{M}(A)$ clearly takes T^* into the conjugate of $\widehat{T}$, and so the only thing which remains to be done is to prove that the mapping above is an isometry.
For any $T \in M(A)$ we have that

$$\langle Te, f\rangle = \langle T(\|e\|e, f\rangle \quad \text{for all } e, f \in E$$

and

$$\langle Tx, e\rangle = \left\langle \left(\sum_{f\in E} \langle x, f\rangle f\right), e\right\rangle = \langle x, e\rangle\langle Te, e\rangle.$$

Therefore

$$\|Tx\|^2 = \sum_{e\in E} |\langle Tx, e\rangle|^2 = \sum_{e\in E} |\langle x, e\rangle|^2 \ |\langle Te, e\rangle|^2 \le \|\widehat{T}\|^2_\infty \|x\|,$$

so that $\|T\| \le |\widehat{T}\|_\infty$. Since the inverse inequality is still valid we can now conclude that the mapping $T \in M(A) \to C(\Delta(A))$ is an isometry. ∎

Note that the last theorem implies that for a compact Abelian group G, the algebra $M(L_2(G))$ is isometrically *-isomorphic to $C(\widehat{G})$.

Theorem 4.50. *Let A be a commutative H^*-algebra. Then the ideal of all compact multipliers $K_M(A)$ is isometrically $*$-isomorphic to $C_0(\Delta(A))$.*

Proof By the previous theorem the mapping

$$\varphi : T \in K_M(A) \to \widehat{T}$$

is a $*$-isomorphism of $K_M(A)$ onto a subalgebra B of $C(\Delta(A)$. We need only to prove that $B = C_0(\Delta(A))$. As before we identify $\Delta(A)$ with the set E defined in (130).

Let us suppose that the Wang function $\widehat{T}$ of $T \in M(A)$ belongs to $C_c(\Delta(A))$, where $C_c(\Delta(A))$ denotes the subalgebra of $C(\Delta(A))$ of all continuous functions on $\Delta(A)$ having a compact support. Since $\Delta(A)$ is discrete the function $\widehat{T}$ has a finite support, for instance $\{e'_{\beta_1}, \dots, e'_{\beta_n}\}$, where

$$e'_{\beta_k} := \frac{e_{\beta_k}}{\|e_{\beta_k}\|} \quad (k = 1, \dots, n),$$

are irreducible self-adjoint idempotents.

From the equality

$$\langle Tx, e'_\beta\rangle = \langle x, e'_\beta\rangle \widehat{T}(e'_\beta)e'_\beta,$$

we then obtain

$$Tx = \sum_\beta \langle Tx, e'_\beta\rangle e'_\beta = \sum_{k=1}^{n} \langle x, e'_{\beta_k}\rangle \widehat{T}(e'_{\beta_k}) \tag{131}$$

for each $x \in A$. This implies that the image space $T(A)$ is contained in the direct sum $\bigoplus e'_{\beta_k}A$ of one-dimensional ideals $e'_{\beta_k}A$, thus T is finite-dimensional and therefore compact. Hence the image of the ideal $K_M(A)$ under φ contains $C_c(\Delta(A))$.

Clearly, $\varphi(K_M(A))$ is closed in the uniform topology, thus we have

$$\varphi(K_M(A)) \supseteq \overline{C_c(\Delta(A))} = C_0(\Delta(A)).$$

Next we want show the inverse inclusion $\varphi(K_M(A)) \subseteq c_0(\Delta(A))$.

Let $T \in K_M(A))$ and $\varphi(T) = \widehat{T} \notin C_0(\Delta(A))$. Since E is discrete there exists an $\varepsilon > 0$ and a sequence (β_k) such that

$$\beta_k \neq \beta_j \quad \text{for } k \neq j \quad \text{and} \quad |\widehat{T}(e'_{\beta_k})| \geq \varepsilon \;\; \text{for all integer } k \in \mathbb{N}.$$

By using the equality (131), for all $k \neq j$ we then obtain the following estimates

$$\begin{aligned}
\|Te'_{\beta_k} - Te'_{\beta_j}\| &= \|\widehat{T}(e'_{\beta_k})e'_{\beta_k} - \widehat{T}(e'_{\beta_j})e'_{\beta_j}\| \\
&= (|\widehat{T}(e'_{\beta_k})|^2 + (|\widehat{T}(e'_{\beta_j})|^2)^{\frac{1}{2}} \\
&\geq \sqrt{2}\varepsilon.
\end{aligned}$$

From this it follows that the sequence (Te'_{β_k}) has no convergent subsequence and this contradicts the compactness of T. Hence $\widehat{T} \in C_0(\Delta(A))$ and therefore $\varphi(K_M(A)) = C_0(\Delta(A))$ ∎

6.1. Comments. The concept of a multiplier on a semi-simple commutative Banach algebra A was originally introduced by Helgason [**156**] as a bounded continuous function φ defined on the regular maximal ideal space $\Delta(A)$ such that $\varphi\widehat{A} \subseteq \widehat{A}$, where $\widehat{A} = \{\widehat{x} : x \in A\}$. The general theory of multipliers on a faithful Banach algebras in the form presented here has been developed successively by Wang [**314**], Birtel [**68**]. A good reference for this theory is the monograph of Larsen [**198**], in which the reader may find applications to abstract harmonic analysis.

One of the most general approaches to the theory of multipliers of non-commutative Banach algebras is owed to Johnson [**175**], whose starting points are the properties of certain mappings on associative semi-groups. Multipliers are also called *centralizers* by some authors. Developments of this theory, with some applications to specific topological algebras, may be found in Johnson [**176**], [**177**], Johnson and Sinclair [**178**], Reid [**278**], Rieffel [**280**], Foiaş [**119**], Kellog [**186**], and Maté [**224**].

The main characterization of multipliers of a faithful commutative Banach algebra as bounded continuous functions on $\Delta(A)$, established in Theorem 4.14, is owed to Wang [**314**]. This result, although not explicitely stated in this abstract context, appeared in some other papers, for instance Helgason ([**156**]). The characterizations of multipliers of commutative Banach algebras given in Theorem 4.12 and Theorem 4.15 are owed to Birtel [**68**].

The algebra $M_0(A)$ of those multipliers which correspond to continuous functions φ_T which vanish at infinity on $\Delta(A)$ was studied by Birtel [**68**]. In particular, the material from Theorem 4.16 to Theorem 4.25 is essentially based on the papers of Birtel [**68**] and Wang [**314**].

The subsequent section on the spectral properties of multipliers contains a sample of results from Aiena [**5**], [**6**], [**7**], [**8**] and [**9**]. Theorem 4.33 and Example 4.35 are taken from Aiena, Colasante, and González [**16**]. The result of Corollary 4.34 has been established first by Laursen and Mbekhta in [**206**], whilst Theorem 4.38 is from Laursen [**203**]. Note that the property of $M(A)$ being an inverse closed algebra of $L(A)$ first appeared in Wang [**314**] who observed, by in his own words, the *curious fact* that if a multiplier T is bijective then its inverse T^{-1} is also a multiplier. This chapter, as well as the subsequent chapters, shows that actually we have much more than this curious fact. Multipliers have a very rich spectral theory.

The classical property that the multiplier algebra of the group algebra $L_1(G)$, G a locally compact Abelian group, may be identified with the measure algebra $\mathcal{M}(G)$ has been proved by Wendel [**320**]. The multiplier algebra for the group algebra $L_1(G)$ has also been investigated by several authors, for instance by Edwards [**104**], Helson [**157**], Gaudry [**122**]). For a complete proof of Theorem 4.40 we refer to the book of Larsen [**198**, Chapter 0], which also gives informations on the problem of describing the isomorphism of group algebras. The characterization of the compact multipliers of $L_1(G)$, G a compact Abelian group is owed to Akemann [**45**], Gaudry [**122**] and Kitchen [**187**].

The study of properties of the multiplier algebra of a Banach algebra with an orthogonal basis presented here is modeled after Aiena [**7**]. A study of these properties may also be found in Husain [**168**], Husain and Watson [**169**], [**170**]. Theorem 4.49 and Theorem 4.50 which characterize the multiplier algebra $M(A)$ and the ideal $K_M(A)$ of a commutative H^* algebra have been first proved by Kellog [**186**]. Both theorems have been successively extended to non-commutative H^* algebras by Ching and Wong [**77**]. In this case, instead of Gelfand's representation theory the authors have used the Ambrose structure theorem, see [**46**], for H^* algebras.

CHAPTER 5

Abstract Fredholm theory

In this chapter we shall develop the abstract Fredholm theory on a complex Banach algebra with a unit and apply this theory to the concrete case of multipliers of commutative Banach algebras. Our approach to the Fredholm theory of Banach algebras will use some ideas of Aupetit [**53**] starting from the concept of inessential ideal. This is a two-sided ideal J of a complex unital Banach algebra having the property that every element $x \in J$ has a finite or a denumerable spectrum which clusters at 0.

Every inessential ideal J of a Banach algebra $\mathcal{A}$ determines a Fredholm theory. In the case of the Banach algebra $L(X)$ of all bounded operators on a Banach space X, the Fredholm theory determined by the inessential ideal of all compact operators $K(X)$ is the classical Fredholm theory introduced in the previous chapters.

A first important inessential ideal of a semi-prime Banach algebra $\mathcal{A}$ is given by the socle of $\mathcal{A}$. The properties of this ideal are developed in a purely algebraic setting in the second section, whilst the third section addresses the characterization of the socle of a semi-prime Banach algebra $\mathcal{A}$ as the largest ideal of $\mathcal{A}$ each of whose elements are algebraic. Generally an arbitrary Banach algebra does not admit a socle. For this reason we shall introduce for an arbitrary algebra the so called pre-socle. This may be characterized as the largest ideal for which every element has a finite spectrum.

In this chapter we shall also introduce the concept of Riesz algebra. These algebras have been introduced by Smyth [**304**] and may be thought of *grosso modo*, as algebras which are very close to their pre-socle, more precisely, which do not admit primitive ideals containing the pre-socle. These algebras may also be described in terms of operator theory. In fact, these algebras are strictly related to the class of Riesz operators on a Banach space, in the sense that if $\mathcal{A}$ is a Riesz algebra then the wedge operators $x \wedge x$, $x \in \mathcal{A}$, acting on $\mathcal{A}$, are Riesz operators. We shall also see that this property characterizes the semi-simple Riesz Banach algebras. In view of the applications to multiplier theory, we shall focus our attention to the particular case of Riesz commutative Banach algebras and will show that these are precisely the commutative Banach algebras having a discrete maximal ideal space.

The concept of a Fredholm element of a Banach algebra is introduced in the fourth section, where we develop the Fredholm theory of the multiplier algebra $M(A)$ with respect to some inessential ideals of $M(A)$, in particular with respect to the ideal $K_M(A)$ of all compact multipliers. This section also contains a description of the ideal $K_M(A)$ for several commutative Banach algebras, in particular for group algebras, for commutative integral domains as well as for commutative C^* algebras.

In the fifth section, in order to give more information on the Fredholm theory of multipliers we shall consider preliminarily the problem of finding conditions for which the range of a bounded operator T on a Banach space X, or some iterate T^n, has closed range. In these conditions the finiteness of the ascent plays a central role, so that these results apply to multipliers of semi-prime commutative Banach algebras. A main result of this chapter is that a sort of Atkinson theorem holds for multipliers: the Fredholm multipliers having index zero may be *intrinsically* characterized as the multipliers invertible modulo the compact multipliers. This result also shows that the multiplier algebra of a commutative semi-simple Banach algebra is sufficiently large to represent all the results concerning Fredholm theory in terms only of multipliers.

Subsequently other results on semi-Fredholm multipliers of A are obtained by adding some other assumptions on the regular maximal ideal space $\Delta(A)$, for instance assuming that A is a regular Tauberian Banach algebra or assuming that the multiplier algebra is regular. We shall also consider the Fredholm theory of a not necessarily commutative $C^\star$ algebra.

A part of this chapter concerns the applications of the theory developed in this chapter to some concrete algebras studied in Chapter 3. The most significant of these applications leads to the characterization of Fredholm convolution operators of a group algebra $L^1(G)$.

In the last section we establish several formulas for the Browder spectrum of multipliers of semi-simple Banach algebra, and in particular for the Browder spectrum of a convolution operator of the group algebra $L^1(G)$ of a compact Abelian group G. Some of these characterizations are given in terms of compressions.

1. Inessential ideals

In the previous chapter we have defined the radical of an algebra $\mathcal{A}$ as the intersection of all primitive ideals of $\mathcal{A}$. The radical may be described in several ways. In the sequel we shall list several characterizations of the radical, each of which is useful in certain situations. Recall first that an element x of an algebra $\mathcal{A}$ is called to be *quasi-invertible* if there exists an element $y \in A$ such that $y \circ x = x \circ y = 0$, where for every $x, y \in \mathcal{A}$ the *circle product* $\circ$ is defined by $x \circ y := x + y - xy$. The element y is called the *quasi-inverse* of x. If $\mathcal{A}$ has a unit element u then x has quasi-inverse y if and only if $u - x$ has inverse $u - y$.

Let q-inv $\mathcal{A}$ denote the set of all quasi-invertible elements of $\mathcal{A}$. Then

$$\begin{aligned} \operatorname{rad}\mathcal{A} &= \{x \in \mathcal{A} : \mathcal{A}x \subseteq \text{q-inv } \mathcal{A} \ \text{ for each } \ a \in \mathcal{A}\} \\ &= \{x \in \mathcal{A} : x\mathcal{A} \subseteq \text{q-inv } \mathcal{A} \ \text{ for each } \ a \in \mathcal{A}\}, \end{aligned} \tag{132}$$

see Bonsall and Duncan [**72**, Proposition 24.16 and Corollary 24.17]). If $\mathcal{A}$ has unit u and inv $\mathcal{A}$ denotes the set of all invertible elements of $\mathcal{A}$ then

$$(133)\qquad \begin{aligned}\operatorname{rad}\mathcal{A} &= \{x \in \mathcal{A} : u - ax \in \text{inv }\mathcal{A} \ \text{ for each } \ a \in \mathcal{A}\} \\ &= \{x \in \mathcal{A} : u - xa \in \text{inv }\mathcal{A} \ \text{ for each } \ a \in \mathcal{A}\}.\end{aligned}$$

Moreover,

$$(134)\qquad \operatorname{rad}\mathcal{A} = \{x \in \mathcal{A} : x + \text{inv }\mathcal{A} \subset \text{inv }\mathcal{A}\}.$$

If $\mathcal{A}$ is a Banach algebra and $Q(\mathcal{A})$ denotes the set of all quasi-nilpotent elements of $\mathcal{A}$ we also have

$$(135)\qquad \begin{aligned}\operatorname{rad}\mathcal{A} &= \{x \in \mathcal{A} : xQ(\mathcal{A}) \subseteq Q(\mathcal{A})\} \\ &= \{x \in \mathcal{A} : Q(\mathcal{A})x \subseteq Q(\mathcal{A})\}.\end{aligned}$$

(Bonsall and Duncan [**72**, Proposition 25.1]). Note that by the characterization (134) it follows that if x belongs to the radical of $\mathcal{A}$ then $\lambda u - x$ is invertible for each $\lambda \neq 0$. Therefore every element of the radical is quasi-nilpotent. In the case of commutative Banach algebra the radical is precisely the set of all quasi-nilpotent elements(Bonsall and Duncan [**72**, Corollary 17.7]).

Theorem 5.1. *Let J be a two-sided ideal of the algebra $\mathcal{A}$. Then:*

(i) $\operatorname{rad} J = J \cap \operatorname{rad}\mathcal{A}$. *In particular, if $\mathcal{A}$ is semi-simple then J is semi-simple;*

(ii) $\mathcal{A}/\operatorname{rad}\mathcal{A}$ *is semi-simple.*

Proof (i) For the first statement see Bonsall and Duncan [**72**, Corollary 24.30]. The second statement is obvious.

(ii) See Bonsall and Duncan [**72**, Proposition 24.21]. ∎

Let $\Pi(\mathcal{A})$ denote the set of all primitive ideals of an algebra $\mathcal{A}$. Given a subset E of $\Pi(\mathcal{A})$ and an ideal J of $\mathcal{A}$, the *kernel* of E and the *hull* of J are defined, respectively, by

$$k_{\mathcal{A}}(E) := \bigcap_{P\in\Pi(\mathcal{A})} P \quad \text{and} \quad h_{\mathcal{A}}(J) := \{P \in \Pi(\mathcal{A}) : J \subseteq P\}.$$

It is understood that $k_{\mathcal{A}}(\varnothing) = \mathcal{A}$. If the algebra $\mathcal{A}$ is unambiguous from the context we shall drop the subscript and write simply $k(E)$ and $h(J)$.

The mapping $E \to h(k(E))$ is a Kuratowski closure operation on the subsets of $\Pi(\mathcal{A})$ and the topology on $\Pi(\mathcal{A})$ corresponding to this closure operation is called the *hull-kernel topology.* The set $\Pi(\mathcal{A})$ provided with this topology is called the *strong structure space.*

Note that if $\mathcal{A}$ is a commutative Banach algebra then an ideal is primitive precisely when it is maximal regular, see Rickart [**279**, Corollary 2.2.10]. Hence in this case $\Pi(\mathcal{A})$ coincides with the space $\Delta(A)$ of all maximal regular ideals of $\mathcal{A}$, and the hk-topology on $\Pi(\mathcal{A})$ coalesces with the hk-topology defined in the previous chapter on $\Delta(\mathcal{A})$.

In the following we collect some important properties of the structure space. For a much fuller account of the structure spaces we refer to Rickart [**279**] or Bonsall and Duncan [**72**].

Remark 5.2. Let $\mathcal{A}$ be an algebra.

(i) If $\mathcal{A}' := \mathcal{A}/\mathrm{rad}\,\mathcal{A}$ then the two structure spaces $\Pi(\mathcal{A})$ and $\Pi(\mathcal{A}')$ are homeomorphic. See Bonsall and Duncan [**72**, Proposition 26.6].

(ii) Let $e \neq 0$ be an idempotent in a Banach algebra $\mathcal{A}$ and $\mathcal{B} := e\mathcal{A}e$. Then the mapping $P \in \Pi(\mathcal{A}) \to P \cap \mathcal{B}$ is a homeomorphism of $\Pi(\mathcal{A}) \setminus h(\mathcal{B})$ onto $\Pi(\mathcal{B})$. See [**72**, Theorem 27.14].

(iii) If $\mathcal{A}$ is a semi-simple Banach algebra and $e \in \mathcal{A}$ is idempotent, then $e\mathcal{A}e$ is a closed semi-simple subalgebra of $\mathcal{A}$. In fact, $\mathcal{B} := e\mathcal{A}e$ is closed in $\mathcal{A}$. From the homeomorphism of $\Pi(\mathcal{A}) \setminus h(\mathcal{B})$ onto $\Pi(\mathcal{B})$ it follows that $\mathrm{rad}\,\mathcal{B} = \mathrm{rad}\,\mathcal{A} \cap \mathcal{B}$. The semi-simplicity of $\mathcal{A}$ then implies that $\mathrm{rad}\,\mathcal{B} = \{0\}$.

Now let $\mathcal{A}$ be a complex Banach algebra with a unit u and let J be a two-sided ideal of $\mathcal{A}$.

Definition 5.3. *We say that the ideal J is* inessential *if for every $x \in J$ the spectrum $\sigma_{\mathcal{A}}(x)$ is either finite or is a sequence converging to zero.*

For instance, if $\mathcal{A} = L(X)$, X a Banach space, the ideal $F(X)$ of finite-dimensional operators or its closure, and the ideal $K(X)$ of all compact operators are inessential. Other inessential ideals of $\mathcal{A} = L(X)$ will be introduced in Chapter 7. Later it will be shown that if G is compact Abelian group and $\mu \in L^1(G)$ then the convolution operator T_μ on $L_1(G)$ is compact, and hence the spectrum $\sigma(\mu) = \sigma(T_\mu)$ is finite set or a sequence converging to zero. Therefore $J = L^1(G)$ is an essential ideal of the measure algebra $\mathcal{M}(G)$.

For each two-sided ideal J of $\mathcal{A}$ denote by $k(h(J))$ the intersection of all primitive ideals of $\mathcal{A}$ which contain J. Clearly $k(h(J))$ is a two-sided ideal and

$$J \subseteq \overline{J} \subseteq k(h(J)), \tag{136}$$

where, as usual, $\overline{J}$ denotes the closure of J.

The ideal $k(h(J))$, where J is a closed ideal of $\mathcal{A}$, may be characterized by means of the radical of $\mathcal{A} \to \mathcal{A}/J$. Let us $\Psi : \mathcal{A} \to \mathcal{A}/J$ denote the canonical quotient map. Then

$$k(h(J)) = \Psi^{-1}(\mathrm{rad}\,(\mathcal{A}/J)), \tag{137}$$

see Jacobson ([**174**, see p. 205].

Let σ be a spectral set of an element x of a Banach algebra $\mathcal{A}$ with unit u, i.e., σ is an open and closed subset of the spectrum $\sigma_{\mathcal{A}}(x)$. Let denote by Γ a closed curve which lies in the resolvent $\rho_{\mathcal{A}}(x)$ and contain σ in its interior. The *spectral idempotent* associated with x and σ is defined by

$$p := p(\sigma, x) = \frac{1}{2\pi i} \int_\Gamma (\lambda u - x)^{-1} \, \mathrm{d}\lambda.$$

Lemma 5.4. *Let J be a two-sided ideal of a Banach algebra $\mathcal{A}$ with unit u and let $x \in k(h(J))$. Then the idempotent p associated with x and α, for each isolated point $\alpha \neq 0$ of the spectrum, belongs to J.*

Proof For each $\lambda \in \Gamma$, Γ separating α from the rest of the spectrum, we have

$$(\lambda u - x)^{-1} = \frac{u}{\lambda} + \frac{1}{\lambda} x (\lambda u - x)^{-1}.$$

Hence

$$p = \frac{1}{2\pi i} \int_\Gamma \frac{u}{\lambda} \, \mathrm{d}\lambda + \frac{x}{2\pi i} \int_\Gamma \frac{1}{\lambda} (\lambda u - x)^{-1} \mathrm{d}\lambda.$$

Evidently the first term is equal to zero and the second term belongs to $k(h(J))$, so $p \in k(h(J)) \subseteq k(h(\overline{J}))$.

Now let $\overline{p}$ denote the equivalence class of p in $\mathcal{A}/\overline{J}$. Since $p \in k(h(J))$, from (137) we obtain that $\overline{p} \in \operatorname{rad} \mathcal{A}/\overline{J}$, and hence the spectral radius of $\overline{p}$ is zero. But since p is idempotent this implies that $\overline{p} = \overline{0}$, and hence $p \in \overline{J}$. Moreover, $p\overline{J}p$ is a closed subalgebra of $\mathcal{A}$, hence a Banach algebra with identity p having pJp as a dense two-sided ideal. From this we then conclude that $pJp = p\overline{J}p$, and consequently

$$p = p^3 \in p\overline{J}p = pJp \subseteq J,$$

so the proof is complete. ∎

A consequence of Lemma 5.4 is that the two ideals J and $k(h(J))$ have the same sets of spectral idempotents associated with their elements.

Theorem 5.5. *Let $\mathcal{A}$ be a Banach algebra with unit u and suppose that K and J are two inessential ideals of $\mathcal{A}$ having the same sets of spectral idempotents. Then the equivalence class $x + K$ of x in $\mathcal{A}/K$ is invertible if and only if the equivalence class $x + J$ of x in $\mathcal{A}/J$ is invertible. Moreover, if K and J are closed then*

$$\sigma_{\mathcal{A}/K}(x + K) = \sigma_{\mathcal{A}/J}(x + J) \quad \textit{for all } x \in \mathcal{A}. \tag{138}$$

Proof Suppose that $x + J$ is invertible in $\mathcal{A}/J$ but $x + K$ not invertible in $\mathcal{A}/K$. We may suppose without loss of generality that $x + K$ is not right invertible. Then there exists $y \in \mathcal{A}$ such that $a = xy - u \in J$. If $u + a$ were invertible in $\mathcal{A}$ we would have $xy(u + a)^{-1} = u$, thus $x + K$ would be right invertible in $\mathcal{A}/K$. Hence $-1 \in \sigma_{\mathcal{A}}(a)$. Since the ideal J is inessential the point -1 is an isolated point of the spectrum. By Lemma 5.4 the corresponding spectral idempotent p belongs to J, so by hypothesis, it is also in K. By means of the elementary functional calculus it is easy to check that $-1 \in \sigma_{\mathcal{A}}(a - ap)$, so that $xy - ap = u + a - ap$ has an inverse z in $\mathcal{A}$. From this it follows that

$$(x + K)(yz + J) = u + J,$$

and this is a contradiction.

Analogously, a symmetric argument shows that $x+K$ is invertible if and only if $x+J$ is invertible. Replacing x by $\lambda u - x$ we then conclude that the equality (138) holds. ∎

The following result will play an important role in this chapter. It is an immediate consequence of the fact that the two ideals J and $k(h(J))$ have the same spectral idempotents.

Corollary 5.6. *Let $\mathcal{A}$ denote an unital Banach algebra and J an inessential ideal. Then $x \in \mathcal{A}$ is invertible in $\mathcal{A}$ modulo J if and only if x is invertible modulo $k(h(J))$.* ∎

2. The socle

A first important example of inessential ideal is given by the socle of a semi-prime unital Banach algebra. In this section we give some preliminary informations on the socle in a purely abstract setting. We first review some standard facts and definitions.

Let $\mathcal{A}$ be any complex algebra. A *minimal idempotent* element of $\mathcal{A}$ is a non-zero idempotent e such that $e\mathcal{A}e$ is a division algebra. Let us denote by $\operatorname{Min}\mathcal{A}$ the set of all minimal idempotents of $\mathcal{A}$. Recall that a minimal left (right) ideal of an algebra $\mathcal{A}$ is a left ideal $J \neq \{0\}$ such that $\{0\}$ and J are the only left (right) ideals contained in J.

In a semi-prime algebra the minimal left ideals and the minimal right ideals are strictly related to the minimal idempotents. In the following theorem we collect some of these relationships, see Bonsall and Duncan [**72**, §30].

Lemma 5.7. *Let $\mathcal{A}$ be semi-prime algebra and J a left ideal of $\mathcal{A}$.*

(i) *J is a minimal left ideal if and only if there exists $e \in Min\,\mathcal{A}$ such that $J = \mathcal{A}e$.*

(ii) *If J is a minimal left ideal and $x \in \mathcal{A}$ then either $\mathcal{A}x = \{0\}$ or $\mathcal{A}x$ is a minimal left deal.*

Similar statements hold for right ideals.

Note that if $e \in \operatorname{Min}\mathcal{A}$ then the left ideal

$$\mathcal{A}(1-e) := \{a - ae : a \in A\}$$

is a maximal regular left ideal in $\mathcal{A}$. Moreover, $\mathcal{A} = \mathcal{A}e \oplus \mathcal{A}(1-e)$. A similar statement holds for the right ideal

$$(1-e)\mathcal{A} := \{a - ea : a \in A\},$$

see [**72**, Proposition 30.11].

Definition 5.8. *Let $\mathcal{A}$ be an algebra which admits minimal left ideals. The* left socle *of $\mathcal{A}$, denoted by* $\operatorname{Lsoc}\mathcal{A}$*, is defined to be the smallest left ideal containing all minimal left ideals. The* right socle $\operatorname{Rsoc}\mathcal{A}$ *of $\mathcal{A}$ is similarly* *defined in terms of right minimal ideals. If $\mathcal{A}$ has both minimal left and minimal right ideals, and if the left socle coincides with the right socle, then the set* $\operatorname{soc}\mathcal{A} := \operatorname{Lsoc}\mathcal{A} = \operatorname{Rsoc}\mathcal{A}$ *is called the* socle *of* $\mathcal{A}$.

It is not difficult to see that, if $\mathcal{A}$ has minimal left ideals, then $\operatorname{Lsoc}\mathcal{A}$ is a two-sided ideal of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is semi-prime and have minimal left ideals (or, equivalently, have minimal idempotents), then the socle does exist ([**72**, Chapter iv]). If the semi-prime algebra $\mathcal{A}$ has no minimal ideals then we shall set $\operatorname{soc}\mathcal{A} = \{0\}$.

Clearly, for a semi-prime algebra we have $\operatorname{Min}\mathcal{A} \subseteq \operatorname{soc}\mathcal{A}$ and

$$\begin{aligned}\operatorname{soc}\mathcal{A} &= \left\{\sum_{k=1}^{n} \mathcal{A}e_k : n \in \mathbb{N}, e_k \in \operatorname{Min}\mathcal{A}\right\}\\ &= \left\{\sum_{k=1}^{m} e_k\mathcal{A} : m \in \mathbb{N}, e_k \in \operatorname{Min}\mathcal{A}\right\}.\end{aligned}$$

As remarked in Theorem 5.1, a two-sided ideal J of a semi-simple algebra $\mathcal{A}$ is also semi-simple, so it admits a socle. Moreover, from the equality $\operatorname{Min} J = \operatorname{Min}\mathcal{A} \cap J$ and from the property that any minimal right ideal of $\mathcal{A}$ has form $e\mathcal{A}$, where $e \in \operatorname{Min}\mathcal{A}$, it follows easily that $\operatorname{soc} J = \operatorname{soc}\mathcal{A} \cap J$.

Example 5.9. The Banach algebra $\mathcal{A} := L(X)$, where X is a Banach space, is semi-simple. Indeed, $\{0\}$ and X are the unique subspaces of X which are invariant under every $T \in L(X)$. Hence the identity I of $\mathcal{A}$ is an injective irreducible representation of $\mathcal{A}$. Since the radical is the intersection of the kernels of all irreducible representation this implies that $\operatorname{rad} L(X) = \{0\}$.

The minimal idempotents of $L(X)$ and the socle of $L(X)$ may be characterized in a very simple way:

$$\operatorname{Min} L(X) = \{P \in L(X) : P = P^2 \text{ and } \dim P(X) = 1\},$$

and $\operatorname{soc} L(X) = F(X)$, the ideal of all finite-dimensional operators.

In fact, let $P \in L(X)$ be projection with $\dim P(X) = 1$. Then there exists an element $0 \neq z \in X$ and a continuous linear functional $0 \neq f \in x^*$ such that $Px = f(x)z$ for all $x \in X$. Since P is a projection we also have $f(z) = 1$. Let (x_α) be a basis of $(I - P)(X)$. Then $f(x_\alpha) = 0$ for every α and every $x \in X$ admits the representation $x = \sum_\alpha \lambda_\alpha x_\alpha + \lambda z$.

If $S \in L(X)$ is arbitrary then

$$PSPx = PS(\lambda x) = \lambda f(Sx)z.$$

From $PSP \neq 0$ we obtain $\mu := f(Sz) \neq 0$, and hence if $T := \mu^{-1}I$

$$PSPPTP = PTPPSP = \mu^{-1}PSP.$$

From this we obtain

$$PSPPTPx = \mu^{-1}PSPx = \lambda\mu^{-1}z = \lambda z = Px.$$

Hence PSP is invertible in $P\mathcal{A}P$ and therefore $P \in \text{Min}\,\mathcal{A}$.

To show that the elements of $\text{Min}\,L(X)$ are exactly the 1-dimensional projections, assume that $P = P^2$ and dim $P(X) \geq 2$. Then there exist linear independent elements $x_1, x_2 \in X$ such that $Px_k = x_k$, $k = 1, 2$, and two elements $f_1, f_2 \in X^*$ such that $f_j(x_k) = \delta_{jk}$. Let $S \in LX)$ be defined by $Sx := f_1(x)x_1$. Then $Sx_1 = x_1$ and $Sx_2 = 0$. From $PSPx_1 = x_1 \neq 0$ it follows that $PSP \neq 0$, and hence, since $x_2 = Px_2 \in \ker PSP$, the element PSP is not invertible in $P\mathcal{A}P$. Therefore $P \notin \text{Min}\,L(X)$.

Finally, from the definition of socle we easily deduce that soc $L(X) = F(X)$.

Definition 5.10. *Let $\mathcal{A}$ be a semi-prime algebra. A left (respectively, right) ideal J of $\mathcal{A}$ is said to have* finite order *if J can be written as the sum of a finite number of minimal left (respectively, right) ideals of $\mathcal{A}$.*

The order *$\theta(J)$ of J is the smallest number of minimal left (respectively, right) ideals of $\mathcal{A}$ which have sum J. If J has not finite order we define $\theta(J) := \infty$. The zero ideal is assumed to have order* 0.

If J is a two-sided ideal of $\mathcal{A}$ the definition of order of J at first glance may appear ambiguous. However, as a consequence of the next Theorem 5.12 the order of J considered as a left ideal is the same of the order of J considered as a right ideal.

Lemma 5.11. *Assume that the left ideal J of a semi-prime algebra $\mathcal{A}$ has finite order n. Suppose that $f_1, \cdots, f_n$ are minimal idempotents such that the sum $\mathcal{A}f_1 \oplus \cdots \oplus \mathcal{A}f_m \subseteq J$. Then $m \leq n$.*

A similar statement holds for right ideals of finite order.

Proof Let $e_1, \cdots, e_n \in \text{Min}\,\mathcal{A}$ be choose such that $J = \mathcal{A}e_1 + \cdots + \mathcal{A}e_n$. Since $f_1 \in J$, there exist elements $x_k \in \mathcal{A}$ such that $f_1 = \sum_{k=1} x_k e_k$.

Assume that $x_j e_j \neq 0$. Since $\mathcal{A}e_j$ is minimal, from the inclusion $\mathcal{A}x_j e_j \subseteq \mathcal{A}e_j$ we know that either $\mathcal{A}x_j e_j = \mathcal{A}e_j$ or $\mathcal{A}x_j e_j = \{0\}$ holds. But $\mathcal{A}$ is semi-prime, hence faithful, so it must be $\mathcal{A}x_j e_j = \mathcal{A}e_j$ and therefore

$$\mathcal{A}e_j = \mathcal{A}x_j e_j \subseteq \mathcal{A}f_1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{A}x_k e_k \subseteq \mathcal{A}f_1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{A}e_k.$$

From this it follows that

$$J = \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{A}e_k + \mathcal{A}e_j \subseteq \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{A}e_k + \mathcal{A}f_1 \subseteq J,$$

and therefore

$$J = \sum_{\substack{k=1 \\ k \neq j}}^{n} \mathcal{A}e_k + \mathcal{A}f_1.$$

Now, $f_2 \in J$ so there exist elements $y_k \in \mathcal{A}$ and $z \in \mathcal{A}$ such that

$$f_2 = zf_1 + \sum_{\substack{k=1\\k\neq j}}^{n} y_k e_k.$$

Since the sum $\mathcal{A}f_1 + \cdots \mathcal{A}f_m$ is direct it follows that $y_i e_i \neq 0$ for some $i \neq j$. Proceeding as before for the ideal J we obtain the representation

$$J = \mathcal{A}f_1 + \mathcal{A}f_2 + \sum_{\substack{k=1\\k\neq j,i}}^{n} \mathcal{A}e_k.$$

By continuing in this manner we can at each successive step replace an ideal $\mathcal{A}e_q$ with an ideal $\mathcal{A}f_p$. If $m > n$, at the end of this process we obtain that $J = \sum_{k=1}^{m} \mathcal{A}f_k$ and this contradicts the assumption that the sum $\sum_{k=1}^{m} \mathcal{A}f_k$ is direct. Therefore $m \leq n$. ∎

Recall that a set $\mathcal{M} := \{e_1, \cdots, e_n\}$ of elements of $\mathcal{A}$ is *orthogonal* if $e_i e_j = 0$ for every $i \neq j$. It is easily seen that if $\mathcal{M} := \{e_1, \cdots, e_n\}$ is an orthogonal set of minimal idempotents then the sum $\mathcal{A}e_1 + \cdots + Ae_n$ is necessarily direct.

Theorem 5.12. *Let J be a non-zero left ideal of a semi-prime algebra $\mathcal{A}$. If $\theta(J) = n$ then every maximal orthogonal set of minimal idempotents in J contains n elements. If $\mathcal{M} = \{e_1, \cdots, e_n\}$ is such a set and $e = \sum_{k=1}^{n} e_k$, then*

$$J = \mathcal{A}e = \mathcal{A}e_1 \oplus \cdots \oplus \mathcal{A}e_n. \tag{139}$$

Moreover, for every $x \in J$ we have

$$x = xe_1 + \cdots + xe_n = xe. \tag{140}$$

A similar statement holds for right ideals of finite order.

Proof Let $\mathcal{M}$ denote a maximal orthogonal set of minimal idempotents in J. By Lemma 5.11 we know that $\mathcal{M}$ is a finite set. Let $\mathcal{M} := \{e_1, \cdots, e_\nu\}$.

We establish first that there are no minimal idempotents $g \in J$ such that $ge_k = 0$ for all $1 \leq k \leq \nu$. Indeed, assume that there exists a minimal idempotent $g \in J$ such that $ge_k = 0$ for all $1 \leq k \leq \nu$. By the maximality of $\mathcal{M}$ we have $e_k g \neq 0$ for some k. By renumbering the elements of $\mathcal{M}$ we can suppose that $e_j g \neq 0$ if $1 \leq j \leq m$, and $e_j g = 0$ if $j > m$. Let us consider the element

$$f := g - \sum_{k=1}^{m} e_k g.$$

Since $gf = g \neq 0$, then $f \neq 0$ and, as is easy to verify, $fe_k = e_k f = 0$ for all $1 \leq k \leq \nu$. Moreover,

$$f^2 = f(g - \sum_{k=1}^{m} e_k g) = fg = f,$$

and hence $\mathcal{A}f = \mathcal{A}fg = \mathcal{A}g$, which implies that f is minimal idempotent.

This contradicts the property of $\mathcal{M}$ being a maximal orthogonal set of minimal idempotents in J. Hence there can be no minimal idempotents $g \in J$ for which the equality $ge_k = 0$ holds for all $e_k \in \mathcal{M}$.

Now, given an arbitrary element $x \in J$ let us define

$$y := x - \sum_{k=1}^{\nu} xe_k.$$

Clearly $ye_k = 0$ for all $1 \leq k \leq \nu$. If $y \neq 0$ then since $\mathcal{A}y \subseteq J \subseteq \operatorname{soc}\mathcal{A}$ there exists a minimal idempotent element g such that $g \in \mathcal{A}y$. But then $ge_k = 0$ for all $1 \leq k \leq \nu$. This implies that $y = 0$, so for any $x \in J$ we have

$$x = \sum_{k=1}^{\nu} xe_k. \tag{141}$$

Hence if $e := \sum_{k=1}^{\nu} e_k$ then $J = \mathcal{A}e = \mathcal{A}e_1 \oplus \cdots \oplus \mathcal{A}e_\nu$.

Next, we show that $\nu = n$. By Lemma 5.11 we have $\nu \leq n$. On the other hand, ν cannot be strictly less that n by the definition of order of an ideal and since $J = \sum_{k=1}^{\nu} \mathcal{A}e_k$. Therefore $\nu = n$.

The last assertion is evident from (141), since $\nu = n$. ∎

Remark 5.13. Let J be a left ideal of finite order n. It not difficult to verify that every left ideal $K \subseteq J$ has finite order $m \leq n$. If K is properly contained in J then m is strictly less than n. Moreover, any maximal orthogonal set of minimal idempotents in K may be extended to a maximal orthogonal set of minimal idempotents in J.

Theorem 5.14. *If $x \in \mathcal{A}$, where $\mathcal{A}$ a semi-prime algebra, the following statements hold:*

(i) *If $x \in \operatorname{soc}\mathcal{A}$ then both $\mathcal{A}x$ and $x\mathcal{A}$ have finite order;*

(ii) *If $\mathcal{A}$ has a unit then $\theta(x\mathcal{A}) = \theta(\mathcal{A}x)$ for every $x \in \mathcal{A}$. Moreover, $x \in \operatorname{soc}\mathcal{A}$ precisely when $\theta(\mathcal{A}x) < \infty$.*

Proof (i) Assume that $x \in \operatorname{soc}\mathcal{A}$. Clearly, by the definition of a socle, the element x belongs to a finite sum of left minimal ideals and hence to a left ideal J of finite order. From the inclusions $\mathcal{A}x \subseteq \mathcal{A}J \subseteq J$ it then follows that also $\mathcal{A}x$ is of finite order. Similarly $x\mathcal{A}$ is of finite order.

(ii) Suppose that $\mathcal{A}$ has a unit u and $\theta(\mathcal{A}x) = n$. By Theorem 5.12 there exists a set of minimal orthogonal idempotents $\{e_1, \cdots, e_n\}$ such that $\mathcal{A}x = \mathcal{A}e_1 \oplus \cdots \oplus \mathcal{A}e_n$ and $z = ze_1 + \cdots + ze_n$ for every $z \in \mathcal{A}x$. In particular, since $x = ux \in \mathcal{A}x$ we have $x = xe_1 + \cdots + xe_n$, and hence

$$x\mathcal{A} \subseteq xe_1\mathcal{A} \oplus \cdots \oplus xe_n\mathcal{A} \subseteq x\mathcal{A},$$

from which we obtain $x\mathcal{A} = xe_1\mathcal{A} \oplus \cdots \oplus xe_n\mathcal{A}$. By part (iii) of Lemma we know that the left ideal $xe_k\mathcal{A}$ either is $\{0\}$ or minimal . According to Lemma

5.7 it follows that there exist some $f_k \in \text{Min}\,\mathcal{A}$, where $1 \leq k \leq m \leq n$, such that

$$x\mathcal{A} = f_1\mathcal{A} + \cdots f_m\mathcal{A}.$$

This shows that $\theta(x\mathcal{A}) \leq m \leq \theta(\mathcal{A}x)$. Analogously, from $\theta(x\mathcal{A}) = n$ we obtain that also $\theta(\mathcal{A}x) \leq \theta(x\mathcal{A})$, thus $\theta(\mathcal{A}x) = \theta(x\mathcal{A})$.

Now, assume that $\mathcal{A}x$ has finite order m. By Theorem 5.12 there exists a set of minimal orthogonal idempotents $\{q_1, \cdots, q_m\}$ such that

$$x = ux = xq_1 + \cdots + xq_m,$$

from which we conclude that $x \in \text{soc}\,\mathcal{A}$. ∎

The following theorem shows the relationship between minimal idempotents and primitive ideals.

Theorem 5.15. *Let $\mathcal{A}$ be a semi-prime algebra. For every $e \in Min\,\mathcal{A}$ there exists an unique primitive ideal $P \in \Pi(\mathcal{A})$ such that $e \notin P$.*

Proof Assume first that $\mathcal{A}$ is semi-simple. If $e \in \text{Min}\,\mathcal{A}$ then $\mathcal{A}(1-e)$ is a maximal regular left ideal and hence

$$P := \{x \in \mathcal{A} : x\mathcal{A} \subseteq \mathcal{A}(1-e)\}$$

is a primitive ideal, see Bonsall and Duncan [**72**, §24, Proposition 12]. Obviously the minimal idempotent $e \notin P$.

In order to prove the P is the unique primitive ideal which does not contain e, let Q be another primitive ideal such that $e \notin Q$. Then $\mathcal{A}e$ being minimal we have $Q \cap \mathcal{A}e = \{0\}$. From this we obtain $Qe = \{0\}$.

Let $q \in Q$ be arbitrary given. Clearly $q\mathcal{A} \subseteq Q$, and therefore $q\mathcal{A}e = \{0\}$. This implies that $q \in P$ and hence $Q \subseteq P$.

On the other hand, $P\mathcal{A}e = \{0\}$ so either the inclusions $\mathcal{A}e \subseteq Q$ or $P \subseteq Q$ hold [**72**, Proposition 24.12]. The first one does not hold since $Q \cap \mathcal{A}e = \{0\}$, so it must be $P \subseteq Q$. Hence $P = Q$.

The result for the non semi-simple case follows from consideration of the semi-simple Banach algebra $\mathcal{A}' := \mathcal{A}/\text{rad}\,\mathcal{A}$. Clearly $p' := p + \text{rad}\,\mathcal{A}$ is a minimal idempotent of $\mathcal{A}'$, and the proof is easily completed by using the homeomorphism between the structure spaces $\Pi(\mathcal{A})$ and $\Pi(\mathcal{A}')$. ∎

3. The socle of semi-prime Banach algebras

In this section we shall establish several characterizations of the socle in the case $\mathcal{A}$ is a semi-prime Banach algebra. Observe first that for a semi-prime Banach algebra $\mathcal{A}$ if $e \in \text{Min}\,\mathcal{A}$, the division algebra $e\mathcal{A}e$ is also a normed algebra over $\mathbb{C}$, and hence $e\mathcal{A}e = \mathbb{C}e$. Moreover, for any complex normed algebra $\mathcal{A}$ if e is idempotent then $\mathcal{A}e$ is a closed left ideal. In fact, if $b_n := a_ne \in J \to b$ as $n \to \infty$, then $b_ne \to be$ and from $b_ne = a_ne^2 = a_ne \to ab$ we deduce that $b = be \in J$.

We need the following important lemma is owed to Kaplansky [**185**]. For the proof see also Aupetit [**52**, Theoreme 1, p.70] or Hirschfeld and Johnson [**163**].

Lemma 5.16. *Let $\mathcal{A}$ be a Banach algebra such that each element $x \in \mathcal{A}$ has a finite spectrum. Then $\mathcal{A}/\mathrm{rad}\,\mathcal{A}$ is finite-dimensional. In particular, if each element of a semi-simple Banach algebra $\mathcal{A}$ has finite spectrum, then $\mathcal{A}$ is finite-dimensional.* ∎

Definition 5.17. *Given an algebra $\mathcal{A}$ and $x, y \in \mathcal{A}$, the* wedge operator *$x \wedge y$ is the linear operator on $\mathcal{A}$ defined by*

$$(x \wedge y)a := xay \quad \textit{for every } a \in \mathcal{A};$$

The next result shows that for a semi-prime Banach algebra, not necessarily unital, the elements of the socle generate finite-dimensional wedge operators.

Theorem 5.18. *Let $\mathcal{A}$ be a semi-prime Banach algebra. We have:*

(i) *If $e, f \in Min\,\mathcal{A}$ then $\dim e\mathcal{A}f \leq 1$;*

(ii) *If e is idempotent and $e\mathcal{A}e$ is finite-dimensional then $e \in \mathrm{soc}\,\mathcal{A}$.*

(iii) *If $x \in \mathrm{soc}\,\mathcal{A}$ then $x \wedge x$ is finite-dimensional.*

Proof (i) Suppose that there exist two linearly independent elements exf and eyf. Then $exf \neq 0$ and $\{0\} \neq \mathcal{A}exf \subseteq \mathcal{A}f$. The minimality of $\mathcal{A}f$ then yields $\mathcal{A}f = \mathcal{A}exf$, so there is an element $z \in \mathcal{A}$ such that $eyf = zexf$. From this we obtain

$$eyf = e(eyf) = ezexf = (eze)exf \in \mathbb{C}exf,$$

and this contradicts the linear independence of exf and eyf. Therefore $\dim\,(e\mathcal{A}f) \leq 1$.

(ii) Suppose that $e \neq 0$ and $e\mathcal{A}e$ finite-dimensional. Then $Ae \neq 0$, since A is faithful. Let J be a left ideal of $\mathcal{A}$ with $J \subseteq Ae$ and $J \neq \{0\}$. Since $\mathcal{A}$ is semi-prime $J^2 \neq \{0\}$, and hence there exist elements be, ce of J such that $bece \neq 0$. Clearly $ece \neq 0$ and $ece \in J \cap e\mathcal{A}e$. Therefore $e\mathcal{A}e$ is a finite-dimensional subspace of $\mathcal{A}$ having intersection with all non-zero left ideals of $\mathcal{A}$ contained in $\mathcal{A}e$.

From this it follows that there exists a non-zero subspace X_1 of $e\mathcal{A}e$ together with a left ideal J_1 of $\mathcal{A}$, where $J_1 \subseteq \mathcal{A}e$ and $J_1 \cap e\mathcal{A}e = X_1$, such that $J_1 \cap X_1$ is either $\{0\}$ or X_1. Let J_2 denote the intersection of all left ideals J of $\mathcal{A}$ such that $J \subseteq \mathcal{A}e$ and $J \cap e\mathcal{A}e = X_1$. Then J_2 is a minimal left ideal of $\mathcal{A}$, and hence by Lemma 5.7 $J_2 = \mathcal{A}f$ for some $f \in \mathrm{Min}\,\mathcal{A}$.

Let $\{a_1, \dots, a_n\}$ be a basis for $e\mathcal{A}e$, with $a_1, \dots, a_m$ in X_1. Clearly $a_j = a_jf$ for all $j = 1, \dots, m$. If b is arbitrarily given in $\mathcal{A}$ then

$$ebe = \sum_{j=1}^{n} \lambda_j a_j \quad \text{and} \quad efbe = \sum_{j=1}^{n} \mu_j a_j.$$

Hence

$$\begin{aligned}(e-ef)b(e-ef) &= ebe - ebef - efbe + efbef \\ &= \sum_{j=1}^{n} \lambda_j(a_j - a_j f) - \sum_{j=1}^{n} \mu_j(a_j - a_j f) \\ &= \sum_{j=m+1}^{n} \lambda_j(a_j - a_j f) - \sum_{j=m+1}^{n} \mu_j(a_j - a_j f).\end{aligned}$$

If we put $b_j := a_j - a_j f$ then

$$e - ef)\mathcal{A}(e - ef) \subseteq \sum_{j=m+1}^{n} \mathbb{C}b_j,$$

and thus the rank of the wedge operator $(e - ef) \wedge (e - ef)$ is strictly less than n.

Now, after a finite number of repetitions of this process we conclude that $(e-g)\mathcal{A}(e-g) = \{0\}$ for some $g \in \operatorname{soc}\mathcal{A}$. But $\mathcal{A}$ is semi-prime, so $e - g = 0$ and hence $e = g \in \operatorname{soc}\mathcal{A}$.

(iii) Let $x \in \operatorname{soc}\mathcal{A}$. Then, as we have already seen in the proof of part (i) of Theorem 5.14, x belongs to a left ideal of finite order, so by Theorem 5.12 there exists a maximal orthogonal subset of minimal idempotents $\{e_1, \cdots, e_n\}$ such that $x = \sum_{k=1}^{n} xe_k$. Therefore

$$\begin{aligned}x\mathcal{A}x &= \left(\sum_{k=1}^{n} xe_k\right)\mathcal{A}\left(\sum_{j=1}^{n} xe_j\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} xe_k\mathcal{A}xe_j \\ &\subseteq \sum_{k=1}^{n}\sum_{j=1}^{n} xe_k\mathcal{A}e_j = \sum_{k=1}^{n}\sum_{j=1}^{n} x\ \mathbb{C}\ b_{kj},\end{aligned}$$

where b_{kj} is a fixed element of $e_k\mathcal{A}e_j$. Thus $\dim\,(x\mathcal{A}x) < \infty$. ∎

Recall that an element x of a Banach algebra $\mathcal{A}$ with unit u is said to be *algebraic* if there exists a polynomial α such that $\alpha(x) = 0$. A first example of algebraic elements is given by each element of a finite-dimensional Banach algebra. In fact, if $\dim \mathcal{A} = n$, for every $x \in \mathcal{A}$ then there exist $\lambda_1, \ldots, \lambda_n$ such that $x^n + \lambda_1 x^{n-1} + \cdots + \lambda_n u = 0$.

From the spectral mapping theorem it easily follows that every algebraic element $x \in \mathcal{A}$ has finite spectrum. Indeed, if $\alpha(x) = 0$ then $\{0\} = \sigma(\alpha(x)) = \alpha(\sigma(x))$ so that every $\lambda \in \sigma(x)$ is a zero of the polynomial α, and hence $\sigma(x)$ is a finite set.

Lemma 5.19. *Let $\mathcal{A}$ be a Banach algebra with unit. If $x \in \operatorname{rad}\mathcal{A}$ is algebraic then x is nilpotent.*

Proof Let α be a polynomial such that $\alpha(x) = 0$. If $x \in \operatorname{rad}\mathcal{A}$ then $\sigma(x) = \{0\}$, so by the spectral mapping theorem

$$\{0\} = \sigma(\alpha(x)) = \alpha(\sigma(x)) = \{\alpha(0)\}.$$

Therefore $\alpha(\lambda) = \lambda^n\beta(\lambda)$ for some positive integer n and some polynomial β with $\beta(0) \neq 0$. Clearly $\beta(x) \notin \operatorname{rad}\mathcal{A}$, and hence $\sigma(\beta(x)) = \{\beta(0)\} \neq \{0\}$. Therefore $\beta(x)$ is invertible and

$$0 = 0 \cdot \alpha(x) \cdot (\beta(x))^{-1} = x^n\beta(x) \cdot (\beta(x))^{-1} = x^n,$$

as desired. ∎

Remark 5.20. A classical result from the theory of Banach algebras establishes that if a two-sided J of a complex semi-prime Banach algebra is *nil*, namely each element of J is quasi-nilpotent then $J = \{0\}$, see Corollary 46.5 of Bonsall and Duncan [**72**]. From this and from Lemma 5.19 we then conclude that if $\mathcal{A}$ be a semi-prime Banach algebra and J a two-sided ideal of $\mathcal{A}$ such that each element of J is algebraic, then $J \cap \operatorname{rad}\mathcal{A} = \{0\}$.

An argument similar to that in the proof of the previous lemma shows that if x is a quasi-nilpotent algebraic element which lies in an ideal J of $\mathcal{A}$ then x is nilpotent.

As an immediate consequence of Lemma 5.19 it follows that every finite-dimensional semi-prime Banach algebra is semi-simple. In fact, if $\mathcal{A}$ is finite-dimensional then every element of $\mathcal{A}$ is algebraic. If $x \in \operatorname{rad}\mathcal{A}$ then by Lemma 5.19 x is nilpotent, so $\operatorname{rad}\mathcal{A}$ is nil and hence is equal to $\{0\}$.

Further information about the ideals for which each element is algebraic are given in the following theorem. We need first to recall that the *Wedderburn theorem* establishes that in every finite-dimensional complex algebra $\mathcal{A}$ the primitive ideals are the maximal ideals and the structure space $\Pi(\mathcal{A})$ is a finite set with the discrete topology; see Proposition 26.7 of Bonsall and Duncan [**72**].

Theorem 5.21. *Let $\mathcal{A}$ be a semi-prime Banach algebra with unit u and J an ideal of $\mathcal{A}$ such that every element of J is algebraic. Then we have:*

(i) *$\overline{J}$ is a semi-simple Banach algebra;*

(ii) *Every idempotent $e \in J$ belongs to $\operatorname{soc}\mathcal{A}$ and $e\mathcal{A}e$ is a finite-dimensional semi-simple Banach algebra with unit e;*

(iii) *$x\mathcal{A} \cap \operatorname{Min}\mathcal{A} \neq \varnothing$ for every $0 \neq x \in J$ and every subset of minimal orthogonal idempotents of $\mathcal{A}x$ is finite. A similar statement holds for $\mathcal{A}x$;*

(iv) *For every $x \in J$ there exists an idempotent $e \in \operatorname{soc}\mathcal{A} \cap \mathcal{A}x$ such that $x = xe$;*

(v) *$J \subseteq \operatorname{soc}\mathcal{A}$.*

Proof (i) By Theorem 5.1 we have $\operatorname{rad}\overline{J} = \overline{J} \cap \operatorname{rad}\mathcal{A}$, so it suffices to show that $\overline{J} \cap \operatorname{rad}\mathcal{A} = \{0\}$. By Remark 5.20 we know that $J \cap \operatorname{rad}\mathcal{A} = \{0\}$, and from the inclusion $J\ \mathrm{rad}\,\mathcal{A} \subseteq J \cap \mathrm{rad}\,\mathcal{A}$ we deduce that $J\ \mathrm{rad}\,\mathcal{A} = \{0\}$.

Suppose now that $x \in \overline{J} \cap \mathrm{rad}\,\mathcal{A}$ and $(x_n) \subset J \cap \mathrm{rad}\,\mathcal{A}$ is a sequence which converges to x. Then $0 = x_n x \to x^2$ as $n \to \infty$, and therefore $x^2 = 0$. This implies that $\overline{J} \cap \mathrm{rad}\,\mathcal{A}$ is nil and therefore $\overline{J} \cap \mathrm{rad}\,\mathcal{A} = \{0\}$.

(ii) Let $0 \neq e = e^2 \in J$. The ideal $\overline{J}$ is semi-simple, by part (i), and hence also $e\overline{J}e$ is semi-simple, as observed before Theorem 5.18. Obviously $e\overline{J}e \subseteq e\mathcal{A}e \subseteq J$ so that

$$e(e\mathcal{A}e)e = e\mathcal{A}e \subseteq eJe \subseteq e\overline{J}e,$$

and hence $e\mathcal{A}e = e\overline{J}e$. The semi-simple Banach algebra $e\mathcal{A}e$ has unit e and every element of it is algebraic and therefore has finite spectrum. By the Kaplansky theorem we conclude that $e\mathcal{A}e$ is finite-dimensional, and hence part (ii) of Theorem 5.18 implies that $e \in \mathrm{soc}\,\mathcal{A}$.

(iii) Let $0 \neq x \in J$. We show first that there exists $b \in \mathcal{A}$ such that bx is not quasi-nilpotent. Indeed, suppose that $\sigma(ax) = \{0\}$ for every $a \in \mathcal{A}$. The characterization (135) of the radical then entails that $x \in \mathrm{rad}\,\mathcal{A}$, and hence, see Remark 5.20, $x \in J \cap \mathrm{rad}\,\mathcal{A} = \{0\}$, which is impossible.

Now, by assumption every element of J is algebraic and hence has a finite spectrum, in particular, since $bx \in J$, $\sigma(bx)$ is a finite set. Let $\mathcal{B}$ denote the commutative Banach algebra generated by bx. Since bx is algebraic, $\mathcal{B}$ is a finite-dimensional algebra which is not radical, and therefore by the Wedderburn theorem possesses a non-zero idempotent $e \in \mathcal{B} \subseteq J$. By part (ii) we then conclude that $e \in \mathrm{soc}\,\mathcal{A}$ and hence $\mathcal{A}x \cap \mathrm{Min}\,\mathcal{A} \neq \{0\}$.

In order to show that every set in $\mathcal{A}x$ of orthogonal minimal idempotents is finite, suppose that there exists an infinite set (e_n) of non-zero orthogonal idempotents of $\mathcal{A}x$ and let $e_n := a_n x$, $a_n \in \mathcal{A}$, for every $n \in \mathbb{N}$. Choose a sequence of distinct points $(\lambda_n) \subset \mathbb{C}$ such that $|\lambda_n| < 2^{-n}\|a_n\|$ for all n. Then the series $\sum_{n=1}^{\infty} \lambda_n a_n$ converges to some $a \in \mathcal{A}$ and $ax = \sum_{n=1}^{\infty} \lambda_n e_n$.

Clearly $e_k ax = \lambda_k e_k$ for every $k \in \mathbb{N}$ and hence $e_k(ax - \lambda_k u) = 0$. From this we infer that $ax - \lambda_k u$ is not invertible for every k, namely $\lambda_k \in \sigma(ax)$ for every k, and this contradicts the finiteness of $\sigma(ax)$.

(iv) Let $x \in J$. If $x = 0$ we take $e = 0$. Suppose that $x \neq 0$. From part (iii) there exists in a finite maximal subset $\Omega = \{e_1, \dots, e_n\}$ of orthogonal minimal idempotents of $\mathcal{A}x$. Note that $xe_k \neq 0$ for some $1 \leq k \leq n$. Let $e := e_1 + \cdots + e_n$. Clearly $e^2 = e \in \mathrm{soc}\,\mathcal{A} \cap \mathcal{A}x$.

We show now that $x = xe$. Suppose $x \neq xe$. The right ideal $(x - xe)\mathcal{A}$ contains a minimal idempotents, f say. Then $f = (x - xe)a$ for some $a \in \mathcal{A}$.

Let us consider the element defined by

$$g := (u - e)afx(u - e).$$

It is easy to verify that

$$g^2 = g = (u - e)afx - (u - e)afxf \in \mathcal{A}x + \mathcal{A}x \subseteq Ax,$$

and $0 = ge_k = e_k g$ for every $k = 1, \cdots, n$. Hence g is a minimal idempotent in $\mathcal{A}x$ orthogonal to all elements e_k, contradicting the maximality of Ω.

(v) This is clear from part (iv). ∎

Corollary 5.22. *The socle of a unital semi-prime Banach algebra $\mathcal{A}$ is the largest ideal of $\mathcal{A}$ whose elements are algebraic.*

Proof If $x \in \operatorname{soc} \mathcal{A}$ then $x\mathcal{A}x$ is finite-dimensional by part (iii) of Lemma 5.18. This implies that $x^2 = xux$ is algebraic and hence also x is algebraic.

On the other hand, if J is an ideal each of whose elements is algebraic then by part (iv) of Lemma 5.21 $J \subseteq \operatorname{soc} \mathcal{A}$, hence $\operatorname{soc} \mathcal{A}$ is the largest ideal of $\mathcal{A}$ whose elements are algebraic. ∎

It should be noted that since each element of $\operatorname{soc} \mathcal{A}$ is algebraic the argument used in the proof of part (i) of Theorem 5.21 shows that for every semi-prime unital Banach algebra we have $\overline{\operatorname{soc} \mathcal{A}} \cap \operatorname{rad} \mathcal{A} = \{0\}$.

Remark 5.23. An inspection of the proof of Theorem 5.21 reveals that the results established there remain valid if we suppose that J is a two-sided ideal of elements having finite spectrum. From this observation it readily follows that in a unital semi-prime Banach algebra $\mathcal{A}$ the socle is also the largest ideal of $\mathcal{A}$ whose elements have finite spectrum.

Theorem 5.24. *Let $\mathcal{A}$ be a unital semi-prime Banach algebra and $x \in \mathcal{A}$. Then*

$$x \in \operatorname{soc} \mathcal{A} \Leftrightarrow x\mathcal{A}x \quad \textit{is finite-dimensional.}$$

Proof By part (iii) of Theorem 5.18 we need only to prove that if $x\mathcal{A}x$ is finite-dimensional then $x \in \operatorname{soc} \mathcal{A}$.

Suppose that $\dim x\mathcal{A}x < \infty$ and let J be the ideal generated by x. Every element y of J admits a representation

$$y = \lambda x + ax + xb + \sum_{k=}^{n} a_k x b_k, \quad a, b, a_k, b_k \in \mathcal{A},\ \lambda \in \mathbb{R}.$$

It is clear that $\dim y\mathcal{A}y < \infty$, so y is algebraic. By Theorem 5.22 we then conclude that $J \subseteq \operatorname{soc} \mathcal{A}$, and in particular $x \in \operatorname{soc} \mathcal{A}$. ∎

Now, let $\mathcal{A}$ be a commutative semi-prime Banach algebra. In this case $e\mathcal{A} = e\mathcal{A}e = \mathbb{C}e$ for every $e \in \operatorname{Min} \mathcal{A}$, so the socle may be characterized in the following simple way:

$$\operatorname{soc} \mathcal{A} = \operatorname{span} \{\operatorname{Min} \mathcal{A}\}.$$

The elements x of the socle in this case may be described by means of the action of the multiplication operators L_x on $\mathcal{A}$.

Corollary 5.25. *Let $\mathcal{A}$ be a semi-prime commutative Banach algebra. If $x \in \operatorname{soc} \mathcal{A}$ then L_x is finite-dimensional. If, additionally, $\mathcal{A}$ is unital then*

$$x \in \operatorname{soc} \mathcal{A} \Leftrightarrow L_x \textit{ is finite-dimensional.}$$

Proof If $x \in \operatorname{soc} \mathcal{A}$ then $x = \sum_{k=1}^{n} \lambda_k e_k$, where $e_k \in \operatorname{Min} \mathcal{A}$ for all $k = 1, \cdots, n$. Since $e_k \mathcal{A} = e_k \mathcal{A} e_k = \mathbb{C} e_k$ the operators L_{e_k} are 1-dimensional, so $L_x = \sum_{k=1}^{n} \lambda_k L_{e_k}$ is finite-dimensional.

Conversely, if $\mathcal{A}$ is unital and L_x is finite-dimensional then $x\mathcal{A}x$ is finite-dimensional, and consequently by Theorem 5.24 x belongs to $\operatorname{soc} \mathcal{A}$. ∎

Remark 5.26. It is easy to see that in a semi-simple commutative Banach algebra $\mathcal{A}$ an idempotent element is minimal if the support of its Gelfand transform is a singleton. From the Shilov idempotent theorem it then follows that there is a one to one correspondence between the minimal idempotents and the isolated points of $\Delta(\mathcal{A})$. In particular, the elements of the socle are precisely the elements having a finite support of its Gelfand transform and $\operatorname{soc} \mathcal{A} = \{0\}$ if and only if $\Delta(\mathcal{A})$ has no isolated points.

The next result shows that in a semi-simple Banach algebra every finite-dimensional left (right) ideal is a subset of the socle.

Theorem 5.27. *Let $\mathcal{A}$ be a semi-simple Banach algebra. If J is finite-dimensional left (respectively, right) ideal of $\mathcal{A}$ then there exists an idempotent $p \in \operatorname{soc} \mathcal{A}$ such that $\mathcal{A}p = J$ (respectively, $J = p\mathcal{A}$).*

Proof The case $J = \{0\}$ is trivial. Suppose $J \neq \{0\}$ and consider a minimal idempotent $e \in J$. Let Γ be the set of all left ideal of the form $J \cap \mathcal{A}(1-q)$, where $q = q^2 \in J \cap \operatorname{soc} \mathcal{A}$. Obviously $\Gamma \neq \varnothing$ because $J \cap \mathcal{A}(1-e) \in \Gamma$. Moreover, since J is finite-dimensional there exists a minimal element of Γ, say $J \cap \mathcal{A}(1-p)$.

We claim that $J \cap \mathcal{A}(1-p) = \{0\}$. To see that, assume $J \cap \mathcal{A}(1-p) \neq \{0\}$. Then there exists $f \in \operatorname{Min} \mathcal{A}$ such that $\mathcal{A}f \subseteq J \cap \mathcal{A}(1-p)$. Let $v := p + f - pf$. From $f \in \mathcal{A}(1-p)$ we obtain $fp = 0$ and hence $vf = f$, $vp = p$. From this it easily follows that $v^2 = v$ and $v \in J \cap \operatorname{soc} \mathcal{A}$.

Now, let us consider an arbitrary element $x \in J \cap \mathcal{A}(1-v)$. We have $xv = 0$ and therefore $0 = xvp = xp$ which implies $x = x - xp \in \mathcal{A}(1-p)$. Hence

$$J \cap \mathcal{A}(1-v) \subseteq J \cap \mathcal{A}(1-p). \tag{142}$$

From

$$fv = fp + f^2 - fpf = f^2 = f \neq 0$$

we infer that $f \notin \mathcal{A}(1-v)$, so the inclusion (142) is proper and this contradicts the choice of p. Hence $J \cap \mathcal{A}(1-p)$ is necessarily $\{0\}$. Finally, if $x \in J$ then $x(1-p) \in J \cap \mathcal{A}(1-p) = \{0\}$, so $x = xp \in \mathcal{A}p$.

This shows that $J \subseteq \mathcal{A}p$ and, since the opposite inclusion $\mathcal{A}p \subseteq J$ is true, because $p \in J$, the proof is now complete. ∎

4. Riesz algebras

In general, given an arbitrary Banach algebra $\mathcal{A}$ the socle does not exist. In this case a Fredholm theory for a Banach algebra may be obtained by replacing the socle with the so called pre-socle. Recall first that, by Theorem 5.1 if $\mathcal{A}$ is an algebra the quotient algebra $\mathcal{A}' := \mathcal{A}/\mathrm{rad}\,\mathcal{A}$ is semi-simple, so its socle exists. Moreover, as in the commutative case a Banach algebra $\mathcal{A}$ is said to be *radical* if $\mathcal{A} = \mathrm{rad}\,\mathcal{A}$.

In the sequel let $x' \in \mathcal{A}'$ denote the equivalence class $x + \mathrm{rad}\,\mathcal{A}$, where $x \in \mathcal{A}$.

Definition 5.28. *Given an algebra $\mathcal{A}$ the* pre-socle *of $\mathcal{A}$ is defined by*

$$\mathrm{psoc}\,\mathcal{A} := \{x \in \mathcal{A} : x' \in soc\,\mathcal{A}'\}.$$

Evidently $\mathrm{psoc}\,\mathcal{A}$ is two-sided ideal of $\mathcal{A}$ which contains $\mathrm{rad}\,\mathcal{A}$, whilst if $\mathcal{A}$ is semi-simple, $\mathrm{rad}\,\mathcal{A} = \{0\}$, so $\mathrm{psoc}\,\mathcal{A} = \mathrm{soc}\,\mathcal{A}$. Moreover, for every radical algebra $\mathcal{A}$ we have $\mathrm{psoc}\,\mathcal{A} = \mathrm{rad}\,\mathcal{A}$.

The next result shows, perhaps not surprisingly, that the equality $\mathrm{psoc}\,\mathcal{A} = \mathrm{soc}\,\mathcal{A}$ characterizes semi-simple algebras amongst unital semi-prime algebras.

Theorem 5.29. *For every semi-prime algebra $\mathcal{A}$ we have* $\mathrm{soc}\,\mathcal{A} \subseteq \mathrm{psoc}\,\mathcal{A}$. *Moreover, a semi-prime algebra $\mathcal{A}$ is semi-simple if and only if* $\mathrm{soc}\,\mathcal{A} = \mathrm{psoc}\,\mathcal{A}$.

Proof If $\mathcal{A}$ does not admit minimal left ideal then $\mathrm{soc}\,\mathcal{A} = \{0\} \subseteq \mathrm{psoc}\,\mathcal{A}$.

Suppose that $\mathcal{A}$ admits a non-trivial minimal left ideal J and consider the two distinct cases $J^2 = \{0\}$ and $J^2 \neq \{0\}$. If $J^2 = \{0\}$, for every $x \in J$ and $a \in \mathcal{A}$ the quasi-product

$$(ax) \circ (-ax) := ax - xa + (xa)^2 = 0,$$

so, by (132) x belongs to rad $\mathcal{A}$ and hence $x \in \mathrm{psoc}\,\mathcal{A}$.

Consider the second case $J^2 \neq \{0\}$. According Lemma 5.7 let $e \in \mathrm{Min}\,\mathcal{A}$ be such that $J = \mathcal{A}e$. It is evident that $e' := e+\mathrm{rad}\,\mathcal{A}$ is minimal idempotent in $\mathcal{A}' = \mathcal{A}/\mathrm{rad}\,\mathcal{A}$, so $J' := \mathcal{A}'e'$ is a minimal left ideal of $\mathcal{A}'$ and therefore is contained in $\mathrm{soc}\,\mathcal{A}'$. Hence $J \subseteq \mathrm{psoc}\,\mathcal{A}$.

In both cases the ideal $\mathrm{psoc}\,\mathcal{A}$ then contains every left minimal ideal of $\mathcal{A}$ and hence $\mathrm{soc}\,\mathcal{A} \subseteq \mathrm{psoc}\,\mathcal{A}$ for every semi-prime Banach algebra.

To prove the last assertion, assume $\mathcal{A}$ semi-prime and $\mathrm{soc}\,\mathcal{A} = \mathrm{psoc}\,\mathcal{A}$. Let $0 \neq x \in \mathrm{rad}\ \mathcal{A} \subseteq \mathrm{soc}\,\mathcal{A} = \mathrm{psoc}\,\mathcal{A}$. Then by Theorem 5.14 $\mathcal{A}x$ is a nontrivial left ideal of finite order, and hence according to Theorem 5.12 there exist minimal idempotents $e_1, \dots e_n$ contained in $\mathcal{A}x$ such that $\mathcal{A}x = \mathcal{A}e_1 + \cdots + \mathcal{A}e_n$. Now, rad $\mathcal{A}$ is an ideal of $\mathcal{A}$, so $e_1 \in \mathcal{A}x \subseteq \mathrm{rad}\ \mathcal{A}$ and hence $e_1 = 0$, since the radical does not contain any non-zero idempotent. This contradicts the property $e_1 \in \mathrm{Min}\ \mathcal{A}$. Therefore $\mathrm{rad}\,\mathcal{A} = \{0\}$ and hence $\mathcal{A}$ is semi-simple. ∎

The following illuminating example shows how in a non-semi-simple semi-prime algebra the difference between $\mathrm{soc}\,\mathcal{A}$ and $\mathrm{psoc}\,\mathcal{A}$ may be extreme.

Example 5.30. Let $\mathcal{A}$ denote the commutative algebra of all formal power series in the indeterminate x, where the sum is defined componentwise and the product is the usual Cauchy product. Clearly, a subset J of $\mathcal{A}$ is an ideal if and only if there exists $k \in \mathbb{N} \cup \{0\}$ such that $J = J_k$, where

$$J_k := \left\{ \sum_{n=k}^{\infty} \lambda_n x^m : \lambda_n \in \mathbb{C} \right\}.$$

Since for every $k \in \mathbb{N} \cup \{0\}$ the ideal $J_k \neq \varnothing$ and J_k does not contain J_{k+1}, the algebra $\mathcal{A}$ does not admit any minimal ideal, and therefore $\operatorname{soc} \mathcal{A} = \{0\}$. Moreover, $\mathcal{A}$ is semi-prime because $J^2 = \{0\}$ implies $J = \{0\}$. The algebra $\mathcal{A}$ has identity $u := x^0$, and the elements which are invertible are all the power series $\sum_{n=0}^{\infty} \lambda_n x^n$ with $\lambda_0 \neq 0$. From this it follows that

$$\operatorname{rad} \mathcal{A} = \left\{ \sum_{n=1}^{\infty} \lambda_n x^n : \lambda_n \in \mathbb{C} \right\}.$$

Hence $\mathcal{A}$ is not semi-simple and $\mathcal{A}' = \mathcal{A}/\operatorname{rad} \mathcal{A} = \{\lambda u' : \lambda \in \mathbb{C}\}$. Evidently $\mathcal{A}'$ is the unique nonzero ideal of $\mathcal{A}'$, so $\operatorname{soc} \mathcal{A}' = \mathcal{A}'$ and therefore $\operatorname{psoc} \mathcal{A} = \mathcal{A}$.

Remark 5.31. Let $x \in \mathcal{A}$, $\mathcal{A}$ a Banach algebra with unit u, and $x' = x + \operatorname{rad} \mathcal{A} \in \mathcal{A}'$. Then $\sigma(x) = \sigma(x')$ for every $x \in \mathcal{A}$. The inclusion $\sigma(x') \subseteq \sigma(x)$ is immediate. The opposite inclusion easily follows from the property that if x' is invertible in $\mathcal{A}'$ then x is invertible in $\mathcal{A}$. Indeed, if x' is invertible then there exists $y \in \mathcal{A}$ and $z \in \operatorname{rad} \mathcal{A}$ such that $yx = u + z$. The characterization (133) of the radical yields that $u + z$ is invertible, and hence x has a left inverse. The same argument shows that x has a right inverse, hence x is invertible.

Theorem 5.32. *Let $\mathcal{A}$ a Banach algebra. Then $\operatorname{psoc} \mathcal{A}$ is the largest ideal in $\mathcal{A}$ each of whose elements has a finite spectrum.*

Proof First we show that every $x \in \operatorname{psoc} \mathcal{A}$ has a finite spectrum.

Suppose first that $\mathcal{A}$ has an identity. Then since $\mathcal{A}' := \mathcal{A}/\operatorname{rad} \mathcal{A}$ is semi-simple $\operatorname{soc} \mathcal{A}'$ is the largest ideal of $\mathcal{A}'$ each of whose elements has finite a spectrum, see Remark 5.23. The identity $\sigma(x) = \sigma(x')$ for all $x' := x + \operatorname{rad} \mathcal{A}$ then completes the proof.

The proof of the case where the algebra $\mathcal{A}$ has no identity easily follows by adjoining an identity to $\mathcal{A}$ in the usual manner and then applying the above argument. ∎

It is well known that if $\mathcal{B}$ is a closed subalgebra of a Banach algebra $\mathcal{A}$ and $x \in \mathcal{B}$ then $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x) \cup \{0\}$, see [**72**, Proposition 5.12].

Corollary 5.33. *Let $\mathcal{B}$ be a closed subalgebra of the Banach algebra $\mathcal{A}$. Then $(\operatorname{psoc} \mathcal{A}) \cap \mathcal{B} \subseteq \operatorname{psoc} \mathcal{B}$.*

Proof It is evident that $(\operatorname{psoc} \mathcal{A}) \cap \mathcal{B}$ is an ideal of $\mathcal{B}$ each of whose elements, by the above remark, has a finite spectrum. Hence by Theorem 5.32 $(\operatorname{psoc} \mathcal{A}) \cap \mathcal{B} \subseteq \operatorname{psoc} \mathcal{B}$. ∎

We now examine psoc $\mathcal{A}$ in relation to the structure space $\Pi(\mathcal{A})$. In the sequel $\partial_{hk}\Pi(\mathcal{A})$ will denote the set of accumulation points of $\Pi(\mathcal{A})$ in the hk-topology. If $\mathcal{A}$ is commutative, the accumulation points of $\Delta(\mathcal{A})$ in the Gelfand topology will be denoted by $\partial_G\Delta(\mathcal{A})$.

Theorem 5.34. *For any Banach algebra $\mathcal{A}$ we have:*

(i) $\partial_{hk}\Pi(\mathcal{A}) \subseteq h(\operatorname{psoc}\mathcal{A})$;

(ii) *If $\mathcal{A}$ is commutative then* $\partial_{hk}\Delta(\mathcal{A}) = \partial_G\Delta(\mathcal{A}) = h(\text{psoc } \mathcal{A})$.

Proof First assume that $\mathcal{A}$ is semi-simple. Then by Theorem 5.29 $\operatorname{psoc}\mathcal{A} = \operatorname{soc}\mathcal{A}$. Let P be a primitive ideal such that $P \notin h(\operatorname{soc}\mathcal{A})$ and choose $x \in \operatorname{soc}\mathcal{A}$ such that $x \notin P$. Then by Theorem 5.14 $\mathcal{A}x$ has finite order, and hence by Theorem 5.12 there exists a maximal set $\{e_1, \dots, e_n\}$ of orthogonal minimal idempotents of $\mathcal{A}x$ such that $x = xe_1 + \dots + xe_n$. Clearly at least one of these idempotents does not belong to P. Let e_j be a such idempotent. By Theorem 5.15 the set $\{P\}$ is the complement of $h(\{e_j\})$ in $\Pi(\mathcal{A})$. Since $h(\{e_j\})$ is closed $\{P\}$ is open and therefore $P \notin \partial_{hk}\Pi(\mathcal{A})$, as required.

The proof of the general case of a non-semi-simple Banach algebra easily follows by using the homeomorphism between the structure spaces of $\mathcal{A}$ and $\mathcal{A}' = \mathcal{A}/\operatorname{rad}\mathcal{A}$.

(ii) If $\mathcal{A}$ is a commutative Banach algebra then $\Pi(\mathcal{A}) = \Delta(\mathcal{A})$ and the Gelfand topology is stronger than the hk-topology, so by part (i) we have

$$\partial_G\Delta(\mathcal{A}) \subseteq \partial_{hk}\Delta(\mathcal{A}) \subseteq h(\operatorname{psoc}\mathcal{A}).$$

Let us consider a multiplicative functional $m_0 \notin \partial_G\Delta(\mathcal{A})$. By the Shilov idempotent theorem there exists an idempotent $e \in \mathcal{A}$ such that $\widehat{e}(m_0) = 1$ and $\widehat{e}(m) = 0$ for every $m \in \Delta(\ A) \setminus \{m_0\}$. Then $\widehat{e}\widehat{\mathcal{A}} = \mathbb{C}\widehat{e}$, hence $e'\mathcal{A}' = \mathbb{C}e'$ and consequently e' is a minimal idempotent of $\mathcal{A}'$. This implies that $e \in \operatorname{psoc}\mathcal{A}$. But $\widehat{e}(m_0) = 1$, hence $m_0 \notin h(\operatorname{psoc}\mathcal{A})$ and therefore $h(\operatorname{psoc}\mathcal{A}) \subseteq \partial_G\Delta(\mathcal{A})$, so the proof is complete. ∎

A situation of particular interest arises when a commutative Banach algebra A itself is an inessential ideal of its multiplier algebra $M(A)$. This is, for instance, the case of the group algebra $L^1(G)$, where G is a compact Abelian group with respect to the measure algebra $\mathcal{M}(G)$. To study this situation within an abstract framework we first introduce a large class of Banach algebras which embraces many special important algebras.

Definition 5.35. *An algebra $\mathcal{A}$ is said to be a* Riesz algebra *if*

$$h(\operatorname{psoc}\mathcal{A}) = \varnothing.$$

Hence for a Riesz algebra $\mathcal{A}$ there exists no primitive ideal containing its pre-socle , so, roughly speaking, an algebra $\mathcal{A}$ is Riesz if it is close to its pre-socle.

The property of being a Riesz algebra may be described in several ways.

Theorem 5.36. *For any algebra $\mathcal{A}$ the following statements are equivalent:*

(i) *$\mathcal{A}$ is a Riesz algebra;*

(ii) *$\mathcal{A} = k(h(\text{psoc}\mathcal{A}))$;*

(iii) *$\mathcal{A}' := \mathcal{A}/\text{rad}\,\mathcal{A}$ is a Riesz algebra;*

(iv) *$\mathcal{A}/\text{psoc}\ \mathcal{A}$ is a radical algebra.*

Proof The equivalence (i) ⇔ (ii) is obvious, whilst the equivalence (ii) ⇔ (iii) easily follows by considering the canonical homeomorphism between $\Pi(\mathcal{A})$ and $\Pi(\mathcal{A}')$. The equivalence (ii) ⇔ (iv) is an immediate consequence of the equality $k(h(\text{psoc}\,\mathcal{A})) = \pi^{-1}(\text{rad}\,(\mathcal{A}/\text{psoc}\ \mathcal{A}))$, see the identity (137), where $\pi : \mathcal{A} \to \mathcal{A}/\text{psoc}\ \mathcal{A}$ denotes the canonical homomorphism. ■

We mention that the Calkin algebra of any infinite-dimensional Hilbert space provides an example of a non-Riesz algebra having a discrete structure space.

The following result shows that a Riesz algebra in the commutative case may be characterized in a very clear way by means of its maximal regular ideal space $\Delta(\mathcal{A})$.

Theorem 5.37. *If $\mathcal{A}$ is a Riesz algebra then $\Pi(\mathcal{A})$ is discrete. Moreover, for a commutative algebra the following statements are equivalent:*

(i) *$\mathcal{A}$ is a Riesz algebra;*

(ii) *$\Delta(\mathcal{A})$ is discrete in the hk-topology;*

(iii) *$\Delta(\mathcal{A})$ is discrete in the Gelfand topology.*

Proof It is immediate from the definition of a Riesz algebra and from Theorem 5.34. ■

Lemma 5.38. *Let J be a two-sided ideal of the algebra $\mathcal{A}$. Then:*

(i) *$\text{psoc}\ J = (\text{psoc}\ \mathcal{A}) \cap J$;*

(ii) *If $x \in \text{psoc}\ \mathcal{A}$ then $\overline{x} := x + J \in \text{psoc}\ (\mathcal{A}/J)$.*

Proof As before, for each $x \in \mathcal{A}$ let x' denote the class $x + \text{rad}\ \mathcal{A}$. Then $J' := \{x' = x + \text{rad}\,\mathcal{A} : x \in J\}$ is a two sided ideal of $\mathcal{A}'$, so by Theorem 5.1 $\text{rad}\ J' = \text{rad}\,\mathcal{A}' \cap J'$. Since $\mathcal{A}'$ is semi-simple it then follows that also J' is semi-simple. Moreover, the semi-simplicity of $\mathcal{A}'$ yields that $\text{soc}\,J' = (\text{soc}\,\mathcal{A}') \cap J'$, see the remark before Example 5.9. From $\text{rad}\ J = (\text{rad}\,\mathcal{A}) \cap J$ we easily deduce that the mapping defined by $\Psi(x') := x + \text{rad}\,J$ is an isomorphism of J' onto $J/\text{rad}\,J$.

Now, in order to show the equality (i) take $x \in (\text{psoc}\ \mathcal{A}) \cap J$. Then $x' \in (\text{soc}\,\mathcal{A}') \cap J' = \text{soc}\,J'$. The isomorphism Ψ then implies that $x + \text{rad}\,J \in \text{soc}\,(J/\text{rad}\,J)$ and therefore $x \in \text{psoc}\ J$.

Conversely, if $x \in \text{psoc}\ J$ then $x + \text{rad}\,J \in \text{soc}(J/\text{rad}\,J)$ and $x \in J$. By using the isomorphism Ψ defined before we then obtain that $x' \in \text{soc}\,J'$. Therefore $x' \in \text{soc}\,\mathcal{A}'$ and hence $x \in \text{psoc}\ \mathcal{A}$. Since $x \in J$ we have $x \in$

$(\mathrm{psoc}\,\mathcal{A}) \cap J$ and this complete the proof of part (i).

To show part (ii) note first that if $\overline{x} := x + J$ then

$$\mathrm{psoc}\,(\mathcal{A}/J) = \{x + J : \overline{x} + \mathrm{rad}\,(\mathcal{A}/J) \in \mathrm{soc}\,((\mathcal{A}/J)/\mathrm{rad}\,(\mathcal{A}/J))\}.$$

Let $K := k(h(J))$. The mapping

$$\Theta : \mathcal{A}/K \to (\mathcal{A}/J)/\mathrm{rad}\,(\mathcal{A}/J)$$

defined by

$$\Theta(x + K) := \overline{x} + \mathrm{rad}\,(\mathcal{A}/J), \quad x \in \mathcal{A},$$

is an isomorphism, see Theorem 2.6.6 of Rickart [**279**], so

$$\mathrm{psoc}\ (\mathcal{A}/J) = \{\overline{x} : x + K \in \mathrm{soc}\,(\mathcal{A}/K)\}.$$

Now, if $x \in \mathrm{psoc}\,\mathcal{A}$ then $x' \in \mathrm{soc}\,\mathcal{A}'$, and from the inclusion $\mathrm{rad}\,\mathcal{A} \subseteq K = h(k(J))$ we can conclude that $x + K \in \mathrm{soc}\,(\mathcal{A}/K)$. Hence $\overline{x} \in \mathrm{psoc}\,(\mathcal{A}/J)$, which completes the proof. ∎

Theorem 5.39. *Let J be an ideal of the Riesz algebra $\mathcal{A}$. Then both J and $\mathcal{A}/J$ are Riesz algebras.*

Proof Clearly, $(J + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A})$ is an ideal of the radical algebra $\mathcal{A}/\mathrm{psoc}\,\mathcal{A}$ and

$$\mathrm{rad}\,((J + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A}) = ((J + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A})) \cap \mathrm{rad}\,(\mathcal{A}/\mathrm{psoc}\,\mathcal{A}),$$

thus $(J + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A}$ is itself a radical algebra. But $(J + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A}$ is isomorphic to $J/(J \cap\ \mathrm{psoc}\,\mathcal{A})$ which is equal by Lemma 5.38 to $J/\mathrm{psoc}\,J$. Hence $J/\mathrm{psoc}\,J$ is a radical algebra and therefore J is a Riesz algebra.

Now assume that $\mathcal{A}/J$ is not a Riesz algebra. Then there exists a primitive ideal P of $\mathcal{A}/J$ such that $\mathrm{psoc}\ (\mathcal{A}/J) \subseteq P$. Again, from Theorem 2.6.6 of Rickart [**279**] it follows that $\{x \in \mathcal{A} : x + J \in P\}$ is a primitive ideal of $\mathcal{A}$, which contains psoc $\mathcal{A}$ by Lemma 5.38. This is impossible since $\mathcal{A}$ is a Riesz algebra. ∎

Note that a subalgebra of a Riesz algebra need not be in general a Riesz algebra. For instance, the algebra $\mathcal{A}$ considered in Example 5.30 is a Riesz algebra since $\mathcal{A} = \mathrm{psoc}\,\mathcal{A}$. The subalgebra $\mathcal{B}$ of $\mathcal{A}$ consisting of the polynomials in x is semi-simple and $\mathrm{psoc}\,B = \mathrm{soc}\,B = \{0\}$. Hence $\mathcal{B}$ is not a Riesz algebra.

Theorem 5.40. *If $\mathcal{A}$ is an algebra then $k(h(\mathrm{psoc}\ \mathcal{A}))$ is the largest ideal which is also a Riesz algebra.*

Proof Let $J := k(h(\mathrm{psoc}\,\mathcal{A}))$. Then by Lemma 5.38 $J/\mathrm{psoc}\,J = J/\mathrm{psoc}\,\mathcal{A}$, which is a radical algebra. Hence J is a Riesz algebra .

Now suppose that K is another ideal of $\mathcal{A}$ which is a Riesz algebra. Let $P \in h(\mathrm{psoc}\,\mathcal{A})$. From Lemma 5.38 we know that the radical algebra $K/\mathrm{psoc}\,K$ is equal to $K/((\mathrm{psoc}\,\mathcal{A}) \cap K)$ which is isomorphic to the ideal $(K + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A}$ of the algebra $\mathcal{A}/\mathrm{psoc}\,\mathcal{A}$. But this is a two-sided ideal of $\mathcal{A}/\mathrm{psoc}\,\mathcal{A}$, so by part (i) of Theorem 5.1 we have $(K + \mathrm{psoc}\,\mathcal{A})/\mathrm{psoc}\,\mathcal{A} \subseteq$

rad $(\mathcal{A}/\text{psoc}\,\mathcal{A})$. However, the set $\{x + \text{psoc}\,\mathcal{A} : x \in P\}$ is a primitive ideal of $\mathcal{A}/\text{psoc}\,\mathcal{A}$, hence it contains $(K + \text{psoc}\,\mathcal{A})/\text{psoc}\,\mathcal{A}$.

This shows that $K \subseteq P$ and therefore $K \subseteq J = k(h(\text{psoc}\,\mathcal{A}))$. ∎

Suppose that a Banach algebra $\mathcal{A}$ does not possess a unit and $\lambda \neq 0$ is an isolated point of the spectrum $\sigma(x)$ of $x \in \mathcal{A}$. Then it still make sense to talk about the spectral idempotent $p(\lambda, x)$ associated with the spectral set $\{\lambda\}$. This is canonically defined as follows: if $\mathcal{A}_u$ denotes the unization $\mathcal{A} \oplus \mathbb{C}u$ of $\mathcal{A}$ then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{A}_u}((x, 0))$, [**72**, Lemma 5.2], and the spectral idempotent $p(\lambda, x)$ is defined to be the spectral idempotent $p(\lambda, (x, 0) \in \mathcal{A}_u$ associated with the isolated point λ of $\sigma_{\mathcal{A}_u}((x, 0))$.

In the sequel by $Z(x)$, $x \in \mathcal{A}$, we shall denote the *centraliser* of x,

$$Z(x) := \{a \in \mathcal{A} : ax = xa \text{ for every } a \in \mathcal{A}\}.$$

Clearly $Z(x)$ is a closed subalgebra of $\mathcal{A}$ which is invariant under $x \wedge x$.

As usual, let $L_x : \mathcal{A} \to \mathcal{A}$ denote the left multiplication operator by x. The next result shows the Riesz algebra s are closely related to Riesz operators.

Theorem 5.41. *Let $\mathcal{A}$ be a Banach algebra for which $x \wedge x$ is a Riesz operator for every $x \in \mathcal{A}$. Then $\mathcal{A}$ is a Riesz algebra.*
Moreover, if $\mathcal{A}$ is semi-simple then

$$\mathcal{A} \text{ is a Riesz algebra } \Leftrightarrow x \wedge x \text{ is a Riesz operator for all } x \in \mathcal{A}.$$

Proof Suppose that $x \wedge x$ is a Riesz operator for all $x \in \mathcal{A}$. From part (iii) of Theorem 3.113 we know that the restriction $(x \wedge x)|Z(x)$ is a Riesz operator. But $(x \wedge x)|Z(x) = (L_x|Z(x))^2$, and hence by part (ii) of Theorem 3.113 $L_x|Z(x)$ is a Riesz operator from which we conclude that the spectrum $\sigma(L_x|Z(x))$ is a finite set or a sequence which converges to 0. The equality $\sigma(x) \setminus \{0\} = \sigma(L_x|Z(x)) \setminus \{0\}$ then implies that every $0 \neq \lambda \in \sigma(x)$ is isolated in $\sigma(x)$.

Let $p := p(\lambda, x)$ be the spectral idempotent associated with $\{\lambda\}$. Since $L_x|\, Z$ is a Riesz operator then λ is a pole of L_x and hence, by part (b) of Remark 3.7, there exists some $m \in \mathbb{N}$ such that $(\lambda - x)^m p = 0$. From this we conclude that there exists some polynomial α such that $p = px\ \alpha(x)$. Clearly,

$$p \wedge p = (x \wedge x)(p\ \alpha(x) \wedge p\ \alpha(x)) = (p\ \alpha(x) \wedge p\ \alpha(x))(x \wedge x),$$

so part (ii) of Theorem 3.112 ensures that $p \wedge p$ is a Riesz operator, and consequently $\ker(I - p \wedge p)$ is finite-dimensional. Obviously $p \wedge p$ is idempotent, so $\ker(I - p \wedge p) = (p \wedge p)(\mathcal{A}) = p\mathcal{A}p$ is finite-dimensional and this implies that also $p'\mathcal{A}'p'$ is finite-dimensional, where $\mathcal{A}' = \mathcal{A}/\text{rad}\,\mathcal{A}$. We may now employ Theorem 5.24 to conclude that $p' \in \text{soc}\,\mathcal{A}'$, and hence $p \in \text{psoc}\,\mathcal{A}$.

Next we want show that $\mathcal{A}/\overline{\text{psoc}\,\mathcal{A}}$ is a radical algebra. Choose $\varepsilon > 0$. Then $\Omega := \{\lambda \in \sigma(x) : |\lambda| \geq \varepsilon\}$ is a finite set, $\{\lambda_1, \cdots, \lambda_n\}$ say. Clearly $q := \sum_{k=1}^n p(\lambda_k, x) \in \text{psoc}\,\mathcal{A}$. It is easily seen that the spectral radius

$r(x - qx) < \varepsilon$. Therefore if $\tilde{x} := x + \overline{\text{psoc}\,\mathcal{A}}$ we have $r(\tilde{x}) \leq \varepsilon$ for every ε. This shows that $\mathcal{A}/\overline{\text{psoc}\,\mathcal{A}}$ is a radical algebra. Since in a Banach algebra all primitive ideals are closed, we then conclude that $h(\text{psoc}\,\mathcal{A}) = \varnothing$. Therefore $k(h(\text{psoc}\,\mathcal{A})) = \mathcal{A}$ so $\mathcal{A}$ is a Riesz algebra.

Conversely, assume that $\mathcal{A}$ is a semi-simple Riesz Banach algebra and $x \in \mathcal{A}$. Then $r(x + \overline{\text{soc}\,\mathcal{A}}) = 0$, so for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $y_n \in \text{soc}\,\mathcal{A}$ such that $\|x^n - y_n\| < \varepsilon^n$ for all $n \geq m$. We have

$$\begin{aligned} \|(x \wedge x)^n - y_n \wedge y_n\| &\leq \|x^n - y_n\|\|x^n\| + \|y_n\|\|x^n - y^n\| \\ &\leq \varepsilon^n(2\|x\|^n + \varepsilon^n) \leq \varepsilon^n(2\|x\| + \varepsilon)^n. \end{aligned}$$

From this we obtain that

$$\|(x \wedge x)^n - y_n \wedge y_n\|^{1/n} < \varepsilon(2\|x\| + \varepsilon),$$

for all $n \geq m$.

Let $K(\mathcal{A})$ denote the ideal of all compact linear operators on $\mathcal{A}$. From Theorem 5.24 we know, since $y_n \in \text{soc}\,\mathcal{A}$, that $y_n \wedge y_n \in K(\mathcal{A})$ for all $n \in \mathbb{N}$. Hence

$$\|(x \wedge x)^n + J\|^{1/n} \leq \varepsilon(2\|x\| + \varepsilon),$$

and therefore

$$r(x \wedge x + J) \leq \varepsilon(2\|x\| + \varepsilon).$$

The Ruston characterization of Riesz operators then implies that $x \wedge x$ is a Riesz operator on $\mathcal{A}$, so the proof of the equivalence is complete. ∎

Lemma 5.42. *Let $\mathcal{A}$ be a semi-simple Banach algebra with unit u. We have:*

(i) *If every $x \in \mathcal{A}$ has spectrum which consists of a single point then $\mathcal{A} = \mathbb{C}u$;*

(ii) *If $e \neq 0$ is not minimal then there exists an element $x \in e\mathcal{A}e$ such that $\sigma_{e\mathcal{A}e}(x)$ contain at least two distinct points.*

Proof (i) Suppose that $\sigma(x)$ is a singleton set for every $x \in \mathcal{A}$. We show first that $\sigma(x) = \{0\}$ implies $x = 0$.

Let $y \in \mathcal{A}$ be arbitrary given. Then $\sigma(xy) = \{0\}$, otherwise if $\sigma(xy) \neq \{0\}$ then xy and yx would both be invertible, because $\sigma(xy) = \sigma(yx)$, and consequently also x would be invertible, a contradiction.

Hence $\sigma(xy) = \{0\}$ for every $y \in \mathcal{A}$ and this implies by the characterization (135) of the radical that $x \in \text{rad}\,\mathcal{A} = \{0\}$. Finally, if $0 \neq z \in \mathcal{A}$ then $\sigma(z) = \{\lambda\}$ where $\lambda \neq 0$, so $\sigma(z - \lambda u) = \{0\}$ and therefore $z = \lambda u$.

(ii) From part (iii) of Remark 5.2 we know that $e\mathcal{A}e$ is a semi-simple unital Banach algebra. Assume that $\sigma_{e\mathcal{A}e}(x)$ is a singleton set for every $x \in e\mathcal{A}e$. By part (i) it follows that $e\mathcal{A}e = \mathbb{C}e$, so e is minimal, a contradiction. ∎

The following result, owed to Smyth [**304**], characterizes a Riesz Banach algebra by means of the spectral structure of its elements.

Theorem 5.43. *A Banach algebra $\mathcal{A}$ is a Riesz algebra if and only if for every $x \in \mathcal{A}$ the spectrum $\sigma(x)$ is a finite set or is a sequence converging to zero.*

Proof Recall that if $x \in \mathcal{A}$ and $x' = x + \mathrm{rad}\,\mathcal{A} \in \mathcal{A}/\mathrm{rad}\,\mathcal{A}$ then $\sigma(x) = \sigma(x')$ and $\mathcal{A}/\mathrm{rad}\ \mathcal{A}$ is semi-simple. Our theorem will be proved without loss of generality if we assume that $\mathcal{A}$ is semi-simple.

Suppose that the semi-simple Banach algebra $\mathcal{A}$ is a Riesz algebra. By Theorem 5.41 we know that $x \wedge x$ is a Riesz operator. The same reasoning used in the first part of the proof of Theorem 5.41 then shows that every $0 \neq \lambda \in \sigma(x)$ is isolated in $\sigma(x)$. This proves the first part of the theorem.

Conversely, assume that $\mathcal{A}$ is not a Riesz algebra. Suppose that for every $x \in \mathcal{A}$ the spectrum $\sigma_{\mathcal{A}}(x)$ is a finite set or is a sequence converging to zero. Then there exists an idempotent $e \notin \mathrm{soc}\,\mathcal{A}$ (otherwise, arguing as in the proof of Theorem 5.41, $\mathcal{A}$ would be a Riesz algebra). Clearly $e\mathcal{A}e$ is a semi-simple Banach algebra with identity e. Moreover, since e is not minimal, part (ii) of Lemma 5.42 ensures that there exists an element $y \in e\mathcal{A}e$ such that the spectrum $\sigma_{e\mathcal{A}e}(y)$ has at least two points. Since $\sigma_{e\mathcal{A}e}(y) \subseteq \sigma(y)$ our assumption implies that 0 is the only possible accumulation point of $\sigma_{e\mathcal{A}e}(y)$. Hence $\sigma_{e\mathcal{A}e}(y)$ contains an isolated point $\mu \neq 0$.

Let $p := p(\mu, y) \in e\mathcal{A}e$ be the spectral projection associated with μ. If $q := e - p$ then we have

$$pe = ep = p \neq e, \quad qe = eq = q \neq e, \quad \text{and } e = p + q,$$

so, at least one of the idempotents p, q does not belong to soc $\mathcal{A}$. At this point, repeating this process, we can find a sequence (e_n) of idempotents such that $e_1 = e$ and $e_n \notin \mathrm{soc}\,\mathcal{A}$.

Let us consider the sequence of nonzero orthogonal idempotents (v_n) defined by $v_n := e_n - e_{n+1}$ for all $n \in \mathbb{N}$. Choose a sequence of distinct points $(\lambda_n) \subset \mathbb{C}$ such that

$$|\lambda_n| < \frac{1}{2^n \|v_n\|} \quad \text{for every } n \in \mathbb{N}.$$

Clearly $v := \sum_{n=1}^{\infty} \lambda_n v_n \in \mathcal{A}$ is a well-defined element of $\mathcal{A}$, and

$$(e + v)v_n = (1 + \lambda_n)v_n \quad \text{for all } n \in \mathbb{N}.$$

Hence $1 + \lambda_n \in \sigma(e + v)$ for all $n \in \mathbb{N}$. Since $\lambda_n \to 0$ as $n \to \infty$ we then conclude that 1 is an accumulation point of $\sigma(e + v)$. This contradicts our assumption that 0 is the only possible accumulation point of the spectrum of an element of $\mathcal{A}$, so the proof is complete. ∎

Corollary 5.44. *Suppose that the left multiplication operator L_x is a Riesz operator on the Banach algebra $\mathcal{A}$ for every $x \in \mathcal{A}$. Then $\mathcal{A}$ is a Riesz algebra. A similar statement holds for the right multiplication operators R_x. Moreover, if $\mathcal{A}$ is commutative and semi-simple then*

$$\mathcal{A} \text{ is a Riesz algebra } \Leftrightarrow L_x \text{ is a Riesz operator for all } x \in \mathcal{A}.$$

Proof Let $x \in \mathcal{A}$. If L_x is a Riesz operator the restriction on the centralizer $L_x \,|Z(x)$ is a Riesz operator and hence $\sigma(x) = \sigma(L_x \,|Z(x))$ is a finite set or a sequence converging to 0. By Theorem 5.43 we then infer that $\mathcal{A}$ is a Riesz algebra.

Assume now that $\mathcal{A}$ is a commutative semi-simple Riesz algebra and $x \in \mathcal{A}$. Then $(L_x)^2 = x \wedge x$ is a Riesz operator on $\mathcal{A}$, by Theorem 5.41, and hence by part (ii) Theorem 3.113, L_x is a Riesz operator also. ∎

Corollary 5.45. *Let J be a two-sided ideal of the Banach algebra $\mathcal{A}$. Then*

$$J \text{ is inessential } \Leftrightarrow J \subseteq k(h(\operatorname{psoc} \mathcal{A})).$$

Moreover, if J is inessential also $k(h(J))$ is inessential.

Proof The first assertion follows from Theorem 5.40 and Theorem 5.43. The second assertion is an immediate consequence of the first: for every inessential ideal J we have

$$k(h(J)) \subseteq k(h(k(h(\operatorname{psoc} \mathcal{A})) = k(h(\operatorname{psoc} \mathcal{A})),$$

and this implies that also $k(h(J))$ is inessential. ∎

We have already observed that a subalgebra of a Riesz algebra need not be a Riesz algebra. A straightforward consequence of Theorem 5.43 shows that a distinguishing property of a Riesz Banach algebra is that every closed subalgebra is itself a Riesz algebra.

Corollary 5.46. *Every closed subalgebra of a Riesz Banach algebra is a Riesz algebra.*

Proof Let $\mathcal{B}$ be a closed subalgebra of the Riesz algebra $\mathcal{A}$. Then 0 is the only possible accumulation point of $\sigma_{\mathcal{A}}(x)$ for every $x \in \mathcal{A}$. From the inclusion $\partial\sigma_{\mathcal{B}} \subseteq \partial\sigma_{\mathcal{A}}$, see [**72**, Proposition 5.12], we see that this is also true for $\sigma_{\mathcal{B}}(x)$, so $\mathcal{B}$ is a Riesz algebra. ∎

5. Fredholm elements of Banach algebras

The Atkinson characterization of Fredholm operators establishes that a bounded operator on a Banach space X is a Fredholm operator precisely when it is invertible in $L(X)$ modulo the ideal $F(X)$ of all finite-dimensional operators. The ideal $F(X)$ is the socle of the semi-simple Banach algebra $L(X)$.

This suggests how to extend Fredholm theory to the more abstract framework of Banach algebras. A natural way of defining a Fredholm element of a Banach algebra $\mathcal{A}$ is that this is an element of $\mathcal{A}$ invertible modulo a fixed ideal J. However, the results of the previous section suggest that in order to obtain a deeper Fredholm theory for Banach algebras which reflects more closely the classical Fredholm operator theory we need to assume that the ideal J is the socle, or more generally that J is inessential.

Definition 5.47. *Given an inessential two-sided ideal J of a Banach algebra $\mathcal{A}$ with a unit u, an element $x \in \mathcal{A}$ is called a J-*Fredholm element *of $\mathcal{A}$ (or also a* Fredholm element of $\mathcal{A}$ relative to J*) if x is invertible in $\mathcal{A}$ modulo J, i.e., there exist $y, z \in \mathcal{A}$ such that $xy - u \in J$ and $zx - u \in J$.*

The set of all J-Fredholm elements of $\mathcal{A}$ will be denoted by $\Phi(J, \mathcal{A})$. Note that we can always assume that J is closed. In fact, from the inclusions $J \subseteq \overline{J} \subseteq k(h(J))$ and from Corollary 5.6 it follows that

$$\Phi(J, \mathcal{A}) = \Phi(\overline{J}, \mathcal{A}) = \Phi(k(h(J), \mathcal{A}). \tag{143}$$

If we take into account that the invertible elements of an algebra form a group and that the factors of an invertible product are either both invertible or both non-invertible, then we immediately obtain that $\Phi(J, \mathcal{A})$ is an open multiplicative semi-group of $\mathcal{A}$. Moreover, from the equalities (143) it easily follows that the set $\Phi(J, \mathcal{A})$ is stable under perturbations by elements of $k(h(J))$, i.e.

$$\Phi(J, \mathcal{A}) + k(h(J)) \subseteq \Phi(J, \mathcal{A}).$$

For any proper closed inessential ideal J of $\mathcal{A}$ let $\Psi : \mathcal{A} \to \mathcal{A}/J$ denote the canonical quotient homomorphism. We have already observed that for every ideal J we have

$$k(h(J)) = \Psi^{-1}(\mathrm{rad}\,\mathcal{A}/J). \tag{144}$$

From the characterization (134) of the radical we readily obtain

$$k(h(J)) = \{x \in \mathcal{A} : x + \Phi(J, \mathcal{A}) \subseteq \Phi(J, \mathcal{A})\}. \tag{145}$$

Let now us consider the particular case of an inessential ideal J which satisfies the inclusions

$$\mathrm{psoc}\ (\mathcal{A}) \subseteq J \subseteq k(h(\mathrm{psoc}\ \mathcal{A})).$$

From Corollary 5.6 we then obtain

$$\Phi(J, \mathcal{A}) = \Phi(\mathrm{psoc}\ \mathcal{A}, \mathcal{A}) = \Phi(k(h(\mathrm{psoc}\ \mathcal{A})), \mathcal{A}),$$

so in the particular case $\mathcal{A}$ is semi-simple we obtain

$$\Phi(J, \mathcal{A}) = \Phi(\mathrm{soc}\,\mathcal{A}, \mathcal{A}) = \Phi(k(h(\mathrm{soc}\,\mathcal{A})), \mathcal{A}). \tag{146}$$

Of course, these considerations find a first important application to the semi-simple Banach algebra $\mathcal{A} := L(X)$ of all bounded operators on a Banach space X. By the Atkinson characterization of Fredholm operators we have $\Phi(X) = \Phi(F(X), L(X))$. Let $\pi : L(X) \to L(X)/F(X)$ denote the canonical quotient map and denote

$$I(X) := \pi^{-1}(\mathrm{rad}\,(L(X)/F(X)).$$

The set $I(X)$ will be called the *inessential ideal or Riesz ideal of operators* of $L(X)$. The equality (144) shows that

$$I(X) = k(h(\mathrm{soc}\,L(X))) = k(h(F(X))$$

and therefore from the inclusions

$$\operatorname{soc} L(X) = F(X) \subseteq K(X) \subseteq I(X) = k(h(F(X)),$$

we infer that

$$\Phi(X) = \Phi(F(X), L(X)) = \Phi(K(X), L(X)) = \Phi(I(X), L(X)).$$

Analogously, the equalities (146) show that $\Phi(X)$ may be described as the class of all operators $T \in L(X)$ invertible in $L(X)$ modulo every ideal of operators which contains $F(X)$ and that is contained in $I(X)$. Some of these ideals and the same ideal $I(X)$ will be studied in Chapter 7.

We next turn to the Fredholm theory of the algebra $\mathcal{A} := M(A)$ with respect to some inessential ideals of $M(A)$.

For any semi-prime Banach algebra A, not necessarily commutative, let us consider the ideals of the multiplier algebra $M(A)$ defined by

$$K_M(A) := \{T \in M(A) : T \text{ is a compact operator on } A\},$$

and by

$$F_M(A) := \{T \in M(A) : T \text{ is finite-dimensional}\}.$$

Clearly, for each $T \in K_M(A)$, or $T \in F_M(A)$, the spectrum $\sigma_{M(A)}(T) = \sigma(T)$ is a finite set or has 0 as its unique accumulation point. Hence $K_M(A)$ and $F_M(A)$ are both inessential ideals of the semi-prime Banach algebra $A = M(A)$. Let us define

$$\Phi_M(A) := \Phi(K_M(A), M(A)).$$

Theorem 5.48. *If $T \in M(A)$, where A is a semi-prime Banach algebra, then*

$$\Phi_M(A) = \Phi(F_M(A), M(A)).$$

Proof It suffices by Theorem 5.5 to show that the two ideals $K_M(A)$ and $F_M(A)$ have the same set of spectral projections. Let $\lambda_0 \in \mathbb{C}$ be an isolated spectral point of $T \in M(A)$. If $\lambda \in \rho(T)$ then $(\lambda I - T)^{-1} \in M(A)$, since $M(A)$ is an inverse closed algebra of $L(A)$. Hence if P_0 is the spectral projection P_0 associated with $\{\lambda_0\}$ we have

$$P_0(T) := \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} d\lambda \in M(A).$$

Now, if P_0 is compact, and hence Riesz, then $\ker(I - P_0) = P_0(A)$ is finite-dimensional, so $P_0 \in F_M(A)$. The converse is obviously true, so $K_M(A)$ and $F_M(A)$ have the same set of spectral projections. ∎

Trivially, by the Atkinson characterization of Fredholm operators we have

$$\Phi_M(A) \subseteq \Phi(A) \cap M(A).$$

The last inclusion may be proper, as the following example shows.

Example 5.49. Let us consider the disc algebra $A = \mathcal{A}(\mathbb{D})$ is a semi-simple unital commutative Banach algebra, so each multiplier of $\mathcal{A}(\mathbb{D})$ is an operator of multiplication T_f for some $f \in \mathcal{A}(\mathbb{D})$, namely, $M(\mathcal{A}(\mathbb{D})) = \mathcal{A}(\mathbb{D})$. Moreover, $\Delta(\mathcal{A}(\mathbb{D})) = \mathbb{D}$ has no isolated points so

$$\mathrm{soc}\,(M(\mathcal{A}(\mathbb{D})) = \mathrm{soc}\,\mathcal{A}(\mathbb{D}) = \{0\}.$$

From this and from the following inclusion

$$K_M(\mathcal{A}(\mathbb{D})) \subseteq k_{(M(A)}(h_{M(A)}(\mathrm{soc}\,(M(\mathcal{A}(\mathbb{D}))) = \{0\},$$

it follows that $K_M(\mathcal{A}(\mathbb{D})) = \{0\}$ (note that this equality is also a consequence of Theorem 4.41). Hence

$$\Phi_M(\mathcal{A}(\mathbb{D})) = \{T_f : f \text{ is an invertible element of } \mathcal{A}(\mathbb{D})\}.$$

Now let us denote by T_g the multiplication operator with $g(z) := z$, for all $z \in \mathbb{D}$. Clearly the operator T_g is injective. Moreover, as is easy to check, the range of T_g is the maximal ideal

$$J := \{h \in \mathcal{A}(\mathbb{D}) : h(0) = 0\}$$

and hence its codimension is equal to 1. Hence we have $T_g \in \Phi(\mathcal{A}(\mathbb{D})) \cap M(A)$ whilst $T_g \notin \Phi_M(\mathcal{A}(\mathbb{D}))$.

Later we shall show that $\Phi_M(A)$ coincides with the set of all multipliers of A which are Fredholm operators having index 0.

Definition 5.50. *A two-sided ideal J of a Banach algebra $\mathcal{A}$ is said to have the* intersection property *if the intersection with any other non-zero two-sided ideal of $\mathcal{A}$ is non-trivial.*

In the literature an ideal with the intersection property is often called an *essential ideal*. The term essential has its own historical origin in the theory of $C^\star$ algebras. We do not adopt this terminology because it could generate a certain confusion of language. In fact the two concepts of inessential ideal and essential ideal are not in any sense opposites.

Lemma 5.51. *Let $\mathcal{A}$ be a commutative semi-prime Banach algebra and suppose that the ideal J of $\mathcal{A}$ has the intersection property. Then* $\mathrm{soc}\,\mathcal{A} = \mathrm{soc}\,J$.

Proof It suffices to prove that $\mathrm{Min}\,\mathcal{A} = \mathrm{Min}\,J$. Suppose first that $e \in \mathrm{Min}\,J$. Obviously $eJ \subseteq e\mathcal{A}$. Since $e \in J$ then $e\mathcal{A} = ee\mathcal{A} \subseteq eJ$. Hence $eJ = e\mathcal{A}$, and consequently $e\mathcal{A}e = eJe = \mathbb{C}e$, hence $e \in \mathrm{Min}\,\mathcal{A}$.

Conversely, suppose that e is a minimal idempotent of $\mathcal{A}$. Then $e\mathcal{A}$ is a minimal ideal, and since eJ is an ideal in $\mathcal{A}$ contained in $e\mathcal{A}$ we have $eJ = \{0\}$ or $eJ = e\mathcal{A}$. Note that $e\mathcal{A} \cap J \subseteq eJ\mathcal{A}$, because if $j \in J$ and $j = ea$ for some $a \in \mathcal{A}$ then $j = ej = eje \in eJ\mathcal{A}$. From this it follows that the first possibility above, $eJ = \{0\}$, entails that $\mathcal{A}e\mathcal{A}J \subseteq \mathcal{A}eJ = \{0\}$, contradicting the assumption that J has the intersection property.

Hence $eJ = e\mathcal{A}$. In particular, there is an element $b \in J$ for which $eb = e^2 = e$, thus $e \in J$. Moreover, from $eJe = e\mathcal{A}e = \mathbb{C}e$ we conclude that e is a minimal idempotent element of J. ∎

In the next theorem we identify, as usual, a commutative algebra A with the set of all multiplication operators $\{L_x : x \in A\}$.

Theorem 5.52. *Suppose that A is a commutative semi-prime Banach algebra. Then* $\operatorname{soc} M(A) = \operatorname{soc} A$.

Proof $M(A)$ is semi-prime so it suffices, by Theorem 5.51, to show that A is an ideal with the intersection property in its multiplier algebra $M(A)$. Let J denote a non-trivial ideal of $M(A)$ and suppose that $A \cap J = \{0\}$.

Let $0 \neq T \in J$. From the inclusion $AJ \subseteq A \cap J = \{0\}$ we infer that $TL_a = L_{Ta} = 0$ for every $a \in A$. Since A is faithful then $Ta = 0$ for every $a \in A$ and therefore $T = 0$, a contradiction. ∎

Recall that for a compact Abelian group G the ideal $P(G)$ of trigonometric polynomials is the set of all finite linear combinations of continuous characters.

Corollary 5.53. *For every locally compact Abelian group G we have* $\operatorname{soc} L^1(G) = \operatorname{soc} \mathcal{M}(G)$. *If G is compact then*

$$F_M(L_1(G)) = soc\, L^1(G) = P(G),$$

whilst if G is non-compact $\operatorname{soc} L^1(G) = \{0\}$.

Proof The equality $\operatorname{soc} L^1(G) = \operatorname{soc} \mathcal{M}(G)$, for every local compact Abelian group G, is clear from Theorem 5.52.

The equality $\operatorname{soc} L^1(G) = P(G)$ for a compact group G is an obvious consequence of the minimal idempotents of $L_1(G)$ being precisely the characters of $\widehat{G}$. Furthermore, by Corollary 5.25, $\mathcal{M}(G)$ being a commutative semi-simple unital Banach algebra, for every $\mu \in M(G)$ the convolution operator T_μ is finite-dimensional if and only if $\mu \in \operatorname{soc} \mathcal{M}(G) = \operatorname{soc} L^1(G)$.

If G is non-compact then $\widehat{G} = \Delta(L_1(G))$ is non-discrete, and hence being a topological group it does not contain isolated points. From Remark 5.26 it then follows that $\operatorname{soc} L^1(G) = \{0\}$. ∎

Note that the equality $\operatorname{soc} \mathcal{M}(G) = P(G)$ holds also for a compact non-Abelian group; see Barnes, Murphy, Smyth, and West [**62**, Lemma A.6.1] or Saxe [**285**].

We have seen in Example 4.30 that multipliers of a commutative semi-simple Banach algebra A may have a non-empty residual spectrum $\sigma_{\mathrm{r}}(T)$. However, the next result shows that, if A has a dense socle, then $\sigma_{\mathrm{r}}(T)$ is empty.

Theorem 5.54. *Suppose that A is a commutative semi-simple Banach algebra having a dense socle. If $T \in M(A)$ then $\sigma_{\mathrm{r}}(T) = \varnothing$ and*

$$\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{se}}(T). \tag{147}$$

Proof Suppose that $A = \overline{\mathrm{soc}\ A}$ and $\sigma_{\mathrm{r}}(T) \neq \varnothing$. Let $\lambda \notin \sigma_{\mathrm{r}}(T)$. Then $\lambda \notin \sigma_{\mathrm{p}}(T)$ and since $\Delta(A)$ is discrete, Theorem 4.31 ensures that $\lambda \notin \widehat{T}(m)$ for every $m \in \Delta(A)$. If $e \in \mathrm{Min}\, A$ is a minimal idempotent of A then there exists $m_0 \in \Delta(A)$ such that $\widehat{e}(m_0) = 1$ and $\widehat{e}$ vanishes identically on the set $\Delta(A) \setminus \{m_0\}$. Taking $z := [\lambda - \widehat{T}(m_0)]^{-1}$ we have

$$(\widehat{\lambda I - T})z(m) = \widehat{e}(m) \quad \text{for each } m \in \Delta(A),$$

and hence $(\lambda I - T)z = x$, so every minimal idempotent belongs to the range $(\lambda I - T)(A)$. Since the subspace generated by the set Min A is dense in A this implies that $\overline{(\lambda I - T)(A)} = A$ and hence $\lambda \notin \sigma_{\mathrm{r}}(T)$, a contradiction. Thus $\sigma_{\mathrm{r}}(T) = \varnothing$.

To prove the equalities (147) observe first that the inclusion $\sigma_{\mathrm{p}}(T) \cup \sigma_c(T) \subseteq \sigma_{\mathrm{ap}}(T)$ holds for each bounded operator on a Banach space. For every $T \in M(A)$ then

$$\sigma_{\mathrm{ap}}(T) \subseteq \sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_c(T) \cup \sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_c(T) \subseteq \sigma_{\mathrm{ap}}(T),$$

thus $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. The equalities $\sigma(T) = \sigma_{\mathrm{su}}(T)$ and $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{se}}(T)$ have been established in Theorem 4.32, part (iii). ∎

In particular, the equalities

$$\sigma_{\mathrm{p}}(T) = \widehat{T}(\Delta(A)) \quad \text{and} \quad \sigma_{\mathrm{r}}(T) = \varnothing$$

hold for every multiplier T of the group algebra $A := L_1(G)$, where G is a compact Abelian group. Later the equalities (147) will be extended to the more general case of regular Tauberian semi-simple commutative Banach algebras.

Theorem 5.55. *Suppose that A is a commutative semi-prime Banach algebra. Then*

$$\Phi_M(A) = \Phi(\mathrm{soc}\, A, A).$$

Proof From Corollary 5.25 and Corollary 5.45 we have

$$\begin{aligned} \mathrm{soc}\, A &= \mathrm{soc}\, M(A) \subseteq K_M(A) \\ &\subseteq k_{M(A)}(h_{M(A)}(\mathrm{soc}\, M(A))) = k_{M(A)}(h_{M(A)}(\mathrm{soc}\ A)). \end{aligned}$$

Consequently by Corollary 5.6, $\Phi_M(A) = \Phi(\mathrm{soc}\, A, M(A))$. ∎

Now let $R_M(A)$ denote the set of all Riesz multipliers, the set of all multipliers which are Riesz operators. Since $M(A)$ is commutative, by Theorem 3.112 the sums of elements of $R_M(A)$ are in $R_M(A)$ as well as the product of every $T \in R_M(A)$ with any $S \in M(A)$. Hence $R_M(A)$ is an ideal of $M(A)$, which is closed, again by Theorem 3.112, and inessential in $M(A)$, by Theorem 5.43.

The following theorem reveals that the Fredholm theory of multipliers of a commutative semi-simple Banach algebra having socle $\mathrm{soc}\, A = \{0\}$ is trivial.

Theorem 5.56. *Let A be a commutative semi-simple Banach algebra. Then the following statements are equivalent:*

(i) $\Delta(A)$ *has no isolated points;*

(ii) $K_M(A) = \{0\}$*;*

(iii) $R_M(A) = \{0\}$.

In such a case $\Phi_M(A) = \mathrm{inv}\, M(A)$, the set of all invertible multipliers of A.

Proof (i) $\Leftrightarrow$ (ii) If $\Delta(A)$ has no isolated points then $\mathrm{soc}\, A = \{0\}$ and hence also $k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)) = \{0\}$. By Corollary 5.45 the inessentiality of $K_M(A)$ yields

$$K_M(A) \subseteq k_{M(A)}(h_{M(A)}(\mathrm{soc}\, M(A))) = k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)) = \{0\},$$

from which we obtain that $K_M(A) = \{0\}$. Conversely, if $K_M(A) = \{0\}$, from Corollary 5.25 we have $\mathrm{soc}\, A \subseteq K_M(A) = \{0\}$, and hence $\Delta(A)$ has no isolated points.

The equivalence (ii) $\Leftrightarrow$ (iii) is proved in a similar way, whilst the last assertion is obvious. ∎

In particular, Theorem 5.56 applies, see Corollary 5.53, to convolution operators on group algebras $L^1(G)$ of non-compact Abelian groups.

A more interesting situation occurs when A has discrete maximal ideal space. Recall that $M_{00}(A) := k_{M(A)}(h_{M(A)}(A))$. The next result shows that in this case $M_{00}(A)$ is the largest essential ideal of $M(A)$.

Theorem 5.57. *Let A be a semi-simple commutative Banach algebra with a discrete maximal ideal space $\Delta(A)$. Then A and $M_{00}(A)$ are inessential ideals of $M(A)$. Precisely,*

$$M_{00}(A) = k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)). \tag{148}$$

Moreover,

$$\Phi_M(A) = \Phi(A, M(A)) = \Phi(M_{00}(A), M(A)). \tag{149}$$

Proof By Theorem 5.37, being $\Delta(A)$ discrete, A is a Riesz algebra, and hence by Theorem 5.43, A is an inessential ideal of $M(A)$. From Corollary 5.45 it then follows that also $M_{00}(A) = k_{M(A)}(h_{M(A)}(A))$ is an inessential ideal of $M(A)$, and hence, again by Corollary 5.45,

$$M_{00}(A) \subseteq k_{M(A)}(h_{M(A)}(\mathrm{soc}\, M(A))) = k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)).$$

On the other hand we also have

$$k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)) \subseteq k_{M(A)}(h_{M(A)}(A)) = M_{00}(A),$$

so the equality (148) is proved.

Finally, $A \subseteq M_{00}(A)$, so from the inclusions

$$\mathrm{soc}\, A \subseteq A \subseteq M_{00}(A) = k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)),$$

from Theorem 5.55 and Corollary 5.6, we conclude that the equalities (149) are verified. ∎

In particular, Theorem 5.57 applies to convolution operators on group algebras $L^p(G)$, with $1 \leq p < \infty$, of compact Abelian groups.

Note that in the situation of Theorem 5.57 we have $R_M(A) \subseteq M_{00}(A)$, $M_{00}(A)$ being the largest essential ideal of $M(A)$. By Theorem 5.43 $M_{00}(A)$ is then a Riesz algebra and consequently by Corollary 5.44 the multiplication operator $L_T : M_{00}(A) \to M_{00}(A)$ for each $T \in M_{00}(A)$ is a Riesz operator. In the next chapter we shall show, always in the case where $\Delta(A)$ is discrete, that if L_T is Riesz on $M_{00}(A)$ then also T is a Riesz operator on A, so the two inessential ideals $R_M(A)$ and $M_{00}(A)$ coincide.

6. Compact multipliers

Of course, it is of interest to describe the ideal $K_M(A)$ in some concrete cases. To do this we first introduce an important class of Banach algebras.

Definition 5.58. *A complex Banach algebra A is said to be a* compact Banach algebra *if the mapping $x \wedge x$ is a compact linear operator for every $x \in A$.*

Any compact Banach algebra may be described by means of a dense subset of it.

Theorem 5.59. *Let A be a Banach algebra and suppose that there exists a dense subset $\Omega \subseteq A$ such that the wedge operator $x \wedge x$ is a compact operator on A for each $x \in \Omega$. Then A is a compact algebra.*

Proof Let $x \in A$ be arbitrarily given and suppose that (x_n) is a sequence of elements in Ω which converges at x. Then for every $a \in A$ we have

$$\begin{aligned} \|x \wedge x(a) - x_n \wedge x_n(a)\| &= \|(x - x_n)ax + x_n a(x - x_n)\| \\ &\leq \|a\|(\|x\| + \|x_n\|)\|x - x_n\|, \end{aligned}$$

so that the operator $x \wedge x$ is the uniform limit of compact operators and hence is itself compact. ∎

Clearly, from Theorem 5.18 every semi-prime Banach algebra, not necessarily unital, with a dense socle is a compact algebra. In particular, for every compact Abelian group G the group algebra $L^1(G)$ is a compact algebra since $L^1(G) = \overline{\text{soc}\ \ L^1(G)} = \overline{P(G)}$. Another example of a compact algebra is provided by $K(X)$, the ideal of all compact operators on a Banach space X, see Bonsall and Duncan [**72**, §33].

Note that every compact algebra A is a Riesz algebra and hence, by Theorem 5.37, has structure space $\Pi(A)$ discrete.

Definition 5.60. *A commutative regular Banach algebra A is said to be* Tauberian *if the elements of A having compact support are norm dense in A, where as usual the* support *of an element $x \in A$ is defined to be the closure in $\Delta(A)$ of the set $\{m \in \Delta(A) : \widehat{x}(m) \neq 0\}$.*

Obviously, if A has an identity then A is Tauberian. Also $C_0(\Omega)$, the algebra of all complex continuous functions which vanish at infinity on a locally compact Hausdorff space Ω, the group algebras $L^1(G)$ for any locally Abelian compact group G, the algebra $L^p(G)$ for a compact Abelian group G, $1 \le p < \infty$, are Tauberian, see Rudin [**282**] or Larsen [**198**].

Theorem 5.61. *Suppose that A is a commutative Banach algebra with a bounded approximate identity. Then the following statements are equivalent:*

(i) $A = K_M(A)$*;*

(ii) A *is a compact algebra.*

Additionally, if A is also semi-simple and Tauberian then the conditions (i) *and* (ii) A *are equivalent to each of the following conditions:*

(iii) $\Delta(A)$ *is discrete, or equivalently A is a Riesz algebra;*

(iv) A *has dense socle.*

Proof Observe first that if A possesses an approximate identity (u_α) then $K_M(A) \subseteq A$, every compact multiplier is a multiplication operator. Indeed, if $T \in M(A)$ is compact then (Tu_α) has a subnet (Tu_β) which converges to an element $x \in A$. For each $a \in A$ we have

$$Ta = T(\lim_\beta au_\beta) = a(T(\lim_\beta u_\beta)) = ax,$$

thus T is the multiplication operator L_x defined by x.

(i) $\Rightarrow$ (ii) Let A be a compact algebra and assume that (u_α) is a bounded approximate identity. By the Cohen factorization theorem, for every $x \in A$ there exist $y, z \in A$ such that $x = yz$. From the equality

$$4yaz = (y+z)a(y+z) - (y-z)a(y-z),$$

we deduce that all the mappings $a \in A \to yaz = xa \in A$ are compact, so $K_M(A) \subseteq A$.

Conversely, if all the mappings $a \in A \to xa \in A$, for every $x \in A$, are compact then also the mappings $a \in A \to xax \in A$ are compact, so A is compact.

(i) $\Rightarrow$ (iii) If $K_M(A) = A$ then L_x is a Riesz operator for each $x \in A$, so by Corollary 5.44 A is a Riesz algebra and hence the maximal ideal space $\Delta(A)$ is discrete by Theorem 5.37.

(iii) $\Rightarrow$ (iv) Since A is Tauberian , for every $a \in A$ there exists a sequence (a_n) of elements of A with compact support $\operatorname{supp} \widehat{a_n}$ such that $a_n \to a$. Since $\Delta(A)$ is discrete, $\operatorname{supp} \widehat{a}_n$ is a finite set, and therefore, see Remark 5.26, $a_n \in \operatorname{soc} A$ for every $n \in \mathbb{N}$. Hence $\operatorname{soc} A$ is dense in A.

(iv) $\Rightarrow$ (ii) Let $A = \overline{\operatorname{soc} A}$. If $x \in A$, let (x_n) be a sequence of $\operatorname{soc} A$ which converges to x. By Corollary 5.25 then L_x is the uniform limit of the finite-dimensional operators L_{x_n}, so is a compact operator and therefore also $(L_x)^2 = x \wedge x$ is compact. ∎

Corollary 5.62. *For every compact Abelian group G, the convolution operator T_μ is compact on $L^p(G)$, $1 \le p < \infty$, if and only if $\mu \in L^p(G)$.* ■

The next result shows that for a multiplier of a commutative Banach algebra with a bounded a bounded approximate identity it turns out that several versions of acting the multiplication operator L_T as a compact operator are equivalent.

Theorem 5.63. *Suppose that A is a commutative Banach algebra with a bounded approximate identity (u_α). Then for each $T \in M(A)$ the following statements are equivalent:*

(i) *The multiplication operator L_T on $M(A)$ is compact;*

(ii) *The multiplication operator L_T on $M_0(A)$ is compact;*

(iii) *The multiplication operator L_T on $M_{00}(A)$ is compact;*

(iv) *$T \in K_M(A)$.*

Proof As already remarked, if A has a bounded approximate identity then the norm on A, as a subset of $M(A)$, and the norm on $M(A)$ are equivalent. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) then follow once it is observed that the restriction of a compact operator on a closed invariant subspace is also compact.

(iv) $\Rightarrow$ (i) Suppose that $T \in K_M(A)$. Then, see the proof of Theorem 5.61, T is the multiplication operator L_x for some $x \in A$. Let (S_n) be any arbitrary bounded sequence in $M(A)$ and choose, for each $n \in \mathbb{N}$, an index $\alpha(n)$ for which $\|S_n x - u_{\alpha(n)} S_n x\| \le 1/n$. Since $T = L_x$ is compact we may suppose, eventually passing to a subsequence, that there is $a \in A$ such that $L_x(S_n u_{\alpha(n)}) \to a \in A$ as $n \to \infty$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\|L_x(S_n) - L_a\| &= \|L_{S_n x} - L_a\| \le \|S_n x - a\| \\ &\le \|u_{\alpha(n)} S_n x - a\| + \frac{1}{n} \\ &= \|L_x(S_n u_{\alpha(n)}) - a\| + \frac{1}{n},\end{aligned}$$

from which we conclude that the sequence $(L_x(S_n))$ converges in $M(A)$ to L_x. This shows that $L_T = L_{L_x}$ is compact, so the implication (iv) $\Rightarrow$ (i) is proved. ■

Another inessential ideal of a faithful commutative Banach algebra A is the set of all quasi-nilpotent multipliers:

$$Q_M(A) := \{T \in M(A) : \sigma(T) = \{0\}\}.$$

Indeed, from the spectral radius formulae it easily follows that the sum of quasi-nilpotent multipliers, as well as the product of a multiplier with a quasi-nilpotent multiplier, is again quasi-nilpotent. Moreover, $Q_M(A)$ is closed, and by Corollary 4.34 $Q_M(A) = \{0\}$ whenever A is semi-simple.

Let us consider the case that the commutative Banach algebra A is an integral domain . Since A is semi-prime the socle of A does exist.

Lemma 5.64. *For every commutative Banach algebra A which is integral domain different from $\mathbb{C}$ we have $\operatorname{soc} A = \{0\}$. If A is also semi-simple then A is infinite-dimensional.*

Proof Suppose that the integral domain A is different from $\mathbb{C}$ and $\operatorname{soc} A \neq \{0\}$. Let e be a minimal idempotent of A. Then $e(x - ex) = 0$ for every $x \in A$ and since $e \neq 0$ and A is an integral domain it follows that $x = ex$ for every $x \in A$. Hence the element e is a unit of A and $A = eA = eAe = \mathbb{C}e$, which contradicts our assumption.

To prove the second assertion assume that A is a semi-simple Banach algebra with $1 < \dim A < \infty$. By the Wedderburn structure theorem $\Delta(A)$ is then a finite set, and by the first part $\operatorname{soc} A = \{0\}$. This implies that $\Delta(A)$ is a singleton set $\{m\}$, and hence by semi-simplicity we obtain that $A = \mathbb{C}$, which contradicts our assumption. Therefore A is infinite-dimensional. ∎

Theorem 5.65. *Let A be an infinite-dimensional commutative Banach algebra which is integral domain . Then $Q_M(A) = R_M(A)$.*
If additionally A is semi-simple then

$$Q_M(A) = R_M(A) = K_M(A) = \{0\}.$$

Proof Note first that if A is an integral domain then also $M(A)$ is an unital integral domain. Indeed, if T and S are non-zero multipliers there exist $x, y \in A$ such that $Sx \neq 0$ and $Ty \neq 0$. Since A is an integral domain then $(ST)xy = SxTy \neq 0$, and therefore $ST \neq 0$.

Moreover, it is easily seen, via the Shilov idempotent theorem, that every unital commutative integral domain has its maximal ideal $\Delta(A)$ connected. Consequently, since for every $T \in M(A)$ we have $\sigma(T) = \widehat{T}(\Delta(M(A))$, see Theorem 7.79, also $\sigma(T)$ is connected.

Now, to show the equality $Q_M(A) = R_M(A)$ observe first that every quasi-nilpotent operator is Riesz. Hence $Q_M(A) \subseteq R_M(A)$.

On the other hand, if we assume that A is infinite-dimensional then $0 \in \sigma(T)$ (otherwise if T were invertible then $\sigma_{\rm f}(T) = \varnothing$ and this is impossible, by part (g) of Remark 1.54). The spectrum of a Riesz operator is a finite set or a sequence which clusters at 0. Since $\sigma(T)$ is connected it then follows that $\sigma(T) = \{0\}$. Therefore $R_M(A) \subseteq Q_M(A)$, so the equality $Q_M(A) = R_M(A)$ is proved.

To show the last assertion, assume that A is semi-simple. By Theorem 4.34 it follows that $Q_M(A) = R_M(A) = \{0\}$. The equality $K_M(A) = \{0\}$ is obvious, since any compact operator is a Riesz operator. ∎

Now we want describe the ideal $K_M(A)$ in the case of a commutative C^* algebra. Recall that a Banach algebra A is a C^* algebra if it possesses an involution $*$ such that $\|xx^*\| = \|x\|^2$ for all $x \in A$ (these algebras are also called B^* algebras). Any C^* algebra A, not necessarily commutative, is semi-simple, see for instance [**279**, Theorem 4.1.19], so the socle of A does exist. Moreover, for every closed ideal J of A we have

$$J = J^\star := \{j^\star : j \in J\},$$

see [**279**, Theorem 4.9.2], and hence $\operatorname{soc} A = (\operatorname{soc} A)^\star$.

Lemma 5.66. *Suppose that $x \in A$, A an unital Banach algebra. If the resolvent $\rho(x)$ is connected and τ is a spectral set for x for which $0 \notin \tau$ then the spectral idempotent $p(\tau, x)$ belongs to the closed subalgebra of A generated by x.*

Proof Immediate. ∎

Theorem 5.67. *Let e be an idempotent of a C^* algebra. Then there exists an unique self-adjoint idempotent $f \in A$ such that $ef = f$ and $fe = e$.*

Proof By using the Gelfand–Naimark theorem, see Bonsall and Duncan [**72**, 38.10], it suffices to prove that to every projection contained in a C^* algebra of operators on a Hilbert space H there corresponds a self-adjoint projection which has the same range.

Suppose then that $P = P^2 \in L(H)$. With respect to the decomposition $H = P(H) \oplus P(H)^\perp$ we may write

$$P = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}.$$

From this we obtain

$$P^\star = \begin{pmatrix} I & 0 \\ T^\star & 0 \end{pmatrix}, \quad \text{and} \quad PP^\star = \begin{pmatrix} I + TT^\star & 0 \\ 0 & 0 \end{pmatrix}.$$

However, $I+TT^\star$ is a bounded operator on $P(H)$ whose spectrum $\sigma(I+TT^\star)$ lies in $\{\lambda \in \mathbb{R} : \lambda \geq 1\}$. Let us consider the operator defined by

$$Q := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, Q is the spectral idempotent associated with the element $PP^\star$ and the set $\sigma(I + TT^\star)$. By Lemma 5.66 it then follows that Q lies in the closed subalgebra generated by $PP^\star$, hence it is in the closed $*$-subalgebra generated by P.

It is easily seen to that Q is an idempotent self-adjoint operator and $PQ = P$, $PQ = Q$. Thus the existence is proved for a C^* algebra of operators. The prove of the uniqueness is trivial. ∎

Corollary 5.68. *Let A be a C^* algebra. If J is a right (left) ideal of finite order there exists an idempotent self-adjoint $f \in \operatorname{soc} A$ such that $J = fA$ ($J = Af$).*

Proof Consider first the case of a minimal right ideal J. Then there exists $e \in \operatorname{Min} A$ such that $J = eA$. According Theorem 5.67 let $f = f^2 = f^\star$ such that $ef = f$ and $fe = e$. Then

$$eA = feA \subseteq fA = efA \subseteq eA,$$

thus $J = fA$.

The proof of the general case of an ideal of finite order easily follows by using a similar argument. ∎

Theorem 5.69. *Let A be a C^* algebra. Then*

(i) $x \in \operatorname{soc} A \Leftrightarrow x^*x \in \operatorname{soc} A$.

(ii) $x \in \overline{\operatorname{soc} A} \Leftrightarrow x^*x \in \overline{\operatorname{soc} A}$.

Proof (i) Obviously, if $x \in \operatorname{soc} A$ then $x^*x \in \operatorname{soc} A$.

Conversely, suppose that $x^*x \in \overline{\operatorname{soc} A}$. If

$$x^*x = \sum k = 1^n a_k e_k \quad \text{with} \quad e_k \in \operatorname{Min} A,\ a_k \in A \text{ for all } k = 1, \cdots, n,$$

then $x \in J$, where the ideal $J := \sum_{k=1}^n e_k A$ has finite order.

By Corollary 5.68 then there exists a self-adjoint element $f \in \operatorname{soc} A$ such that $x^*x \in Af$. Hence $x^*x(1-f) = 0$ and

$$\|x - xf\| = \|x(1-f)\|^2 = \|(1-f)x^*x(1-f)\| = 0,$$

from which we obtain that $x = xf$.

(ii) This is a consequence of the Segal and Kaplansky theorem which states that if J is any closed ideal of A then $J = J^\star$ and A/J is a C^* algebra, see Bonsall and Duncan [**72**, Theorem 38.18]. In this case we have

$$\|x^*x + J\| = \|(x^* + J)(x + J)\| = \|x + J\|^2,$$

so $x^*x \in J \Leftrightarrow x \in J$. ∎

We now characterize the closure of the socle of a C^* algebra A as the set of all elements whose corresponding wedge operators are compact.

Theorem 5.70. *For a C^* algebra A we have:*

(i) $\operatorname{soc} A = \{x \in A : x \wedge x \text{ is finite dimensional}\}$*;*

(ii) $k(h(\operatorname{soc} A)) = \overline{\operatorname{soc} A} = \{x \in A : x \wedge x \text{ is compact}\}$.

Proof (i) Since A is semi-simple, if $x \in \operatorname{soc} A$ then $x \wedge x$ is finite-dimensional by part (iii) of Theorem 5.18.

Conversely, assume that xAx is finite-dimensional. Clearly $xAx^*x \subseteq xAx$, thus $\dim xAx^*x < \infty$ and hence $\dim x^*xAx^*x < \infty$.

To prove that $x \in \operatorname{soc} A$ it suffices to show, by Theorem 5.69, that $xx^* \in \operatorname{soc} A$. For this it suffices to assume that x is selfadjoint.

Let $Z(x)$ be the centralizer of x in A. We have

$$\sigma(x \wedge x | Z(X)) = \sigma_{Z(x)}(x^2),$$

and since $x \wedge x$ is finite-dimensional this set is finite, say $\{\lambda_1, \ldots, \lambda_n\}$. Let $p_k := p(\lambda_k, x)$ be the spectral projection associated with x and λ_k. Then

$$x = \sum_{k=1}^{n} \lambda_k p_k, \quad \text{where } \lambda_k \neq 0,\ p_k x = x p_k = \lambda_k p_k,\ p_k = {p_k}^*$$

for all $k = 1, \cdots, n$ and $p_k p_i = 0$ for $k \neq i$. For each $k = 1, \ldots, n$ we have

$$\dim\, p_k A p_k = \dim\, p_k x A x p_k \leq \dim\, x A x < \infty.$$

We show now that $p_k \in \operatorname{soc} A$ for all $k = 1, \ldots, n$. Since $\dim\, p_k A p_k$ is finite-dimensional it follows that $p_k A p_k$ is a Riesz algebra which, by Theorem 5.43, is equal to its own socle. But if $e \in \operatorname{Min}\,(p_n A p_n)$ and $x \in A$ then

$$exe = e p_n x p_n e = \lambda e \quad \text{for some } \lambda \in \mathbb{C}.$$

Hence $e \in \operatorname{Min} A$, so

$$p_n A p_n = \operatorname{soc}\,(p_n A p_n) \subseteq \operatorname{soc} A.$$

This shows that each $p_n \in \operatorname{soc} A$ and therefore $x = \sum_{k=1}^{n} \lambda_k p_k \in \operatorname{soc} A$.

(ii) First we show the second equality. Suppose that $x \in \overline{\operatorname{soc} A}$ and choose $x_n \in \operatorname{soc} A$ such that $\|x_n\| \leq \|x\| + 1$ and $\|x_n - x\| \to 0$. Clearly

$$\|x_n \wedge x_n - x \wedge x\| \leq (2\|x\| + 1)\|x_n - x\| \to 0,$$

so by Theorem 5.24 the operator $x \wedge x$ is the uniform limit of finite-dimensional operators and therefore is compact.

Conversely, let $x \wedge x$ be compact on A. The operator $x^* x \wedge x^* x$ is compact since it is the composition of the operators $a \to a x^* \to x a x^* x \to x^* x a x^* x$, $a \in A$. Hence from the equivalence (ii) of Theorem 5.69 it suffices to assume that x is self-adjoint.

Let $Z(x)$ denote the commutant of x in A. As in the proof of the first part, $\sigma(x \wedge x | Z(X)) = \sigma_{Z(x)}(x^2)$, and since $x \wedge x$ is compact this set clusters at most at 0, and the same also holds for $\sigma(x)$. Now, if $\varepsilon > 0$ the set

$$\Omega := \{\lambda \in \sigma(x) : |\lambda| > \varepsilon\}$$

is a finite set, say $\Omega = \{\lambda_1, \ldots, \lambda_n\}$.

If $p_k := p(\lambda_k, x)$ is the spectral idempotent associated with x and λ_k, then from the equalities $p_k x = x p_k = \lambda_k p_k$ we obtain

$$p_k \wedge p_k = \frac{1}{\lambda_k^2}(p_k \wedge p_k)(x \wedge x)$$

which implies that $p_k \wedge p_k$ is compact operator. But any compact projection is finite-dimensional, so by Theorem 5.24 $p_k \in \operatorname{soc} A$ for every $k = 1, \ldots, n$.

Hence $p = \sum_{k=1}^{n} p_k \in \operatorname{soc} A$ and, by using that x is self-adjoint, we obtain

$$\|x(1-p)\| = r(x(1-p) < \varepsilon.$$

This shows that $x \in \overline{\operatorname{soc} A}$.

To show the equality $k(h(\operatorname{soc} A)) = \overline{\operatorname{soc} A}$ observe first that $A/\operatorname{soc} A$ is a C^* algebra and hence semi-simple . Since for every two-sided ideal A/J is semi-simple if and only if $J = k(h(J)$, see Jacobson [**174**, p.205] then

$$\overline{\operatorname{soc} A} = k(h(\overline{\operatorname{soc} A})) = k(h(\operatorname{soc} A)),$$

so the proof is complete. ∎

We conclude this section by describing the multiplier algebra and the ideal $K_M(A)$ for a commutative C^* algebra.

Theorem 5.71. *If A is a commutative C^* algebra, then also $M(A)$ is a commutative C^* algebra. Moreover, $M(A)$ is $*$-isomorphic to $C(\Delta M(A))$.*

Proof For every $T \in M(A)$ let us define

$$T^+(x) := (Tx^*)^* \quad \text{for every } x \in A.$$

Clearly T^+ is linear and for all $x, y \in A$ we have

$$\begin{aligned} T^+(xy) &= (T(xy)^*)^* = (T(y^*x^*))^* \\ &= (y^*(Tx^*))^* = (Tx^*)^*y = T^+(x)y, \end{aligned}$$

thus $T^+ \in M(A)$. It easy to verify that the mapping $T \to T^+$ is an involution on $M(A)$. We show now that this involution makes $M(A)$ into a C^* Banach algebra.

Let U denote the closed unit ball of A. U is self-adjoint, thus

$$\begin{aligned} \|T^+\| &= \sup_{x\in U} \|T^+(x^*)\| = \sup_{x\in U} \|(T^+x)^*)\| \\ &= \sup_{x\in U} \|Tx\| = \|T\|, \end{aligned}$$

and therefore $\|TT^+\| \le \|T\|^2$. The C^* condition $\|TT^+\| = \|T\|^2$ then follows by observing that $\|z\| = \sup_{y\in U} \|yz\|$ for every $z \in A$, and hence

$$\begin{aligned} \|TT^+\| &= \sup_{x\in U} \|(TT^+)x\| = \sup_{x\in U} \sup_{y\in U} \|TT^+(xy)\| \\ &\ge \sup_{x\in U} \|TT^+(xx^*)\| = \sup_{x\in U} \|(Tx)(T^+x)\| \\ &= \sup_{x\in U} \|Tx\|^2 = \|T\|^2. \end{aligned}$$

Hence $M(A)$ is a commutative unital C^* algebra, and therefore via the commutative version of the Gelfand–Naimark theorem, see [**72**, Theorem 35.4], $M(A)$ is $*$-isomorphic to $C(\Delta(M(A))$, the Banach algebra of all continuous functions on the compact space $\Delta(M(A))$. ∎

Any C^* algebra A has a bounded approximate identity, see Bonsall and Duncan [**72**, Lemma 39.14], hence, see the first section of Chapter 4, the norm on A is equivalent to the operator norm on $M(A)$. Therefore the closure of $\operatorname{soc} A$ in A is the same of the closure of $\operatorname{soc} A$ with respect to the operator norm of $M(A)$.

Theorem 5.72. *Let A be a commutative C^* algebra. Then $F_M(A) = \operatorname{soc} A$ and $K_M(A) = \overline{\operatorname{soc} A} = k_{M(A)}(h_{M(A)}(\operatorname{soc} A))$.*

Proof $M(A)$ a C^* algebra and hence is semi-simple, so by Theorem 5.52 we have $\overline{\operatorname{soc} A} = \overline{\operatorname{soc} M(A)}$. The inclusion $\overline{\operatorname{soc} A} \subseteq K_M(A)$ is then an obvious consequence of Corollary 5.25, and hence to complete the proof we need only to prove the inclusion $K_M(A) \subseteq \overline{\operatorname{soc} A}$.

Let $T \in M(A)$ be a compact operator. Since A has a bounded approximate identity it follows that T is a multiplication operator L_x for some $x \in A$, see the proof of Theorem 5.61. Since $L_x : A \to A$ is compact $(L_x)^2 = x \wedge x$ is also compact, and hence by Theorem 5.70 $x \in \overline{\operatorname{soc} A}$.

A similar reasoning shows that $F_M(A) = \operatorname{soc} A$.

The equality $\overline{\operatorname{soc} A} = k_{M(A)}(h_{M(A)}(\operatorname{soc} A))$ is a consequence of $M(A)$ being a C^* algebra. Indeed, applying Theorem 5.69 to $M(A)$ and taking into account that the closure with respect to A and $M(A)$ are the same, we have

$$\overline{\operatorname{soc} A} = \overline{\operatorname{soc} M(A)} = k_{M(A)}(h_{M(A)}(\operatorname{soc} A)),$$

as desired. ∎

7. Weyl multipliers

In this section we shall show that the class $\Phi_M(A)$ of a semi-prime Banach algebra coincides with the class of all multipliers which are Weyl operators. We first need to find conditions which ensure that a multiplier T, or some power T^n of it, has closed range.

Before dealing more specifically with multipliers we begin with some general observations about bounded operators on Banach spaces. The following two results relate, for a bounded operator $T \in L(X)$ on a Banach space X, the property of being $T^n(X)$ closed with the finiteness of the ascent.

Lemma 5.73. *For a bounded operator $T \in L(X)$ on a Banach space X, the following conditions hold:*

(i) If $T^n(X)$ is closed for some natural $n \geq 0$ then $T^{n-k}(X) + \ker T^k$ is closed for all integer $0 \leq k \leq n$.

(ii) If T has finite ascent $p := p(T)$ and $T^n(X)$ is closed for some $n > p$, then $T^{n+k}(X)$ is closed for all $k \geq 0$.

Proof (i) Let (u_j) be a sequence in $T^{n-k}(X) + \ker T^k$ which converges to some $u \in X$ as $j \to \infty$. We have to show that $u \in T^{n-k}(X) + \ker T^k$.

Let us consider two sequences (x_j) in X and (y_j) in $\ker T^k$ such that $u_j = T^{n-k}x_j + y_j$. Then

$$T^n x_j = T^k(T^{n-k}x_j + y_j) = T^k u_j \to T^k u,$$

as $j \to \infty$, and hence, since by assumption $T^n(X)$ is closed, $T^k u \in T^n(X)$. Therefore there exists $z \in X$ such that $T^k u = T^n z$. From this it follows that $T^{n-k}z - u \in \ker T^k$, and consequently $u \in T^{n-k}(X) + \ker T^k$, so the proof of the statement (i) is complete.

(ii) Suppose that $T^n(X)$ is closed. Applying the result of part (i) to the case $k := n-1$, we infer that the sum $T(X)+\ker T^{n-1}$ is closed. Since $n-1 \geq p$ we know that $\ker T^p = \ker T^{n-1} = \ker T^n$, hence $T(X) + \ker T^n$ is closed. By Lemma 1.37 we then conclude that also the sum $T^{n+1}(X) = T^n(T(X)+\ker T^n)$ is closed. An inductive argument then shows that $T^{n+k}(X)$ is closed for all $k \geq 0$. ∎

It is easy to find examples of operators $T \in L(X)$ having ascent $p = p(T)$ finite, for which $T^p(X)$ is closed and $T^{p+1}(X)$ is not closed. For instance, if T is the Volterra operator on $X := C[0,1]$ defined in Example 2.35, then the operator $S : X \times X \to X \times X$ defined by

$$S(x,y) := (0, x+Ty), \quad \text{for all } x, y \in X,$$

has closed range $\{0\} \times X$, while S^2 has range $\{0\} \times T(X)$, and the latter subspace is not closed in $X \times X$.

The next result shows that if an operator having finite ascent p the condition that $T^{p+1}(X)$ is closed holds precisely when $T^n(X)$ is closed for all $n \geq p$.

Theorem 5.74. *Suppose that $T \in L(X)$, X a Banach space, has finite ascent $p := p(T)$. Then the following conditions are equivalent:*

(i) *$T^{p+1}(X)$ is closed;*

(ii) *$T^{p+n}(X)$ is closed for every integer $n \geq 0$.*

Proof The implication (ii) ⇒ (i) is clear, so we need only to prove the implication (i) ⇒ (ii).

(i) ⇒ (ii) Suppose that $T^{p+1}(X)$ is closed. Let $q := 2p+1+n$, where $n \geq 0$ is arbitrarily given, and $k := 2p+1$. From part (ii) of Lemma 5.73 we obtain that the condition $T^{p+1}(X)$ closed entails that $T^q(X)$ is closed, because $q \geq p+1$. Since $k > p$ we have $\ker T^p = \ker T^k$, and hence by part (i) of Lemma 5.73

$$T^n(X) + \ker T^p = T^n(X) + \ker T^k = T^{q-k}(X) + \ker T^k,$$

is closed. This for all $n \geq 0$. In particular, $T^{n+p}(X)+\ker T^{n+p} = T^{n+p}(X)+\ker T^p$ is closed for all $n \geq 0$. By Lemma 3.2, the condition $p = p(T) < \infty$ yields that $T^{n+p}(X) \cap \ker T^{n+p} = \{0\}$, so by Theorem 1.14 we may conclude that $T^{n+p}(X)$ is closed. ∎

Now, in order to look more closely the Fredholm theory of multipliers we need first to introduce the concept of generalized inverse of an operator T defined on a Banach space X. Recall that $T \in L(X)$, X a Banach space, is said to be relatively regular if there exists an operator $S \in L(X)$ for which $T = TST$ and $STS = S$. The operator S is also called a *generalized inverse* of T.

Generally a generalized inverse of T, if it exists, is not uniquely determined. But there exists at most one generalized inverse which commutes with a given $T \in L(X)$. In fact, if S and S' are two generalized inverses of T both commuting with T then

$$TS' = TSTS' = STS'T = ST,$$

and therefore

$$S' = S'TS' = S'TS = STS = S.$$

The following result shows that a bounded operator T admits a generalized inverse which commutes with T precisely when T has both ascent and descent less or equal to 1.

Theorem 5.75. *Let $T \in L(X)$, X a Banach space. Then the following conditions are equivalent:*

(i) *T has a generalized inverse commuting with T;*

(ii) *$X = T(X) \oplus \ker T$;*

(iii) *$p(T) = q(T) \leq 1$;*

(iv) *$T = PU = UP$, where $U \in L(X)$ is invertible and $P \in L(X)$ is idempotent (hence a projection);*

(v) *$T = TCT$, where $C \in L(X)$ is invertible and $TC = CT$.*

Moreover, if T satisfies the equivalent conditions (i)–(v) *then $T^n(X)$ is closed for every $n \in \mathbb{N}$.*

Proof (i)⇒(ii) Let us suppose that there exists a generalized S of T such that $TS = ST$. Then we have

$$I = TS + (I - TS) = TS + (I - ST).$$

The operator TS is, as we have observed above, a bounded projection of X onto $T(X)$ and $I - TS = I - ST$ is a bounded projection of X onto $\ker T$. Hence (i) ⇒ (ii).

The equivalence (ii)⇔(iii) has already been observed in Remark 3.7, part (d).

(iii)⇒(iv) Assume that T has ascent $p(T) = q(T) \leq 1$. Then 0 belongs to the resolvent $R(\lambda, T)$ or is a simple pole of $R(\lambda, T)$. In both cases $T(X)$ is closed, and the restriction $T|T(X)$ is bijective, see part (f) and part (d) of Remark 3.7. Define $U \in L(X)$ by $U := T|T(X) \oplus I|\ker T$. Clearly U is invertible, and if P denotes the projection of X onto $T(X)$ with $\ker P = \ker T$ then $T = PU = UP$.

(v)⇒(vi) If we set $C := U^{-1}$ a straightforward calculation shows that $T = TCT$ and $TC = CT$.

(iv)⇒(v) Suppose that (iv) holds. Then $S := C^2T$ is a commuting generalized inverse of T.

The last assertion is clear from Theorem 5.74 since $T(X)$ is closed. ■

If $T = TCT$, where $C \in L(X)$ is invertible, then T is called *regularly decomposable.*

The previous result applies to multipliers of semi-prime Banach algebras, since these operators have ascent $p(T)$ less or equal to 1. The next result shows that if $T \in M(A)$ has a commuting generalized inverse S in $L(A)$, this operator S will necessarily be a multiplier. This is a generalization of the property that if a multiplier has an inverse (as linear operator) then this inverse is necessarily a multiplier.

Theorem 5.76. *Let A be a semi-prime Banach algebra and $T \in M(A)$. Then the following properties are equivalent:*

(i) *T has a commuting generalized inverse in $L(A)$;*

(ii) *T has a generalized inverse $S \in L(A)$ for which $TS \in M(A)$;*

(iii) *T has a generalized inverse $S \in L(A)$ for which TS commutes with T;*

(iv) *T has a generalized inverse $S \in M(A)$;*

(v) *$A = T(A) \oplus \ker T$, or equivalently $p(T) = q(T) \leq 1$;*

(vi) *$T = PU = UP$ where $U \in M(A)$ is invertible and $P \in M(A)$ is idempotent;*

(vii) *T is regularly decomposable in $M(A)$;*

(viii) *$T^2(A) = T(A)$, or equivalently T has descent $q(T) \leq 1$.*

If the equivalent conditions (i)–(viii) *hold then $T^n(A)$ is closed for all $n \in \mathbb{N}$.*

Proof (i)⇒(ii) Suppose that S is a commuting generalized inverse in $L(A)$ of $T \in M(A)$. If we put $P := TS$ then by Remark **??** P projects A onto $P(A) = T(A)$ along $\ker P = \ker T$. Thus both $\ker P$ and $P(A)$ are two-sided ideals in A. By Theorem 4.9 $\ker T$ is orthogonal, in the sense of algebras, to $T(A)$, so $A = P(A) \oplus \ker P$ is an orthogonal decomposition. By Theorem 4.10 then $P = TS$ is a multiplier of A.

(ii)⇒(iii) This implication is trivial because $M(A)$ is a commutative algebra.

(iii)⇒(v) Since for $P := TS$ we have $P(A) = T(A)$, it suffices to show that $\ker P \subseteq \ker T$. If $Px = 0$ then $Tx = Pz$ for a suitable $z \in A$, hence

$$Tx = P^2 z = PTx = TPx = 0,$$

and so $(I - P)(A) = \ker P \subseteq \ker T$.

The equivalence of (i), (ii), (iii) and (v) then follows by Theorem 5.75.

(v)⇒(vi) If we assume (v) then the projection P of A onto $T(A)$ along $\ker P = \ker T$ is, by (ii), a multiplier. Consequently $U := T + I - P \in M(A)$. Observe that U is the operator which appears in the proof of Theorem 5.75 and hence is invertible. Since $T = UP = PU$ the implication (v)⇒ (vi) is proved.

(vi)⇒(iv) Clearly the operator $S := PU^{-1} \in M(A)$ is a generalized inverse of T.

(iv)⇒(v) Since M(i) is commutative this is a consequence of the implication (i) ⇒ (ii) of Theorem 5.75.

(vi)⇒(vii) If $T = PU = UP$, where $U \in M(A)$ is invertible and $P \in M(A)$ is a projection, then we have

$$TU^{-1}T = TUU^{-1}T = PT = T.$$

(vii)⇒(i) Clearly, if $T = TCT$, where C is an invertible multiplier, then $S := CTC \in M(A)$ is a commuting generalized inverse of T.

The equivalence (v)⇔(viii) follows immediately from part (i) of Theorem 4.32, as soon as we have observed that the ascent $p(T)$ and the descent $q(T)$ must coincide whenever are both finite.

The last assertion is clear from Theorem 5.75. ∎

Remark 5.77. From Theorem 5.76 we know that for a multiplier $T \in M(A)$ of a semi-prime Banach algebra the condition $T^2(A) = T(A)$ implies that $T(A)$ is closed. The reverse inclusion is not true, in general, the closedness of $T(A)$ does not implies $T^2(A) = T(A)$, or, equivalently, $q(T) = 1$.

For instance, if $A = \mathcal{A}(\mathbb{D})$ is the disc algebra and T_g is the multiplication operator by $g(z) := z$, $z \in \mathbb{D}$, then T_g is a Fredholm operator having index ind $T_g = -1$, see Example 5.49, and both the operators T_g and T_g^2 have closed range (precisely, T_g^n has closed range for every $n \in \mathbb{N}$). Clearly $q(T_g) = \infty$, otherwise the finiteness of $q(T_g)$ would imply that $p(T_g) = q(T_g) = 1$ and therefore $\operatorname{ind}(T_g) = \dim \ker T_g$ - $\operatorname{codim} T_g(A) = 0$.

Suppose that for a multiplier $0 \in \sigma(T)$. In this case $p(T)$ is not equal to 0 and hence $p(T) = 1$, so the condition (v) of Theorem 5.76 is equivalent to saying that 0 is a simple pole of the resolvent. An immediate consequence of Theorem 4.36 is that in the case A is a semi-simple Banach algebra we can add another condition to the equivalent conditions (i)–(viii) of Theorem 5.76.

Corollary 5.78. *Let $T \in M(A)$, where A is a semi-simple Banach algebra and $0 \in \sigma(T)$. Then the conditions* (i)–(viii) *of Theorem 5.76 are equivalent to the following condition:*

(ix) 0 *is isolated from the spectrum $\sigma(T)$.* ∎

Remark 5.79. Note that the hypothesis of semi-simplicity is essential in Corollary 5.78. To see that, let T_a be the injective operator defined on the semi-prime radical algebra A_ω of Example 4.35. It is easily seen that T_a is not surjective, since $a \notin T_a(A_\omega)$. Hence $A_\omega \neq T_a(A_\omega) = T_a(A_\omega) \oplus \ker T_a$. On the other hand, 0 is isolated in $\sigma(T)$ since T_a is quasi-nilpotent.

Recall that by Cohen's factorization theorem if A possesses a bounded approximate identity then it admits a factorization. We shall use this fact in the proof of the following result.

Theorem 5.80. *Let A be a commutative semi-simple Banach algebra with a minimal approximate identity. If $T \in M(A)$ then the conditions* (i)–(viii) *of Theorem 5.76 are equivalent to the following property:*

(a) $T(A)$ *is a closed ideal with a bounded approximate identity.*

Proof Assume (vi) of Theorem 5.76, there is a factorization $T = PU$, where $P \in \Phi_M(A)$ is idempotent and $U \in M(A)$ is invertible. Let (e_μ) denote a bounded approximate identity of A. Clearly $T(A)$ is an ideal in A, and from $T(A) = PU(A) = P(A)$ we infer that $T(A)$ is closed. Moreover, the bounded net (Pe_μ) is a subset of $T(A)$, and for every $x \in T(A)$ we have

$$\lim_\mu xPe_\mu = \lim_\mu P(xe_\mu) = Px = x.$$

Therefore (vi) of Theorem 5.76 implies (a).

On the other hand, assume that $T(A)$ is a closed ideal with a bounded approximate identity. If $x \in A$, by Cohen's factorization applied to $T(A)$ there exist two elements $y, z \in A$ for which

$$Tx = (Ty)(Tz) = T(yTz) = T^2(yz).$$

This shows that $T(A) \subseteq T^2(A)$, and consequently $T(A) = T^2(A)$ since the reverse inclusion is trivial. Therefore the condition (viii) of Theorem 5.76 is satisfied. ∎

We show now that if A is a semi-prime Banach algebra the class $\Phi_M(A)$ of the multipliers invertible in $M(A)$ modulo $K_M(A)$ is precisely the class of all Fredholm multipliers having index zero.

Theorem 5.81. *Let A be a semi-prime Banach algebra and let $T \in M(A)$. Then the following statements are equivalent:*

(i) T *is Weyl;*

(ii) T *has a generalized inverse in $M(A)$ and has defects $\alpha(T)$ and $\beta(T)$ both finite;*

(iii) $T = R + K$, *where $R \in M(A)$ is bijective, $K \in F_M(A)$;*

(iv) $T = R + K$, *where $R \in M(A)$ is bijective, $K \in K_M(A)$;*

(v) $T \in \Phi_M(A)$;

(vi) $T = PU$, *where $P \in \Phi_M(A)$ is idempotent and $U \in M(A)$ is invertible.*

Proof (i)⇒ (ii) Suppose that $T \in M(A)$ is a Weyl operator. Since $p(T) \le 1$ by Theorem 4.32, the condition $\alpha(T) = \beta(T)$ entails, see Theorem 3.4, that $p(T) = q(T) \le 1$. From Theorem 5.76 it then follows that T has a generalized inverse in $M(A)$.

(ii)⇒(iii) Suppose that $T = TST$ for some $S \in M(A)$. If $P := TS = ST \in M(A)$ then by Remark **??** P is a projection of A onto $T(A)$ along $\ker T$. Hence $K := I - P \in M(A)$ is a finite-dimensional projection of A onto $\ker T$ along $T(A)$.

Let us define $R := T - K \in M(A)$. We claim that R is invertible. In fact, if $Rx = 0$ then $Tx = Kx$. Since $Kx \in \ker T$ it follows that $Tx = Kx \in \ker T \cap T(A)$. Hence $Kx = (I - P)x = 0$. Thus $Px = x \in T(A)$, and from $Tx = 0$ we also obtain that $x \in \ker T$. Hence $x \in \ker T \cap T(A)$, and consequently $x = 0$, by Theorem 4.32, so R is injective.

Now, K is finite-dimensional, and hence by part (f) of Remark 1.54, $T - K \in \Phi(A)$ and

$$-\beta(T) = \text{ind } R = \text{ind } (T - K) = \text{ind } T = 0,$$

so R is also surjective. Therefore R is bijective and $T = R + K$.

(iii)⇒(iv) Obvious.

(iv)⇒(v) Let $T = R+K$, where $R \in M(A)$ is bijective and $K \in K_M(A)$. Then $R^{-1} \in M(A)$, and by the commutativity of $M(A)$ we have

$$R^{-1}T = TR^{-1} = I + R^{-1}K.$$

Since $R^{-1}K \in K_M(A)$, this implies that T is invertible in $M(A)$ modulo $K_M(A)$.

(v)⇒(i) Let $T \in \Phi_M(A)$. Then there exist operators $W \in M(A)$ and $K \in K_M(A)$ such that $WT = I + K$. Since T and $I + K$ are Fredholm operators W is also a Fredholm operator, and hence by the index theorem

$$\text{ind } (WT) = \text{ind } W + \text{ind } T = \text{ind } (I + K) = 0.$$

Therefore ind $T = -$ind W. Since every multiplier has the SVEP, from Corollary 3.19 it follows that the index of T and of W is non-positive, and this implies that ind $T =$ ind $W = 0$. Therefore T is Weyl.

(ii)⇒(vi) Suppose that (ii) holds (and hence also (i)). Then by Theorem 5.76, $T = PU = UP$ where $P \in M(A)$ is idempotent and $U \in M(A)$ is invertible. From the equalities $T(A) = PU(A) = P(A)$ and $\ker P = \ker T$ we conclude that $P \in \Phi_M(A)$. Finally, U is a Fredholm operator of index 0 because U is invertible.

(vi)⇒(i) If $T = PU$, where $P, U \in \Phi_M(A)$, then the index formula entails that ind $T =$ ind $P +$ ind $U = 0$. Hence T is Weyl. ∎

Lemma 5.82. *Let A be a semi-prime Banach algebra and $T \in M(A)$. Then* $\operatorname{soc} A \subseteq T(A) \oplus \ker T$.

Proof The inclusion soc $A \subseteq T(A) + \ker T$ easily follows from each minimal idempotent e being an eigenvector of T. Indeed,

$$Te = T(e^3) = e(Te)e \in \mathbb{C}e.$$

This shows the inclusion soc $A \subseteq T(A) + \ker T$, and the last sum is direct since $T(A) \cap \ker T = \{0\}$, by Theorem 4.32. ∎

The fact that each multiplier T of a semi-prime Banach algebra has ascent $p(T) \leq 1$ implies by Theorem 3.4 that $\alpha(T) \leq \beta(T)$. This inequality also entails that $T \in M(A)$ is lower semi-Fredholm precisely when T is Fredholm. Consequently we have

$$\Phi_M(A) \subseteq \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) \subseteq \Phi_+(A) \cap M(A).$$

We show now that all these sets coincide in a very special case:

Theorem 5.83. *If A is a semi-prime Banach algebra with a dense socle then*

$$\Phi_M(A) = \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) = \Phi_+(A) \cap M(A).$$

Proof If $T \in \Phi_+(A) \cap M(A)$ the subspace $T(A) + \ker T$ is the sum of a closed and a finite-dimensional subspace of A, therefore it is closed. Moreover, since A has a dense socle $\operatorname{soc} A$, by Lemma 5.82 we have

$$A = \overline{\operatorname{soc} A} = \overline{T(A) \oplus \ker T} = T(A) \oplus \ker T.$$

Thus $\dim \ker T = \operatorname{codim} T(A)$, so T is a Weyl operator, and hence by Theorem 5.81 $T \in \Phi_M(A)$. ∎

8. Multipliers of Tauberian regular commutative algebras

A recurrent assumption in this section will be, for a multiplier $T \in M(A)$, the closedness of the ideal $T^2(A)$. In particular, we shall see that if A is a semi-simple regular commutative Banach algebra the closedness of $T^2(A)$ has some important consequences.

It has been observed in the previous section that the closedness of the range of T^2 for a bounded operator $T \in L(X)$ on a Banach space implies the closedness of $T(X)$, whilst the converse generally is not true. However, the first result of this section establishes that for a multiplier $T \in M(A)$ the property of $T^2(A)$ being closed is equivalent to the property that $T(A) \oplus \ker T$ is closed.

Theorem 5.84. *Let A be a semi-prime (not necessarily commutative) Banach algebra and $T \in M(A)$. Then*

$$T^2(A) \quad \textit{is closed} \ \Leftrightarrow T(A) \oplus \ker T \quad \textit{is closed.}$$

Proof If $T^2(A)$ is closed then by Lemma 5.73 $T(A) + \ker T$ is closed and the sum is direct, since $T(A) \cap \ker T = \{0\}$, by Theorem 4.32.

Conversely, if $T(A) + \ker T$ is closed then $T^2(A)$ is closed by Lemma 1.37. ∎

If we add to the closedness of $T^2(A)$ the assumption of regularity of the algebra A, we can say much more. Note that at this point in the proof of the following result we apply the results of local spectral theory developed in the previous chapters.

Theorem 5.85. *Suppose that A is a semi-simple regular commutative Banach algebra. If $T \in M(A)$ has the property that $T^2(A)$ is closed then*

$\varphi_T = \widehat{T}|\Delta(A)$ is bounded away from 0 on the set $\Delta(A) \setminus h_A(T(A))$, to be precise there is a $\delta > 0$ for which $|\widehat{T}(\varphi)| \geq \delta$ for all $\varphi \in h_A(T(A))$.

Proof Suppose that $T^2(A)$ is closed. Let $B := \overline{T(A)}$ be the closure of $T(A)$. Observe that B is a commutative regular semi-simple Banach algebra having the maximal ideal space $\Delta(B) = \Delta(A) \setminus h_A(T(A))$, see Rickart [**279**, Theorem 2.6.6].

Clearly the restriction $S := T|B$ is an injective multiplier. Moreover, S has a closed range because $S(B) = T^2(A)$. In fact, the inclusion $S(B) \subseteq T^2(A)$ is clear by the continuity of T and because $T^2(A)$ is closed. The reverse inclusion easily follows from $T^2(A) = T(T(A)) \subseteq T(B) = S(B)$. Hence S is bounded below. Consequently by Theorem 4.32 $0 \notin \sigma_{\rm se}(S)$.

Now, by Theorem 2.73 and Theorem 1.36 there exists $\varepsilon > 0$ such that for every open disc $\mathbb{D}(0, \delta)$ with $0 < \delta \leq \varepsilon$ we have

$$B_s(\mathbb{C} \setminus \mathbb{D}(0, \delta)) = \bigcap_{n \in \mathbb{N}} S^n(B),$$

where, as usual, $B_s(\mathbb{C} \setminus \mathbb{D}(0, \delta))$ denotes the local spectral subspace of B relative to the closed set $\mathbb{C} \setminus \mathbb{D}(0, \delta)$. This implies, again by Theorem 2.73, that the subspaces $\widehat{S}^{-1}\overline{(\mathbb{D}(0, \delta))}$ are all equal for every $0 < \delta \leq \varepsilon$.

Next, we want show that $\widehat{S}^{-1}(\mathbb{D}(0, \delta) \setminus \{0\})$ is empty. In fact, if $0 \neq \alpha \in \mathbb{D}(0, \varepsilon) \cap \widehat{S}(\Delta(B))$ choose $0 < \delta_1 < |\alpha| < \varepsilon$. We have then

$$\begin{aligned} \varnothing \;\neq\; & \widehat{S}^{-1}(\{\alpha\}) \subseteq \widehat{S}^{-1}(\mathbb{D}(0, \varepsilon)) \setminus \widehat{S}^{-1}(\overline{\mathbb{D}(0, \delta_1)}) \\ \subseteq\; & \widehat{S}^{-1}(\mathbb{D}(0, \varepsilon)) \setminus \overline{\widehat{S}^{-1}(\mathbb{D}(0, \delta_1))} \\ \subseteq\; & \overline{\widehat{S}^{-1}(\mathbb{D}(0, \varepsilon))} \setminus \overline{\widehat{S}^{-1}(\mathbb{D}(0, \delta_1))}, \end{aligned}$$

which gives a contradiction.

This implies that

$$\widehat{S}^{-1}(\mathbb{D}(0, \varepsilon)) = \widehat{S}^{-1}(\{0\}),$$

and hence $\widehat{S}^{-1}(\{0\})$ is an open and closed subset of $\Delta(A)$. But this implies that $\widehat{S}^{-1}(\{0\}) = \varnothing$, because otherwise, since B is a regular algebra, we can find an element $b \in B$ for which $\operatorname{supp} \widehat{b} \subseteq \widehat{S}^{-1}(\{0\})$. Clearly, for this element b we obtain $\widehat{(Sb)} = 0$, and therefore $Sb = 0$, contradicting the injectivity of S. Hence $\widehat{S}^{-1}(\mathbb{D}(0, \varepsilon) = \varnothing$, and consequently

$$|\widehat{S}(m)| \geq \varepsilon \quad \text{for all } m \in \Delta(B).$$

Since $\widehat{S}$ coincides with the restriction of $\widehat{T}$ on $\Delta(A) \setminus h_A(T(A))$ the proof is complete. ∎

Theorem 5.86. *Suppose that A is a commutative semi-simple regular Tauberian Banach algebra and $T \in M(A)$. Then the condition $T^2(A)$ closed is equivalent to the conditions of Theorem 5.76. In particular, we have*

$$T^2(A) \text{ is closed } \Leftrightarrow A = T(A) \oplus \ker T.$$

Proof Assume that $T^2(A)$ is closed. As we have observed in Theorem 5.84, the property of $T^2(A)$ being closed is equivalent to the property that $T(A) \oplus \ker T$ is closed. We shall show that under the assumption that A is a regular Tauberian Banach algebra $T(A) \oplus \ker T$ is then dense in A, and this trivially implies that $A = T(A) \oplus \ker T$.

To show that $T(A) \oplus \ker T$ is dense in A it suffices to prove that, since A is Tauberian, the subspace $T(A) \oplus \ker T$ contains every element $x \in A$ with compact support $K := \operatorname{supp} \widehat{x}$. Let such x be given and define $z \in A$ so that

$$\widehat{z} = 0 \text{ on } \widehat{T}^{-1}(0) \quad \text{and} \quad \widehat{z} = \widehat{x} \text{ on } K \cap \widehat{T}^{-1}(\mathbb{C} \setminus \mathbb{D}(0, \varepsilon),$$

where $\mathbb{D}(0, \varepsilon)$ is as in the proof of Theorem 5.85. Let $y := x - z$. Obviously

$$\operatorname{supp} \widehat{z} \subseteq K \cap \widehat{T}^{-1}(\mathbb{D}(0, \varepsilon)),$$

so by compactness we conclude from Theorem 6.70 that

$$\sigma_T(z) \subseteq \widehat{T}(\widehat{T}^{-1}(\mathbb{D}(0, \varepsilon)) \subseteq \mathbb{C} \setminus \mathbb{D}(0, \varepsilon).$$

Therefore

$$z \in A_T(\mathbb{C} \setminus \mathbb{D}(0, \varepsilon)) \subseteq T(A).$$

On the other hand, $\operatorname{supp} \widehat{y} \subseteq \widehat{T}^{-1}(\{0\})$ hence $\widehat{Ty} = 0$. This shows, because of the semi-simplicity of A, that $x = y + z \in T(A) \oplus \ker T$. Thus the closedness of $T^2(A)$ implies $A = T(A) \oplus \ker T$.

The converse is clearly true, again because of Theorem 5.84, so the proof is complete. ∎

Corollary 5.87. *If A is a commutative semi-simple regular Tauberian Banach algebra then a multiplier $T \in M(A)$ with closed range is injective if and only if it is surjective.*

Proof If $T \in M(A)$ is surjective then since $T(A) \cap \ker T = \{0\}$ we have $\ker T = \{0\}$ and therefore T is injective.

Conversely, let us suppose $T(A)$ closed and $\ker T = \{0\}$. Trivially, the subspace $T(A) \oplus \ker T$ is closed, and hence combining Theorem 5.84 and Theorem 5.86 we have

$$T(A) = T(A) \oplus \ker T = A.$$

Therefore, T is surjective. ∎

The next result extends to commutative regular Tauberian Banach algebras the result established in Theorem 5.54 under the stronger condition that the Banach algebra A has dense socle.

Corollary 5.88. *Let A be a commutative semi-simple regular Tauberian Banach algebra and $T \in M(A)$. Then*

$$\sigma(T) = \sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{se}}(T).$$

Proof The equality $\sigma(T) = \sigma_{\rm su}(T)$ and $\sigma_{\rm se}(T) = \sigma_{\rm ap}(T)$ have been established in Theorem 4.32. The equality $\sigma_{\rm su}(T) = \sigma_{\rm ap}(T)$ is an obvious consequence of Corollary 5.87. ∎

A direct proof of Corollary 5.88 which does not involve the machinery of local spectral theory may be found in [**214**, p. 416].

Corollary 5.88 reveals that the analogy, already observed, between some aspects of spectral theory of normal operators on Hilbert spaces and multipliers of commutative Banach algebras becomes more evident in the case where A is a Tauberian regular commutative Banach algebra. In fact, for every normal operator T on a Hilbert space H the semi-regular spectrum $\sigma_{\rm se}(T)$ coincides with the whole spectrum, since both T and $T^\star$ have the SVEP and hence Corollary 2.45 applies.

The next theorem is a version of Theorem 5.86 for elements of the ideal $M_0(A)$. Note that in such a case the condition of Tauberianess is not needed.

Theorem 5.89. *Let A be a semi-simple regular commutative Banach algebra and $T \in M_0(A)$. Then the conditions* (i)–(viii) *of Theorem 5.76 are equivalent to the following statements:*

(a) *$T^2(A)$ is closed;*

(b) *There is an idempotent $e \in A$ and an invertible multiplier S such that $T = L_{Se}$, where L_{Se} is the multiplication operator on A defined by the element Se.*

Proof Assume the equivalent conditions (i)-(viii) of Theorem 5.76, in particular that the condition (v) holds. Then $T(A) \oplus \ker T = A$ is closed, hence by Theorem 5.84 also $T^2(A)$ is closed.

(a)⇒(b) Suppose that $T^2(A)$ is closed and $T \in M_0(A)$. Let us consider the set

$$K := \{m \in \Delta(A) : \widehat{T}(m) \neq 0\}.$$

By Theorem 5.85 there exists a $\delta > 0$ for which

$$K = \{m \in \Delta(A) : |\widehat{T}(m)| > \delta\}.$$

Since $T \in M_0(A)$ the set K is compact as well as open. The Shilov idempotent theorem ensures that there exists an element $e \in A$ such that $\widehat{e}(m) = 1$ for each $m \in K$ and $\widehat{e}(m) = 0$ for each $m \in \Delta(A) \setminus K$. Clearly the element e is an idempotent, and taking into account that for any $a \in A$ and $m \in \Delta(A)$ we have $\widehat{(Ta)}(m) = \widehat{T}(m)\widehat{a}(m)$ we easily obtain $(\widehat{(Ta)} = \widehat{(eTa)}$. From the semi-simplicity of A we then obtain that

$$Ta = e(Ta) = (Te)a \quad \text{for any } a \in A,$$

so T is the multiplication operator on A defined by the element Te.

Next, let us define $Sa := (Te)a + a - ea$. Clearly $S \in M(A)$ and

$$\begin{aligned} S(ea) &= e(Sa) = e(Ta + a - ea) = e(Ta) \\ &= (Te)a = Ta, \end{aligned}$$

which shows that $T = L_{Se}$.

It only remains to prove the invertibility of S. To do this assume that 0 belongs to the spectrum $\sigma(S)$. Clearly this happens if and only if there is a sequence (m_k) in $\Delta(A)$ for which $(\widehat{Te - e})(m_k) \to 0$ as $k \to \infty$. Evidently this convergence enforces the equality $\widehat{e}(m_k) = 1$ for all large $k \in \mathbb{N}$, hence $0 \neq \widehat{T}(m_k) \to 0$. But because $\widehat{T}(m_k) \neq 0$ we have $|\widehat{T}(m_k)| > \delta$, a contradiction.

We now show that the statement (b) implies that $T^2(A) = T(A)$, which is the condition (viii) of Theorem 5.76.

Set $T := L_{Se}$, where $e = e^2$ and S is invertible. Then

$$T(A) = (Se)A = eS(A) = eA$$

and from that it follows that

$$\begin{aligned} T^2(A) &= T(T(A)) = T(eA) = e(T(A)) \\ &= e^2A = eA = T(A), \end{aligned}$$

so the proof is complete. ∎

Corollary 5.90. *If A is a commutative semi-simple regular Banach algebra then a multiplier $T \in M_0(A)$ with closed range is neither injective nor surjective.*

Proof $M_0(A)$ is a proper ideal in $M(A)$ so a multiplier $T \in M_0(A)$ cannot be bijective. Now, if $T \in M_0(A)$ with closed range were injective or surjective, by arguing as in the proof of Corollary 5.87, just replacing Theorem 5.86 with Theorem 5.89, we would obtain that T is bijective and this is impossible. ∎

A semi-simple commutative Banach algebra A with a dense socle is regular since its maximal ideal space $\Delta(A)$ is discrete. Moreover, such an algebra is also Tauberian because the elements of the socle have a finite support. These considerations show that the next theorem extends the result of Theorem 5.83 in the commutative case.

Theorem 5.91. *Let A be a commutative semi-simple regular Tauberian Banach algebra. Then*

$$\Phi_M(A) = \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) = \Phi_+(A) \cap M(A).$$

Proof By Theorem 5.81 it suffices to show that if $T \in \Phi_+(A) \cap M(A)$ then T has index 0.

Let $T \in \Phi_+(A) \cap M(A)$. Since $\ker T$ is finite-dimensional the subspace $T(A) \oplus \ker T$ is closed, and hence by Theorem 5.84 and Theorem 5.86 $A =$

$T(A) \oplus \ker T$. The last equality trivially implies $\alpha(T) = \beta(T)$ and hence T has index 0, and this, by Theorem 5.81, is equivalent to saying that $T \in \Phi_M(A)$. ∎

The semi-Fredholm theory of multipliers becomes trivial in the case of a semi-simple regular Tauberian commutative Banach algebra for which the maximal ideal space $\Delta(A)$ contains no isolated points.

Corollary 5.92. *Let A be a commutative semi-simple regular Tauberian Banach algebra having a maximal ideal space $\Delta(A)$ which contains no isolated points. For $T \in M(A)$ the following conditions are equivalent:*

(i) *T is semi-Fredholm;*

(ii) *T is invertible in $M(A)$.*

Proof The implication (ii)⇒ (i) is trivial. To prove (i) ⇒(ii) we observe that by Theorem 5.91 $T \in \mathit{\Phi}_M(A)$. Hence T is invertible in $M(A)$ modulo $K_M(A)$. Since by hypothesis $\Delta(A)$ contains no isolated points then by Theorem 4.41 $K_M(A) = \{0\}$, and consequently T is invertible in $M(A)$. ∎

The additional assumption that $\Delta(A)$ be connected allows us to fit the condition of closedness of $T(A)$ quite neatly into the equivalence of the last corollary.

Theorem 5.93. *Let A be a semi-simple regular commutative Tauberian Banach algebra having a connected maximal ideal space $\Delta(A)$. If $0 \neq T \in M(A)$ then the following properties are equivalent:*

(i) *T is semi-Fredholm;*

(ii) *$T^2(A)$ is closed;*

(iii) *T is invertible in $M(A)$.*

Proof If A is finite-dimensional then $A = \mathbb{C}$ since $\Delta(A)$ is assumed to be connected ([**72**]). In this case the theorem is trivially true. So we can suppose that the algebra A is infinite-dimensional.

By Corollary 5.92 the properties (i) and (iii) are equivalent. Since (iii) trivially implies (ii) it suffices to prove the implication (ii)⇒(iii).

By Theorem 5.86 $A = T(A) \oplus \ker T$, from which we obtain, since $\Delta(A)$ is assumed to be connected and $T \neq 0$, that $A = T(A)$. This shows that T is invertible in $M(A)$. ∎

Next we shall introduce a technical condition (**I**) formulated in [**126**] by Glicksberg. This condition has obvious relations to the existence of bounded relative units in the unitization of A.

(**I**) *There is a constant $\delta > 1$ with the property that for each neighborhood $\mathcal{U}$ of any element $m \in \Delta(A)$ there exists an element $z \in A$ such that*

$$\operatorname{supp} \widehat{z} \subseteq \mathcal{U}, \quad \widehat{z}(m) = 1, \quad \text{and} \quad \|z\| \leq \delta.$$

A commutative Banach algebra A for which the condition (**I**) holds is said to be *boundedly regular.* Observe that this condition is satisfied by the group algebra $L^1(G)$, whenever G is a locally compact Abelian group (see [**282**, Theorem 2.6.8]), as well as by $C_0(\Omega)$ for every locally compact Hausdorff space Ω.

The following statement is analogous to that of Theorem 5.85. Here the condition (**I)** allows us to replace the assumption of $T^2(A)$ closed with the weaker assumption that $T(A)$ is closed.

Theorem 5.94. *Let A be a semi-simple regular commutative Banach algebra which satisfies condition* (**I**)*. Suppose that $0 \neq T \in M(A)$ and $T(A)$ closed. Then $\widehat{T}$ is bounded away from zero on $\Delta(A) \setminus h_A(T(A))$.*

Proof Let us consider a multiplicative functional $m \in \Delta(A) \setminus h_A(T(A))$ and a compact neighborhood V of m such that $h_A(T(A)) \cap V = \varnothing$.

Invoking condition (**I)** we may choose an element $z \in A$ such that the set $\{m \in \Delta(A) : \widehat{z}(m) = 0\}$ is a neighborhood of $h_A(T(A))$ and $\widehat{z}(m) = 1$ whilst $\|z\| \leq \delta$. Since A is regular we have $z \in T(A)$. Moreover, from the assumption of closedness of $T(A)$, it follows that the map $T : A \to T(A)$ is open and hence that there exists a constant $K > 0$ and an element $u \in A$ for which $Tu = z$ and for which $\|u\| \leq K\|z\|$.

From that we obtain the estimate

$$\begin{aligned} 1 &= \widehat{z}(m) = \widehat{T}(m)\widehat{u}(m) \leq |\widehat{T}(m)|\|u\| \\ &\leq |\widehat{T}(m)|K\|z\| \leq |\widehat{T}(m)|\delta K. \end{aligned}$$

Thus $|\widehat{T}(m)| \geq 1/\delta K$, and since $m \in \Delta(A) \setminus h_A(T(A))$ is arbitrary the proposition follows. ∎

With the assumption that $\widehat{T}$ vanishes at infinity on $\Delta(A)$, condition (**I)** allows to include in the equivalences of Theorem 5.89, also the property of $T(A)$ being closed.

Corollary 5.95. *Let A be a semi- simple regular commutative Banach algebra which verifies condition* (**I**)*. Suppose that $T \in M_0(A)$. Then the conditions* (i)–(viii) *of Theorem 5.76 and the conditions* (i)–(ii) *of Theorem 5.89 are equivalent with the condition that $T(A)$ is closed.*

Proof The case $T = 0$ is trivial. If $T \neq 0$ then by Theorem 5.94 the closedness of $T(A)$ entails that the transform $\widehat{T}$ is bounded away from 0 on the set $\{m \in \Delta(A) : \widehat{T}(m) \neq 0\}$. At this point proceed exactly as in the proof of the implication (i)$\Rightarrow$ (ii) of Theorem 5.89. ∎

9. Some concrete cases

The most significant applications of the theory developed in the previous sections are to group algebras and measure algebras of locally compact Abelian group. We recall, once more, that $L^1(G)$ is a regular semi-simple

Tauberian commutative Banach algebra which satisfies condition (**I**).

The following important result, owed to Host and Parreau [**166**], characterizes in a very simple way the convolution operators on $L^1(G)$ which have closed range.

Theorem 5.96. *Let G be a locally compact Abelian group and let $\mu \in \mathcal{M}(G)$. Then the corresponding convolution operator $T_\mu : L^1(G) \to L^1(G)$ has closed range if and only if μ is the product of an idempotent measure and an invertible measure.*

Note that the result of Host and Parreau extends, in the case where $A = L^1(G)$ for a locally compact group G the result of Corollary 5.95, established for multipliers which vanish at infinity, to *any* multiplier of $L^1(G)$. The proof of this deep result is beyond the scope of this monograph. However, it is an open problem whether the result of Host and Parreau extends to any multiplier of an arbitrary semi-simple regular Tauberian commutative Banach algebra. Clearly, in this general setting the result of Theorem 5.94 concerning multipliers $T \in M_0(A)$ may be viewed as a partial analog of Theorem 5.96.

If we collect the results proved for the case of a regular semi-simple Tauberian Banach algebra, and express them in terms of $A = L^1(G)$ and $M(A) = \mathcal{M}(G)$, and take into account Host–Parreau's theorem we obtain a complete description of semi-Fredholm convolution operators:

Theorem 5.97. *Let G be a locally compact Abelian group and let $\mu \in \mathcal{M}(G)$ be a non zero regular complex Borel measure on G. For the corresponding convolution operator $T_\mu : L^1(G) \to L^1(G)$ consider the following statements:*

(i) *μ is invertible in $\mathcal{M}(G)$;*

(ii) *T_μ is upper semi-Fredholm;*

(iii) *T_μ is lower semi-Fredholm;*

(iv) *T_μ is Fredholm;*

(v) *T_μ is Fredholm of index zero;*

(vi) *μ is invertible in the measure algebra $\mathcal{M}(G)$ modulo the ideal of all compact multipliers;*

(vii) *$\mu = \nu \star \tau$ where $\nu \in \mathcal{M}(G)$ is idempotent and $\tau \in \mathcal{M}(G)$ is invertible;*

(viii) *$\mu \star L^1(G)$ is closed;*

(ix) *$\mu \star \mu \star L^1(G)$ is closed;*

(x) *$\mu \star \mu \star L^1(G) = \mu \star L^1(G)$.*

Then the following implications hold:
All the statements (ii)–(x) *are equivalent, and* (i) *implies every other statement on the list.*

If the group G is not compact, or if the dual group $\widehat{G}$ is connected, then all the statements (i)–(x) *are equivalent.*

Proof That the statement (i) implies the others is trivial. Since $\mathcal{M}(G) = M(L^1(G))$, the equivalence of the statements (ii), (iii), (iv), (v), and (vi) is just a transcription of Theorem 5.91. The equivalence of (vi) and (vii) follows from Theorem 5.81, while the equivalence of (vii) and (viii) is the result established by Host and Parreau. Moreover, the statement (vii) is equivalent to the statement (viii) by Theorem 5.86, whilst the equivalence of (vii) and (x) follows from Theorem 5.76.

If G is not compact then the dual group $\widehat{G}$ has no isolated point, so the equivalence of (i) and (ii) follows from Corollary 5.92.

Finally, if $\widehat{G}$ is connected then by Theorem 5.93 the statements (i) and (ix) are equivalent. ∎

In the case that G is a compact Abelian group the situation become even more clear.

Theorem 5.98. *Suppose that G is a compact Abelian group and that $\mu \in \mathcal{M}(G)$. For the convolution operator $T_\mu : L^1(G) \to L^1(G)$ each of the following statements is equivalent to the conditions* (ii)–(x) *listed in Theorem 5.97:*

(a) *μ is invertible in $\mathcal{M}(G)$ modulo $L^1(G)$;*

(b) *μ is invertible in $\mathcal{M}(G)$ modulo the ideal $P(G)$ of all trigonometric polynomials on G;*

(c) *$\mu = \psi + \varphi$ where $\psi \in \mathcal{M}(G)$ is invertible and $\varphi \in L^1(G)$;*

(d) *$\mu = \psi + \varphi$ where $\psi \in \mathcal{M}(G)$ is invertible and $\varphi \in P(G)$.*

Proof We know by Corollary 5.62 and Corollary 5.53 that for a compact Abelian group G we have

$$K_M(L^1(G)) = L^1(G) \quad \text{and} \quad F_M(L^1(G)) = \operatorname{soc} L^1(G) = P(G).$$

In this case the assertion (a) is *mutatis mutandis*, the statement (v) of Theorem 5.97. By Theorem 5.52 the assertion (a) is equivalent to the assertion (b).

The equivalence of the statements (a), (c), and (d) follows from Theorem 5.81. ∎

Also a Banach algebra A with an orthogonal basis (e_k) has a dense socle, since the set of all minimal idempotents coincides with the set $\{e_k : k \in \mathbb{N}\}$. Hence the equalities of Theorem 5.91 are valid for any Banach algebra with an orthogonal basis. If the basis (e_k) is unconditional we can say much more.

Theorem 5.99. *Let A be a Banach algebra with an orthogonal unconditional basis (e_k) and let $(\lambda_k(Te_k))$ be the sequence associated with $T \in M(A)$. Then T is a Fredholm operator if and only if there exists a bounded sequence (ξ_k) such that $\xi_k \lambda_k(Te_k) \to 1$ as $k \to \infty$.*

Proof We know by Theorem 4.44 that the mapping $T \to (\lambda_k(Te_k))$ is an isomorphism of $M(A)$ onto l^∞. Moreover, if $K_{M_0}(A)$ is the set

$$\{T \in K(A) : (\lambda_k(Te_k)) \in c_0\},$$

as noted before Theorem 4.45 we have $F_M(A) \subseteq K_{M_0}(A) \subseteq K_M(A)$. These inclusions entail that the set $\Phi_{M_0}(A)$ of all Fredholm elements of $M(A)$ relative to $K_{M_0}(A)$ coincides with $\Phi_M(A)$. Since by Theorem 4.45 $K_{M_0}(A)$ is isomorphic to c_0 then T is Fredholm if and only if the corresponding sequence $(\lambda_k(Te_k))$ is invertible in l^∞ modulo c_0. ∎

We now consider the case A is commutative semi-simple Banach algebra for which its multiplier algebra $M(A)$ is regular. Note that if $M(A)$ is regular then A is regular, see Theorem 4.25, whilst the converse need not be true. An immediate example is given by the group algebra $A = L^1(G)$ of any non discrete locally compact abelian group G, since in this case $M(A) = \mathcal{M}(G)$ and the last algebra is not regular [**242**]. Of course, the results of Theorem 5.89 and Corollary 5.95 apply to this particular case. Anyway, Theorem 5.91 does not apply to this case because we do not assume that A is Tauberian.

We need first to make some general remarks about Banach algebras. Let $\mathcal{B}$ be any (not necessarily commutative) Banach algebra with identity u. Suppose that $\mathcal{A}$ is a commutative subalgebra of $\mathcal{B}$ with identity $u \in \mathcal{A}$ such that $\mathcal{A}$ is also a Banach algebra (however, $\mathcal{A}$ need not be a closed subalgebra of $\mathcal{B}$).

We recall that given a commutative Banach algebra $\mathcal{A}$ we shall say that a subset $J \subseteq \mathcal{A}$ *strongly separates* the points of $\Delta(\mathcal{A})$ if for every $m \in \Delta(\mathcal{A})$ there exists an element $x \in J$ such that $\widehat{x}(m) \neq 0$.

Theorem 5.100. *Let $\mathcal{A}$ satisfy either of the following properties:*

(i) $\Delta(\mathcal{A})$ *is totally disconnected;*

(ii) *There exists a semi-simple commutative regular Banach algebra $\mathcal{D}$ and an algebra homomorphism φ of $\mathcal{D}$ into $\mathcal{A}$ which maps the identity of D onto the identity of $\mathcal{A}$ and is such that $\varphi(\mathcal{D})$ strongly separates the points of $\Delta(\mathcal{A})$.*

Then $\mathcal{A}$ is an inverse closed subalgebra in $\mathcal{B}$.

Proof Let $\mathcal{W}$ be a maximal commutative subalgebra of $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{W}$. It is well known that $\mathcal{W}$ is inverse closed in $\mathcal{B}$. Hence to prove our theorem we need only to show that $\mathcal{A}$ is inverse closed in $\mathcal{W}$. Thus we may assume that $\mathcal{B}$ is commutative.

Let $m^\star \in \Delta(\mathcal{A})$ denote the restriction to $\mathcal{A}$ of an arbitrary multiplicative functional $m \in \Delta(\mathcal{B})$, and set

$$\Omega := \{m^\star : m \in \Delta(\mathcal{B})\}.$$

We claim that under the assumptions (i) or (ii), Ω is dense in $\Delta(\mathcal{A})$.

In fact, let $\overline{\Omega}$ denote the closure of Ω in $\Delta(\mathcal{A})$ and assume that (i) holds, and suppose that $\overline{\Omega} \neq \Delta(\mathcal{A})$. Then there exists a non-empty open and closed subset Γ disjoint from $\overline{\Omega}$, and via the Shilov idempotent theorem an idempotent $e \in \mathcal{A}$ such that the Gelfand transform $\widehat{e}$ is equal to 1 on Γ and $\widehat{e}$ is equal to 0 on $\Delta(\mathcal{A}) \setminus \Gamma$. In particular,

$$\widehat{e}(m) = 0 \quad \text{for all } m \in \Delta(\mathcal{B}),$$

which implies $e \in \operatorname{rad} \mathcal{B}$, but this is impossible. Hence if the condition (i) holds Ω is dense in $\Delta(\mathcal{A})$.

Now assume that (ii) holds and, again, suppose that $\overline{\Omega} \neq \Delta(\mathcal{A})$. Then the mapping defined by

$$\Psi : m \in \Delta(\mathcal{A}) \to m \circ \varphi \in \Delta(\mathcal{D})$$

is continuous. Moreover, since $\varphi(\mathcal{D})$ strongly separates the points of $\Delta(\mathcal{A})$ Ψ is also injective. Set

$$\Lambda := \Psi(\overline{\Omega}) = \{m \circ \varphi : m \in \overline{\Omega}\},$$

and fix $m_1 \notin \overline{\Omega}$. Clearly we can choose two elements $v, w \in \mathcal{D}$ such that

$$\widehat{v}(\Lambda) = \{0\}, \quad \widehat{w}(m_1 \circ \varphi) \neq 0 \quad \text{and } (u - v)w = 0.$$

From this it follows that

$$(u - \varphi(\mathrm{v}))\varphi(w) = 0 \quad \text{and} \quad \varphi(w) \neq 0.$$

Also $\varphi(\mathrm{v})$ belongs to the radical and this is impossible, see Theorem 2.3.2 of Rickart [**279**]. So in either cases Ω is dense in $\Delta(\mathcal{A})$.

Finally, let x be an element of A with inverse $x^{-1} \in \mathcal{B}$. Then there exists $\delta > 0$ such that

$$|\widehat{x}(m)| \geq \delta \quad \text{for every } m \in \Delta(\mathcal{B}).$$

But then

$$|\widehat{x}(m^\star)| \geq \delta \quad \text{for every } m^\star \in \Omega.$$

Hence x is also invertible in $\mathcal{A}$. ∎

Corollary 5.101. *Suppose that J is a closed ideal of $\mathcal{B}$, $\mathcal{A} \cap J$ a closed ideal of $\mathcal{A}$, and that $\mathcal{A}/\mathcal{A} \cap J$ satisfies condition* (i) *or condition* (ii) *of Lemma 5.100. Then $x \in \mathcal{A}$ is invertible modulo $\mathcal{A} \cap J$ if and only if x is invertible modulo J.*

Proof Clearly, the mapping

$$x + (\mathcal{A} \cap J) \to x + J$$

is an isomorphic embedding of the algebra $\mathcal{A}/\mathcal{A} \cap J$ onto a subalgebra of $\mathcal{B}/J$. Thus it suffices to apply Theorem 5.100. ∎

Now, let $\mathcal{A}$ be a commutative subalgebra of $L(X)$, X a Banach space, such that the operator identity $I \in \mathcal{A}$. If $\mathcal{A}$ satisfies the condition (i) or the condition (ii) of Theorem 5.100 then $\mathcal{A}$ is a closed inverse subalgebra of $L(X)$ and this trivially implies that $K := \mathcal{A} \cap K(X)$ is an inessential ideal of $\mathcal{A}$.

Corollary 5.102. *Let $\mathcal{A}$ be a closed commutative subalgebra of $L(X)$, X a Banach space. Suppose that $\mathcal{A}$ verifies condition* (i) *or condition* (ii) *of Theorem 5.100. Then $\Phi_K(\mathcal{A}) = \Phi(X) \cap \mathcal{A}$, where $\Phi_K(\mathcal{A})$ is the set of all K-Fredholm elements of $\mathcal{A}$.*

Proof First note that if the algebra $\mathcal{A}$ verifies the conditions (i) or (ii) of Theorem 5.100 then so does the quotient algebra $\mathcal{A}/K$. The equality $\Phi_K(\mathcal{A}) = \Phi(X) \cap \mathcal{A}$ then follows immediately from Corollary 5.101. ■

Theorem 5.103. *Let A be a commutative semi-simple Banach algebra and suppose that $M(A)$ is regular. Then*

$$\Phi_M(A) = \Phi(A) \cap M(A).$$

Proof The inclusion $\Phi_M(A) \subseteq \Phi(A) \cap M(A)$ is clear, so we need to prove the reverse inclusion $\Phi_M(A) \supseteq \Phi(A) \cap M(A)$.

By assumption $M(A)$ is regular, and by Theorem 4.25 $M(A)$ is also semi-simple. Therefore if we take $\mathcal{A} = \mathcal{D} = M(A)$ and denote by φ the identity mapping of $M(A)$ onto $M(A)$, the condition (ii) of Theorem 5.100 is trivially verified. Since $M(A)$ is a closed commutative subalgebra of $L(A)$ we may now employ Corollary 5.102 to conclude that the equality $\Phi_M(A) = \Phi(A) \cap M(A)$ hold. ■

In Remark 5.77 we have observed that for a semi-prime commutative Banach algebra the condition that $T(A)$ is closed is generally weaker than the condition $T(A) = T^2(A)$, which is equivalent by Theorem 5.76 to the condition $A = T(A) \oplus \ker T$. Moreover, the example considered in Remark 5.77 shows that the condition $A = T(A) \oplus \ker T$ generally cannot be relaxed to the condition that $T(A) \oplus \ker T$ is closed. In fact, in this example $\ker T_g = \{0\}$ and therefore $T_g(A) \oplus \ker T_g$ is closed.

An important case in which these conditions are equivalent is the case of a (not necessarily commutative) C^* algebra.

Theorem 5.104. *Let A be a C^* algebra and let $T \in M(A)$. Then the following statements are equivalent to the conditions* (i)–(viii) *of Theorem 5.76:*

(a) *$T(A)$ is closed;*

(b) *$T(A) \oplus \ker T$ is closed.*

Proof (b)⇒(a) If $T(A) \oplus \ker T$ is closed then $T^2(A)$ is closed by Theorem 5.84. Since for every multiplier T the ascent $p(T)$ is less than or equal to 1, the closedness of $T^2(A)$ implies by Theorem 5.74 that $T(A)$ is closed.

(a)⇒(b) Since $T(A)$ is a closed two-sided ideal in a C^* algebra, $T(A)$ has a bounded approximate identity, so that Cohen's factorization theorem entails that $T(A) = [T(A)]^2$. If $z \in T(A)$ then there exists $x, y \in A$ such that $z = TxTy = T^2(xy) \in T^2(A)$, from which we obtain that $T(A) \subseteq T^2(A)$. Since the reverse inclusion is satisfied for each operator we then conclude that $T(A) = T^2(A)$. The last equality is equivalent by Theorem 5.76 to the condition $A = T(A) \oplus \ker T$, which trivially implies (b). ■

Corollary 5.105. *Let A be a C^* algebra and suppose that $T \in M(A)$ has closed range. Then T is injective if and only if T is surjective.*

Proof If $T(A)$ is closed then $A = T(A) \oplus \ker T$ and this decomposition implies that $T(A) = A$ precisely when $\ker T = \{0\}$. ■

Corollary 5.106. *Let A be a C^* algebra and suppose that $T \in M(A)$. Then $\rho(T) = \rho_{\rm se}(T)$ and $\sigma(T) = \sigma_{\rm ap}(T)$.*

Proof Argue exactly as in the proof of Theorem 5.88. ■

Corollary 5.107. *Let A be a C^* algebra. Then*

$$\Phi_M(A) = \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) = \Phi_+(A) \cap M(A).$$

Proof Argue exactly as in the proof of Theorem 5.91. ■

10. Browder spectrum of a multiplier

In this section we shall describe the Browder spectrum $\sigma_{\rm b}(T)$ of a multiplier of a semi-prime Banach algebra, in particular the Browder spectrum of a convolution operator acting on the group algebra $L^1(G)$. Recall that by Corollary 3.53 the Browder spectrum $\sigma_{\rm b}(T)$ of a multiplier of a semi-prime Banach algebra coincides with the Weyl spectrum $\sigma_{\rm w}(T)$, since every multiplier T has the SVEP. Note that for multipliers of semi-prime Banach algebras the inclusion $\sigma_{\rm f}(T) \subseteq \sigma_{\rm w}(T)$ may be proper. This is essentially owed to the fact that a Fredholm multiplier may have index not equal to 0. This, for instance, is the case of the multiplication operator $(T_z f)(z) := zf(z)$ defined on the disc algebra $\mathcal{A}(\mathbb{D})$. In this case $\sigma_{\rm f}(T_z) = \partial\mathbb{D}$, whilst $\sigma_{\rm w}(T_z) = \sigma_{\rm b}(T_z) = \mathbb{D}$.

For every multiplier $T \in M(A)$ let us denote

$$\Lambda(T) := \{\lambda \in \text{iso } \sigma(T) : \alpha(\lambda I - T) = \infty\}.$$

Theorem 5.108. *Let A be a semi-simple Banach algebra and $T \in M(A)$. Then*

$$\sigma_{\rm b}(T) = \sigma_{\rm w}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is a limit point of } \sigma(T)\} \cup \Lambda(T).$$

Proof Suppose that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{\rm w}(T) = \sigma_{\rm b}(T)$. Then $p(\lambda I - T) = q(\lambda I - T) < \infty$ and hence λ is isolated in $\sigma(T)$. Obviously $\lambda \notin \Lambda(T)$.

Conversely, assume that $\lambda \in \mathbb{C}$ is an isolated point of the spectrum $\sigma(T)$ for which the kernel $\ker(\lambda I - T)$ is finite-dimensional. By Theorem 4.36 λ is then a simple pole of the resolvent, hence $A = (\lambda I - T)(A) \oplus \ker(\lambda I - T)$ and this easily implies that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda \notin \sigma_{\rm w}(T)$. ■

By Corollary 3.41 we know that for every bounded operator $T \in L(X)$ on a Banach space X the Weyl spectrum may be described in terms of spectra of compact perturbations, as well as in terms of spectra of finite-dimensional perturbations:

$$\sigma_{\mathrm{w}}(T) = \bigcap_{K \in F(X)} \sigma(T+K) = \bigcap_{K \in K(X)} \sigma(T+K),$$

whilst, from Corollary 3.49 we also know that the Browder spectrum is given by the formulas

$$\sigma_{\mathrm{b}}(T) = \bigcap_{K \in F(X),\, KT=TK} \sigma(T+K) = \bigcap_{K \in K(X),\, KT=TK} \sigma(T+K).$$

Evidently in the case of a multiplier T the compact perturbations, or the finite-dimensional perturbations, which appear in the equalities above are not, in general, multipliers. The characterizations of Fredholm multipliers given in the previous sections illustrate that the multiplier algebra $M(A)$ of a semi-prime Banach algebra is sufficiently large to represent all the results concerning Fredholm theory in terms *only* of elements of $M(A)$. The results of this section once more confirm this property. Indeed, we shall show now that for multipliers one can restrict the intersections above to compact multipliers or to finite-dimensional multipliers.

To see this, we first need to extend the concepts of Weyl spectrum and Browder spectrum to elements of a semi-prime Banach algebra $\mathcal{A}$ with respect to a fixed inessential ideal of $\mathcal{A}$.

Definition 5.109. *Let J be a closed inessential ideal J of a Banach algebra $\mathcal{A}$ with unit u. The* Fredholm spectrum of an element $x \in \mathcal{A}$ *relative to J is the set*

$$\sigma_{\mathrm{f}}(x, J) := \{\lambda \in \mathbb{C} : \lambda - x \notin \Phi(\mathcal{A}, J)\}.$$

The Weyl spectrum of an element $x \in \mathcal{A}$ *relative to J is the set*

$$\sigma_{\mathrm{w}}(x, J) := \bigcap_{y \in J} \sigma(x+y).$$

A complex λ is said to be a Riesz point *of $x \in \mathcal{A}$ if either $\lambda \in \rho(x)$ or if λ is a J-Fredholm point of x which is an isolated point of $\sigma(x)$.*

The Browder spectrum of an element $x \in \mathcal{A}$ *with respect to J is the set*

$$\sigma_{\mathrm{b}}(x, J) := \{\lambda \in \mathbb{C} : \lambda \textit{ is not a } J\textit{-Riesz point}\}.$$

Of course, the classical Fredholm, Weyl, and Browder spectra of a bounded operator on a Banach space X may be viewed as spectra originating from the Fredholm theoy of $L(X)$ relative to the inessential ideal $K(X)$. Clearly the three spectra above defined are all closed subsets of $\sigma(x)$ and hence compact subsets of $\mathbb{C}$. Moreover, for every $x \in \mathcal{A}$ and inessential ideal J the following inclusions

$$\sigma_{\mathrm{f}}(x, J) \subseteq \sigma_{\mathrm{w}}(x, J) \subseteq \sigma_{\mathrm{b}}(x, J), \tag{150}$$

hold. The proof of the first inclusion is immediate. Indeed, from the definition of Fredholm elements we have

$$\sigma_{\mathrm{f}}(x, J) = \sigma_{\mathcal{A}/J}(x + J) \subseteq \sigma(x + y),$$

for every $x, y \in \mathcal{A}$. The proof of the second inclusion of (150) is not immediate and requires the concept of abstract index of elements of Banach algebras, which is beyond the scope of this monograph (for a proof see [**62**, Theorem R.2.2]). Moreover, by Theorem R.5.1 of Barnes, Murphy, Smyth, and West [**62**], for any commutative and unital Banach algebra and any inessential ideal J we have

$$\sigma_{\mathrm{f}}(x, J) = \sigma_{\mathrm{w}}(x, J) = \sigma_{\mathrm{b}}(x, J).$$

Let us consider the case $\mathcal{A} := M(A)$, the multiplier algebra of a semi-prime Banach algebra A and let $J := K_M(A)$. Evidently for each $T \in M(A)$ we have $\sigma_{\mathrm{f}}(T, K_M(A)) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_M(A)$, and since $M(A)$ is a commutative unital Banach algebra,

$$\sigma_{\mathrm{f}}(T, K_M(A)) = \sigma_{\mathrm{w}}(T, K_M(A)) = \sigma_{\mathrm{b}}(T, K_M(A)). \tag{151}$$

Moreover, from Theorem 5.81 we easily obtain that

$$\sigma_{\mathrm{f}}(T, K_M(A)) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T). \tag{152}$$

Theorem 5.110. *Let A be a semi-prime Banach algebra and $T \in M(A)$. Then*

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \bigcap_{K \in F_M(A)} \sigma(T + K) = \bigcap_{K \in K_M(A)} \sigma(T + K). \tag{153}$$

Moreover, if A is a semi-simple commutative Banach algebra then

$$\sigma_{\mathrm{b}}(T) = \bigcap_{x \in \overline{\operatorname{soc} A}} \sigma(T + L_x). \tag{154}$$

where the closure of $\operatorname{soc} A$ *is taken with respect to the operator norm of* $M(A)$.

Proof The equalities (153) easily follow from (151) and (152), taking into account that the ideals $K_M(A)$ and $F_M(A)$ generate the same Fredholm elements in $M(A)$.

Now suppose that A is a semi-simple commutative Banach algebra and consider the inessential ideal $J := \overline{\operatorname{soc} A} = \overline{\operatorname{soc} M(A)}$, see Theorem 5.52. Again, $M(A)$ being a commutative unital Banach algebra we deduce that

$$\sigma_{\mathrm{f}}(T, \overline{\operatorname{soc} A}) = \sigma_{\mathrm{w}}(T, \overline{\operatorname{soc} A}) = \bigcap_{x \in \overline{\operatorname{soc} A}} \sigma(T + L_x),$$

for every $T \in M(A)$. By Theorem 5.55 we also know that $\sigma_{\mathrm{f}}(T, \overline{\operatorname{soc} A}) = \sigma_{\mathrm{f}}(T, K_M(A))$ so that from the equalities (152), we may conclude that (154) holds. ∎

Corollary 5.111. *Let A be a commutative semi-simple Banach algebra with a bounded approximate identity. If $T \in M(A)$ then*

$$\sigma_{\mathrm{b}}(T) = \bigcap_{x \in \overline{\operatorname{soc} A}} \sigma(T + L_x),$$

where the closure of $\operatorname{soc} A$ *is taken with respect to the norm of A.*

Proof As already observed, if a commutative Banach algebra A has a bounded approximate identity then A is closed in its multiplier algebra $M(A)$ since the norm of A is equivalent to the operator norm of $M(A)$. ■

In particular, Corollary 5.111 applies to multipliers T of C^* algebras. In this case $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, since by Corollary 5.107 we have $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{f}}(T, K_M(A))$.

Much more we can say under some assumptions on the maximal ideal space $\Delta(A)$.

Theorem 5.112. *Suppose that A is a semi-simple commutative Banach algebra and $T \in M(A)$. Then the following statements hold:*

(i) *If A has a bounded approximate identity and $\Delta(A)$ is discrete then*

$$\sigma_{\mathrm{b}}(T) = \bigcap_{x \in A} \sigma(T + L_x) = \bigcap_{K \in M_{00}(A)} \sigma(T + K);$$

(ii) *If A is a Tauberian regular Banach algebra, then*

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{f}}(T);$$

(iii) *If $\Delta(A)$ has no isolated points then*

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) = \sigma(T).$$

Proof (i) If A has a discrete maximal ideal space $\Delta(A)$ then A and $M_{00}(A)$ are inessential ideals of $M(A)$ by Theorem 5.57. Moreover, A is closed in the operator norm of $M(A)$ since A has a bounded approximate identity. Theorem 5.57 also ensures that the inessential ideals $K_M(A)$, A and $M_{00}(A)$ generate the same set of Fredholm elements in $M(A)$, so for every $T \in M(A)$ we have

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{f}}(T, K_M(A)) = \sigma_{\mathrm{f}}(T, A) = \sigma_{\mathrm{f}}(T, M_{00}(A)).$$

The commutativity of the unital Banach algebra $M(A)$ yields

$$\sigma_{\mathrm{f}}(T, A) = \sigma_{\mathrm{w}}(T, A) = \bigcap_{x \in A} \sigma(T + L_x)$$

and

$$\sigma_{\mathrm{f}}(T, M_{00}(A)) = \sigma_{\mathrm{w}}(T, M_{00}(A)) = \bigcap_{K \in M_{00}(A)} \sigma(T + K),$$

from which we conclude that the statement (i) is true.

(ii) The equality $\sigma_{\rm f}(T) = \sigma_{\rm w}(T)$ is an obvious consequence of Theorem 5.91 and Theorem 5.81.

(iii) The equalities follow from the equalities (152), once it is observed that if $\Delta(A)$ has no isolated points then $K_M(A) = \{0\}$, see Theorem 4.41.
■

Theorem 5.110 and Theorem 5.112 apply to the group algebra case $A := L^1(G)$ of a locally compact abelian group G. Recall that $L^1(G)$ possesses a bounded approximate identity, and by Corollary 5.62, for every compact abelian group G we have $K_M(L^1(G)) = L^1(G)$ and $F_M(L^1(G))$ coincides with the ideal $P(G)$ of all trigonometric polynomials on G.

Corollary 5.113. *Let $\mu \in \mathcal{M}(G)$, where G is a locally compact abelian group. For the convolution operator $T_\mu : L^1(G) \to L^1(G)$ we then have:*

(i) *$\lambda \in \sigma_{\rm b}(T_\mu)$ precisely when either λ is not isolated in $\sigma(\mu)$ or $\widehat{\mu}^{-1}(\{\lambda\})$ is an infinite set of $\widehat{G}$. Moreover,*

$$\sigma_{\rm f}(T_\mu) = \sigma_{\rm w}(T_\mu) = \sigma_{\rm b}(T_\mu);$$

(ii) *If G is compact then*

$$\sigma_{\rm b}(T_\mu) = \bigcap_{\nu \in P(G)} \sigma(\mu + \nu) = \bigcap_{\nu \in L^1(G)} \sigma(\mu + \nu) = \bigcap_{\nu \in \mathcal{M}_{00}(G)} \sigma(\mu + \nu).$$

(iii) *G is not compact then $\sigma_{\rm f}(T_\mu) = \sigma_{\rm w}(T_\mu) = \sigma_{\rm b}(T_\mu) = \sigma(\mu)$.*

Proof All the statements are consequences of Theorem 5.112 and Theorem 4.38. ■

In Chapter 3 we have seen that the Browder spectrum of a bounded operator on a Banach space may be expressed by means of spectra of compressions. Now we show that the Browder spectrum of a multiplier T of a semi-prime (not necessarily commutative) Banach algebra may be expressed by means of spectra of compressions relative to projections which are multipliers.

Let $\mathcal{P}(A)$ be the set of all bounded projections $P \in L(A)$ such that codim $P(A) < \infty$. Recall that the compression generated from a projection $P \in \mathcal{P}(A)$ is the bounded linear operator $T_P : P(A) \to P(A)$ defined by $T_P y := PTy$ for every $y \in P(A)$.

Theorem 5.114. *Suppose that A is a semi-prime Banach algebra and $T \in M(A)$. Then*

$$\sigma_{\rm b}(T) = \bigcap_{P \in \mathcal{P}(A) \cap M(A)} \sigma(T_P). \tag{155}$$

Proof Suppose that $T \in M(A)$. From the equality (82) established in Chapter 3, taking into account that the equality $PT = TP$ is automatically ensured for multipliers on semi-prime Banach algebras, we obtain

$$\sigma_{\mathrm{b}}(T) = \bigcap_{P\in\mathcal{P}(A)} \sigma(T_P) \subseteq \bigcap_{P\in\mathcal{P}(A)\cap M(A)} \sigma(T_P).$$

To establish the opposite inclusion we need only to prove that if $\lambda \notin \sigma_{\mathrm{b}}(T)$ then there is a a projection $P \in \mathcal{P}(A)$ which is a multiplier of A such that $\lambda \notin \sigma(T_P)$.

Now, if $\lambda \notin \sigma_{\mathrm{b}}(T)$ then either $\lambda I - T$ is invertible or $\lambda \in \sigma(T)$. Trivially, in the first case, taking $P := I \in M(A)$ we have $\lambda \notin \sigma(T_P)$. In the second case the condition $\lambda \notin \sigma_{\mathrm{b}}(T)$ entails that λ is an isolated point of $\sigma(T)$.

Let Q denote the spectral projection associated with $\{\lambda\}$. Then $\ker Q = (\lambda I - T)(A)$, by part (f) of Remark 3.7, and $\sigma(T|\ker Q) = \sigma(T) \setminus \{\lambda\}$. Clearly $Q \in M(A)$, and if we put $P := I - Q$ then $P(A) = \ker Q$ is finite-codimensional. Thus $P \in \mathcal{P}(A) \cap M(A)$. Finally, from the commutativity of $M(A)$ we obtain that

$$T_P(y) = (PT)y = (TP)y = Ty \quad \text{for every } y \in P(A).$$

This means that T_P coincides with the restriction of T on $P(A) = \ker Q$, hence $\lambda \notin \sigma(T_P)$. ∎

In a semi-simple commutative Banach algebra A the ideal $P(A)$, where $P \in \mathcal{P}(A) \cap M(A)$, may be represented in a very precise way. To see that first consider for an arbitrary idempotent $e \in A$ the *annihilator* $\mathrm{an}\{e\}$. Clearly $\mathrm{an}\{e\} = (1 - e)A$, where $(1 - e)A := \{a - ea : a \in A\}$ and hence $A = eA \oplus (1 - e)A$.

Now let $P \in \mathcal{P}(A) \cap M(A)$, A being a semi-simple commutative Banach algebra A. Since $\ker P$ is a finite-dimensional ideal in A, by Theorem 5.27, there exists an idempotent $p \in \operatorname{soc} A$ such that $\ker P = pA$. If $x = Pa \in P(A)$ then

$$px = p(Pa) = P(pa) = 0,$$

which implies that $P(A) \subseteq \mathrm{an}\{p\} = (1 - p)A$. From the decomposition $A = P(A) \oplus \ker P = P(A) \oplus pA$ we then conclude that $P(A) = (1 - p)A$.

Conversely, if $p \in \operatorname{soc} A$ and $p = p^2$ the multiplication operator L_p is an idempotent multiplier of A, and by Corollary 5.25 pA is a finite-dimensional ideal of A. If we let $P := I - L_p$ then P is an idempotent element of $M(A)$. Moreover,

$$P(A) = (I - L_p)(A) = (1 - p)A$$

and

$$\dim \ker P = \dim L_p(A) = \dim pA < \infty.$$

Thus $P \in \mathcal{P}(A) \cap M(A)$.

The argument above shows that there is a correspondence

$$P \in \mathcal{P}(A) \cap M(A) \Leftrightarrow p \in \operatorname{soc} A \text{ such that } P(A) = (1 - p)(A).$$

Now let $T_p := T \mid (1-p)(A)$ denote the restriction of T to $(1-p)A$. Combining the result of Theorem 5.114 with the correspondence proved above we then obtain the following formula for $\sigma_{\mathrm{b}}(T)$.

Theorem 5.115. *Let A be a semi-simple commutative Banach algebra with socle $\operatorname{soc} A$. Then*

$$\sigma_{\mathrm{b}}(T) = \bigcap_{p=p^2 \in \operatorname{soc} A} \sigma(T_p).$$

Let us consider again the case $A := L^1(G)$, where G is a locally compact abelian group. If $\delta_0 \in \mathcal{M}(G)$ is the Dirac measure concentrated at the identity and $\nu \in \mathcal{M}(G)$ then $(1-\nu) \star L^1(G) = (\delta_0 - \nu) \star L^1(G)$.

Let us denote by $T_{\mu,\nu}$ the restriction of T_μ on the ideal $(\delta_0 - \nu) \star L^1(G)$. As already observed, for a compact group G we have $\operatorname{soc} L^1(G) = P(G)$, the ideal of all trigonometric polynomial on G. As a particular case of Theorem 5.115 we obtain the following result.

Corollary 5.116. *Let G be a compact abelian group and $\mu \in \mathcal{M}(G)$. For the convolution operator $T_\mu : L^1(G) \to L^1(G)$ we have*

$$\sigma_{\mathrm{b}}(T_\mu) = \bigcap_{\nu=\nu\star\nu \in P(G)} \sigma(T_{\mu,\nu}).$$

The next result shows that every multiplier, as every normal operator on a Hilbert space, satisfies Weyl's theorem.

Theorem 5.117. *Suppose that $T \in M(A)$, A a commutative semi-simple Banach algebra. Then Weyl's theorem holds for T and T^*. If T^* has SVEP, then a-Weil's theorem holds for T and T^*.* ■

Proof By Theorem 4.33 every multiplier on a semi-simple Banach algebra has property (H), so by Corollary 3.97 Weyl's theorem holds for T. The second assertion is clear from Corollary 3.109. ■

We now show that under certain assumptions on the algebra A every multiplier obey's a-Weyl theorem.

Theorem 5.118. *Let A be a commutative semi-simple regular Tauberian Banach algebra and $T \in M(A)$. Then a-Weil's theorem holds for T. If T^* has SVEP then a-Weil's theorem holds for T^*.*

Proof If A is regular and Tauberian then $\sigma_{\mathrm{ap}}(T) = \sigma(T)$ by Corollary 5.88. From this it follows that $\pi^a_{00}(T) = \pi_{00}(T)$.

To show that a-Weyl's theorem holds for T assume that

$$\sigma_{\mathrm{uf}}(T) \cap \pi^a_{00}(T) = \sigma_{\mathrm{uf}}(T) \cap \pi_{00}(T) \neq \varnothing.$$

Let $\lambda \in \sigma_{\mathrm{uf}}(T) \cap \pi^a_{00}(T)$. From the definition of $\pi^a_{00}(T)$ we know that $H_0(\lambda I - T) = \ker(\lambda I - T)$ is finite-dimensional, and hence since λ is an isolated point of $\sigma(T)$, see Theorem 3.96, it follows that $\lambda I - T$ is semi-Fredholm. Since

$p(\lambda I - T) < \infty$ it follows that $\alpha(\lambda I - T) \leq \beta(\lambda I - T)$ by Theorem 3.4. The last inequality obviously implies that $\lambda I - T$ is upper semi-Fredholm, a contradiction.
Therefore $\sigma_{\mathrm{uf}}(T) \cap \pi_{00}^{a}(T) = \varnothing$, so from Theorem 3.105 we may conclude that a-Weyl' s theorem holds for T. The second assertion is clear by Theorem 5.117. ∎

Corollary 5.119. *Let G be a locally compact abelian group and $\mu \in M(G)$. If T_μ is a convolution operator on a group algebra $L^1(G)$ then a-Weil's theorem holds for T_μ.* ∎

Theorem 5.120. *Let A be a Banach algebra with an orthogonal basis and $T \in M(A)$. Then a-Weil's theorem holds for both T and T^*.*

Proof We have by Theorem 4.46 $\sigma_{\mathrm{ap}}(T) = \sigma(T)$, and reasoning as in the proof Theorem 5.118, it follows that a-Weil's theorem holds for both T and T^*. The fact that a-Weyl's theorem holds for T^* follows from Corollary 5.117, since every multiplier of a Banach algebra with orthogonal basis is decomposable, see Proposition 4.8.11 of [**214**]. ∎

Theorem 5.121. *Let A be a $C^\star$ algebra and let $T \in M(A)$. Then a-Weil's theorem holds for T.*

Proof It is well known that a C^* algebra is semi-simple. Also in this case we have $\sigma_{\mathrm{ap}}(T) = \sigma(T)$, see Corollary 5.107, and hence reasoning as in the proof Theorem 5.118 it follows that a-Weil's theorem holds for T. ∎

We conclude this chapter with some remarks about the Browder and the Weyl spectra of a convolution operator $T_{\mu,p}$ on $L^p(G)$ for $1 < p \leq \infty$. Contrary to the case $p = 1$, for $1 < p$ the spectrum of the operator $T_{\mu,p}$ need not to be equal to the spectrum of the measure μ in the Banach algebra $\mathcal{M}(G)$. Moreover, if the group G is *amenable*, see §43 of Bonsall and Duncan [**72**] for the definition, the spectrum of μ is equal to the *order spectrum* of the regular operator $T_{\mu,p}$, $1 \leq p \leq \infty$, the spectrum in the subalgebra of all regular operators, see Arendt [**49**] for definitions.

Let $\mathcal{A} := L^r(X)$ be the Banach algebra of all regular operators on the complex Banach lattice X. The set $K^r(X)$ of all compact regular operators is a closed inessential ideal of $L^r(X)$. The Fredholm spectrum and the Weyl spectrum of an operator $T \in L^r(X)$ in the Banach algebra $\mathcal{A} := L^r(X)$ with respect to $K^r(X)$ are called the *order Fredholm spectrum* (also called *order essential spectrum*) and the *order Weyl spectrum* of T, respectively, see Arendt and Sourour [**50**]. The *order Browder spectrum* of T in $L^r(X)$ with respect to $K^r(X)$ has been investigated by Saxe [**285**], see also Weis [**318**].

Now, if $\mu \in \mathcal{M}(G)$ and J is any closed inessential ideal of the measure algebra $\mathcal{M}(G)$ we can consider the Fredholm spectrum $\sigma_{\mathrm{f}}(\mu)$, the Weyl spectrum $\sigma_{\mathrm{w}}(\mu)$ and the Browder spectrum $\sigma_{\mathrm{b}}(\mu)$ of the element μ in $\mathcal{M}(G)$

with respect to J.

In [**285**] Saxe has shown that if G is a compact (not necessarily abelian) group, $\mu \in \mathcal{M}(G)$, and $1 \leq p \leq \infty$, then for the operator $T_{\mu,p} : L^p(G) \to L^p(G)$ the following equalities hold:

$$\sigma_{\mathrm{f}}(\mu) = \sigma_{\mathrm{w}}(\mu) = \sigma_{\mathrm{b}}(\mu) = \sigma_{\mathrm{of}}(T_{\mu,p}) = \sigma_{\mathrm{ow}}(T_{\mu,p}) = \sigma_{\mathrm{ob}}(T_{\mu,p}).$$

If $p = 1$ or $p = \infty$ this result reduces to

$$\sigma_{\mathrm{f}}(\mu) = \sigma_{\mathrm{w}}(\mu) = \sigma_{\mathrm{b}}(\mu) = \sigma_{\mathrm{f}}(T_{\mu,p}) = \sigma_{\mathrm{w}}(T_{\mu,p}) = \sigma_{\mathrm{b}}(T_{\mu,p}), \tag{156}$$

where the last three sets are, respectively, the ordinary Fredholm spectrum, the Weyl spectrum $\sigma_{\mathrm{w}}(\mu)$, and the Browder spectrum of the operator $T_{\mu,p}$. Moreover, $\sigma_{\mathrm{b}}(T_{\mu,p}) = \sigma_{lf}(T_{\mu,p})$, $T_{\mu,p}$ is Browder if and only if $T_{\mu,p}$ is lower semi-Fredholm [**285**, Theorem 2.2]. Note that the equalities (156) do not hold in general if $1 < p < \infty$, see Saxe [**285**] and Arendt and Sourour [**50**].

Comments An excellent treatment of the abstract Fredholm theory on a Banach algebra may be found in the book by Barnes, Murphy, Smyth, and West [**62**]. In this book Fredholm theory is developed in a primitive Banach algebra, and then extended to the general case, whilst Riesz theory follows as a consequence. Our approach, based on the concept of an inessential ideal, to the abstract Fredholm theory follows some ideas of Aupetit [**53**].

The Fredholm theory in an algebraic setting was pioneered by Barnes [**58**], [**59**] in the context of a semi-prime ring. In particular, Barnes uses the concept of an ideal of finite order to replace the finite dimensionality of the kernel and the finite codimensionality of the range of a Fredholm operator.

The original setting for Fredholm theory on Banach algebras, developed by Barnes [**58**], was a semi-simple Banach algebra and this theory was relative to the socle. The extension of Fredholm theory to semi-prime Banach algebras was also developed by Barnes [**59**]. Since the socle does not always exist in the general case, Smyth [**301**] and Veselić [**310**] developed a Fredholm theory relative to the algebraic kernel of an algebra, precisely relative to the maximal algebraic ideal of the algebra. Note that the algebraic kernel of a semi-prime Banach algebra is the socle. This fact has been first proved by Smyth [**302**] in the case of a semi-simple Banach algebra. The extension of this result to semi-prime Banach algebras, here established in Corollary 5.22, is taken from Giotopoulos and Roumeliotis [**124**]. Theorem 5.18 is modeled after Alexander [**43**], whilst Theorem 5.27 is owed to Barnes [**57**].

The Fredholm theory from semi-simple Banach algebras was carried to general Banach algebras by Smyth [**301**] who first introduced the pre-socle. It is remarkable that also in the context of abstract Fredholm theory an index function may be defined and a punctured neighbourhood may be established, see for instance Barnes, Murphy, Smyth and West [**62**, Theorem F.3.8], Smyth [**301**], Pearlman [**257**].

The study of the properties of the pre-socle of a Banach algebra, developed in the fourth section, as well as the part concerning the Riesz algebras developed in the fifth section, is modeled after Smyth [**302**]. The results related to the socle of the multiplier algebra, in the beginning of the sixth section, elaborate some ideal of Aiena and Laursen [**27**], [**28**]. In particular, Theorem 5.61 is a new abstract version of a result (here given in Corollary 5.62) owed to Kitchen [**187**] and Akemann [**45**]. The subsequent theory on the socle of C^* algebras is taken from Smyth [**302**], for further results see also [**62**].

All the results concerning the Fredholm theory of multipliers of commutative regular Tauberian Banach algebras are owed to Aiena and Laursen[**27**]. This paper extends to regular Tauberian Banach algebras previous results obtained in Aiena [**9**] in the simpler case of a commutative semi-simple Banach algebra having a dense socle. The Aiena and Laursen's work [**27**] was strongly inspired by a previous article of Glicksberg [**126**] who considered a factorization problem of the convolution operator of the group algebra $L^1(G)$, G a locally compact abelian group. At the end of his paper Glicksberg pointed out that his method of proof really applies to any commutative semi-simple regular Tauberian Banach algebra A.

In particular, he established the following three results:

(a) If the maximal ideal space $\Delta(A)$ is connected and A satisfies the condition **(I)** after Theorem 5.93, then a nonzero multiplier T of A has closed range if and only if T is invertible.

(b) If A satisfies condition **(I)** and $T = L_a$ is the multiplication operator on A by the element $a \in A$ then $T(A)$ is closed if and only if T is the product of an idempotent and an invertible multiplier.

(c) If A satisfies condition **(I)** and $T \in M(A)$ then $T^2(A)$ is closed if and only if T is the product of an idempotent and an invertible multiplier.

The main motivating question of the paper of Glicksberg was whether the closedness of the range $T(A)$, for any convolution operator T_μ on $L^1(G)$, is equivalent to the statement that μ is the product of an idempotent and an invertible measure. Obviously, if this factorization holds then the closedness of $T(A)$ easily follows from Theorem 5.76 (see Remark 5.77). But the proof of the converse of this property is decidedly a very complicated task.

The equivalence of these two properties in $L^1(G)$ was finally proved in 1978 in an impressive paper by Host and Parreau [**166**]. Anyway, the general question remains open in the case of a commutative semi-simple regular Tauberian Banach algebra. Clearly, in this general setting Theorem 5.86 is a partial analog of Host and Parreau's result. Moreover, this theorem generalizes a result in Dutta and Tewari [**103**] obtained only for Segal algebras.

Corollary 5.87 was first noted by Ransford in an unpublished manuscript (the original proof given by Ransford may be found in Laursen and Neumann [**214**]). This result was a generalization of the same result for isometric multipliers on a regular Tauberian Banach algebra noted by several authors, Dutta and Tewari [**103**] and Eschmeier, Laursen, and Neuman [**112**].

The description of semi-Fredholm convolution operators established in

Theorem 5.97 is taken from Aiena and Laursen [**27**], whilst the results concerning the Fredholm theory of multipliers of a semi-simple Banach algebra having a regular multiplier algebra $M(A)$ follows some ideas of Barnes [**60**].

The part concerning the Fredholm theory of multipliers of C^* algebras is a sample of results proved by Laursen and Mbektha [**206**] and Aiena [**10**].

The formulas of the Browder spectrum in the special case of multipliers is modeled after Aiena [**11**], see also Barnes, Murphy, Smyth, and West [**62**, §A.6] for convolution operators on (not necessarily commutative) group algebras. That for a multiplier the Browder spectrum and the Weyl spectrum coincide was observed, by using different methods, by Laursen [**204**]. The results on Weyl's theorem and a-Weyl's theorem for multipliers are taken from Aiena and Villafãne [**35**].

CHAPTER 6

Decomposability

The most important concept introduced in this chapter is that of decomposability for operators on Banach spaces. A modern and more complete treatment of this class of operators may be found in the recent book of Laursen and Neumann [**214**]. This book also provides a large variety of examples and applications to several concrete cases. Our study is concentrated to the elements of this theory needed for applications to Riesz theory for multipliers of commutative semi-simple Banach algebras.

Roughly speaking, a bounded operator T on a Banach space X is decomposable if every open covering of $\mathbb{C}$ produces a certain splitting of the spectrum $\sigma(T)$ and of the space X by means of T-invariant closed subspaces. In this splitting X is the sum of two T-invariant closed subspaces. Originally, in the definition of decomposable operators given by Colojoară (see [**83**], it was required that the two subspaces were spectral maximal [**83**]. These definitions are equivalent, as will be shown in the second section. Thanks to work by Albrecht [**38**], and independently by Lange [**195**] and by Nagy [**241**], the definition of decomposability may be described as above.

The first section of this chapter is devoted to some basic properties of spectral maximal subspaces, whilst the second section addresses the notion of decomposability. In this section we shall also introduce the property (β) introduced by Bishop in [**70**] and another decomposition property, the property (δ). The decomposability of an operator T on a Banach space may be described as the conjuction of two weaker properties (β) and (δ). Furthermore, another kind of decomposability is considered, in the third section, the so called super-decomposability and we shall see that, if an operator $T \in L(X)$ is super-decomposable and if there is no non-trivial divisible subspace for T, then the local spectral subspaces are precisely the algebraic spectral subspaces introduced in Chapter 2.

We shall prove in the fourth section that every operator having a totally disconnected spectrum, and in particular every Riesz operator, is super-decomposable. The Riesz operators may be characterized as the decomposable operators for which the local spectral subspaces $X_T(\Omega)$ are finite-dimensional for each closed set Ω which does not contain 0. Section four also addresses the characterization of decomposable weighted unilateral shift operators on $\ell^p(\mathbb{N})$, with $1 \leq p < \infty$. We shall also explore appropriate conditions in the special case where the abstract shift condition $T^\infty(X) = \{0\}$ holds, and under which 0 is a point isolated from the rest of the spectrum of a bounded operator T. As a consequence of these results we shall see that for a weighted right shift operator T the decomposability is equivalent to T being a Riesz operator, and this happens only in the trivial case that T is quasi-nilpotent.

In the fifth section we shall return to consider multipliers of commutative Banach algebras. We shall investigate the various properties of decomposability, previously introduced, within the theory of multipliers of commutative semi-simple Banach algebras. We shall see that in the framework of multiplier theory, the Gelfand theory, spectral theory and harmonic analysis are closely interwinded. An example of this interaction is given by the relationships between the various kind of decomposability of a multiplier T, the hull-kernel continuity of the Gelfand transform $\widehat{T}$ and the property of T to have a natural spectrum. The decomposability of a multiplier is studied in several situations, for instance in the very special case that $M(A)$ is regular then every $T \in M(A)$ is decomposable, whilst for an arbitrary semi-simple commutative Banach algebra this is no longer true in general. We shall see that if $T \in M_0(A)$ or if A possesses a bounded approximate identity then the decomposability of T, or the super-decomposability of T, is equivalent to the formally weaker property (δ).

In the sixth section we shall compare certain local spectral properties of a multiplier T with those of the corresponding multiplication operator L_T defined on the multiplier algebra $M(A)$ or in $M_0(A)$. The main result of this section establishes that if $T \in M_0(A)$ and $\Delta(A)$ is discrete then T is decomposable precisely when T is a Riesz operator, or, equivalently, when T has a countable spectrum. Finally, in the last section we specialize some of these results to the case of group algebras $A = L^1(G)$, where G is a locally compact Abelian group, or G is a compact Abelian group.

1. Spectral maximal subspaces

We first introduce the following notion of spectral maximal subspace which plays a relevant role in invariant subspace theory.

Definition 6.1. *Given $T \in L(X)$, where X is a Banach space, a closed subspace M of X is said to be* spectral maximal *for T if for every T-invariant closed subspace Y of X the inclusion $\sigma(T|Y) \subseteq \sigma(T|M)$ implies $Y \subseteq M$.*

In the following theorem we establish some important basic properties of spectral maximal subspaces. We recall first that if $\lambda \in \rho(T)$ we denote the resolvent $(\lambda I - T)^{-1}$ by $R(\lambda, T)$.

Theorem 6.2. *Let $T \in L(X)$ be a bounded operator on a Banach space X. Then the following assertions hold:*

(i) *Any spectral maximal subspace M of T is T-hyper-invariant, M is invariant for every bounded operator that commutes with T;*

(ii) *For every spectral maximal subspace M of T and $\lambda \in \rho(T)$ we have $R(\lambda, T)(M) \subseteq M$. Moreover, $\sigma(T|M) \subseteq \sigma(T)$;*

(iii) *If M_1, M_2 are spectral maximal subspaces of T, then $M_1 \subseteq M_2$ if and only if $\sigma(T|M_1) \subseteq \sigma(T|M_2)$.*

Proof Assume that $ST = TS$ and choose $\lambda \in \rho(S)$. Denote by M_λ the closed subspace $R(\lambda, S)(M)$. Clearly $R(\lambda, S)T = TR(\lambda, S)$, so that from $T(M) \subseteq M$ we obtain that $T(M_\lambda) \subseteq M_\lambda$. Since $T|M_\lambda$ is the restriction of $R(\lambda, S)(T|M_\lambda)(\lambda I - S)$ to M_λ we also have

$$T|M_\lambda = [(\lambda I - S)|M_\lambda]^{-1}(T|M)((\lambda I - S)|M_\lambda),$$

which implies that $\sigma(T|M_\lambda) = \sigma(T|M)$. Being M spectral maximal for T we then conclude that $M_\lambda \subseteq M$. Therefore, $R(\lambda, S)x \in M$ for all $x \in M$. From the Riesz functional calculus, for any $x \in M$ we have

$$Sx = \frac{1}{2\pi i} \int_\Gamma \lambda \; R(\lambda, S)x \; \mathrm{d}\lambda,$$

where, as usual, Γ is a closed rectifiable Jordan curve which surrounds the spectrum $\sigma(S)$. This evidently implies that $Sx \in M$, since M is a closed linear subspace of X.

(ii) Using the same argument as in the proof of part (i), we obtain $R(\lambda, T)M \subseteq M$ for all $\lambda \in \rho(T)$. From this it follows that $R(\lambda, T|M) = R(\lambda, T|M)$ exists for each $\lambda \in \rho(T)$, so $\rho(T) \subseteq \rho(T|M)$.

(iii) If $M_1 \subseteq M_2$, proceeding as in the proof of part (ii) we obtain that $\sigma(T|M_1) \subseteq \sigma(T|M_2)$. The opposite implication is clear from the definition of spectral maximal subspace. ■

A first example of spectral maximal subspace is provided by the range of spectral projections.

Theorem 6.3. *Let $T \in L(X)$, where X is a Banach space. If $\sigma \subseteq \sigma(T)$ is a spectral set and P_σ is the spectral projection associated with σ, then $P_\sigma(X)$ is a spectral maximal subspace of T.*

Proof If we set $M := P_\sigma(X)$, from functional calculus we know that $\sigma(T|M) = \sigma$. Let Y be a closed linear subspace T-invariant such that $\sigma(T|Y) \subseteq \sigma(T|M) = \sigma$. If $\lambda \notin \sigma$ then $(\lambda I - T)|Y$ admits inverse and for each $y \in Y$ we have

$$\begin{aligned} P_\sigma y &= \frac{1}{2\pi i} \int_\Gamma \lambda \; R(\lambda, T)y \; \mathrm{d}\lambda = \frac{1}{2\pi i} \int_{|\lambda| = \|T\|+1} \lambda \; R(\lambda, T|Y)y \; \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda| = \|T\|+1} \lambda \; R(\lambda, T)y \; \mathrm{d}\lambda = y, \end{aligned}$$

so $y \in P_\sigma(X)$. Therefore, $Y \subseteq P_\sigma(X)$. ■

We recall that if $T \in L(X)$ has the SVEP and $x \in X$ then there exists a maximal analytic extension $R(\cdot, T)x$ to $\rho_T(x)$, see Remark 2.4, part (a). This maximal analytic extension in the sequel is denoted by $\widetilde{f}_x$.

Theorem 6.4. *Suppose that a bounded operator $T \in L(X)$ on a Banach space X has the SVEP. For every closed T-invariant subspace M then $\sigma_T(x) = \sigma_{T|M}(x)$ for all $x \in M$ if and only if $\widetilde{f}_x(\lambda) \in M$ for every $\lambda \in \rho_T(x)$.*

Proof Suppose that for all $x \in M$ we have $\rho_T(x) = \rho_{T|M}(x)$. Let λ be an arbitrary point of $\rho_T(x) = \rho_{T|M}(x)$ and denote by $\widetilde{g}_x : \rho_{T|M}(x) \to M$ the maximal analytic extension of $R(\cdot, T|M)$ to $\rho_{T|M}(x)$. Obviously $\widetilde{f}_x(\lambda) = \widetilde{g}_x(\lambda) \in M$ for every $\lambda \in \rho_T(x)$. Conversely, if $\widetilde{f}_x(\lambda) \in M$ for all $\lambda \in \rho_T(x)$, with $x \in M$, then

$$(\lambda I|M - T|M)\widetilde{f}_x(\lambda) = (\lambda I - T)\widetilde{f}_x(\lambda) = x,$$

thus $\rho_T(x) \subseteq \rho_{T|M}(x)$. Since also the opposite inclusion holds, see Remark 2.4, part (b), the proof is complete. ∎

Choose $x \in X$, in the sequel we shall denote by $\mathcal{F}_x(T)$ the linear closed subspace generated by all the vectors $\widetilde{f}_x(\lambda)$ with $\lambda \in \rho_T(x)$.

Theorem 6.5. *Suppose that $T \in L(X)$, where X is a Banach space, has the SVEP and let M be a spectral maximal subspace for T. Then $\mathcal{F}_x(T) \subseteq M$ for every $x \in M$.*

Proof Let $x \in M$ and let $\lambda_0 \in \rho_T(x)$ be arbitrarily fixed. To prove the theorem it suffices to prove that $\widetilde{f}_x(\lambda) \in M$. If $\lambda_0 \in \rho(T|M)$ then $\widetilde{f}_x(\lambda_0) = R(\lambda_0, T)x \in M$, since M is invariant under $R(\lambda_0, T)$, so it remains to prove that $\widetilde{f}_x(\lambda_0) \in M$ whenever $\lambda_0 \in \sigma(T|M)$. With this aim, suppose that $\lambda_0 \in \rho_T(x) \cap \sigma(T|M)$ and $\widetilde{f}_x(\lambda_0) \notin M$.

Let Z denote the linear subspace spanned by M and $\widetilde{f}_x(\lambda_0)$. Clearly, Z is closed. Let $z := y + \alpha \widetilde{f}_x(\lambda_0) \in Z$, with $y \in M$. From the equality $(\lambda_0 I - T)\widetilde{f}_x(\lambda_0) = x$ we obtain that

$$Tz = (Ty - \alpha x) + \alpha\lambda_0 \widetilde{f}_x(\lambda_0) \in Z.$$

This shows that Z is T-invariant.

Next we want show that $\rho(T|M) \subseteq \rho(T|Z)$. Let $\mu \in \rho(T|M)$ and us consider an element $z = y + \alpha\widetilde{f}_x(\lambda_0) \in Z$ such that $(\mu I - T)z = 0$. We have

$$\begin{aligned} 0 &= (\mu I - T)z = (\mu I - T)y + \alpha(\mu I - T)\widetilde{f}_x(\lambda_0) \\ &= (\mu I - T)y + \alpha[(\mu - \lambda_0)I + (\lambda_0 I - T)]\widetilde{f}_x(\lambda_0) \\ &= (\mu I - T)y + \alpha x + \alpha(\mu - \lambda_0)\widetilde{f}_x(\lambda_0). \end{aligned}$$

Since $\mu \in \rho(T|M)$ and $y \in M$, it follows that $(\mu I - T)y \in M$, so

$$\alpha(\mu - \lambda_0)\widetilde{f}_x(\lambda_0) = -(\mu I - T)y - \alpha x \in M.$$

But $\mu \neq \lambda_0$ and $\widetilde{f}_x(\lambda_0) \neq 0$, since $\widetilde{f}_x(\lambda_0) \notin M$, so $\alpha = 0$ and hence $(\mu I - T)y = 0$. Since, by assumption, $\mu \in \rho(T|M)$ it follows that $y = 0$, and therefore the restriction $(\mu I - T)|Z$ is injective.

To prove the surjectivity of the restriction $(\mu I - T)|Z$ let us consider an arbitrary element $z = y + \alpha\widetilde{f}_x(\lambda_0)$ and define

$$y_0 := R(\lambda, T|M)(y - \frac{\alpha}{\lambda - \lambda_0}x) \in M.$$

Define

$$z_0 := y_0 + \frac{\alpha}{\lambda - \lambda_0}\widetilde{f}_x(\lambda_0) \in Z.$$

An easy estimate yields that $(\mu I - T)z_0 = z$, so $(\mu I - T)|Z$ is surjective and therefore bijective. Hence $\sigma(T|Z) \subseteq \sigma(T|M)$ and this implies, since M is spectral maximal, that $Z \subseteq M$, contradicting the assumption $\widetilde{f}_x(\lambda_0) \notin M$. Therefore $\widetilde{f}_x(\lambda_0) \in M$ for all $\lambda_0 \in \sigma(T|M)$, so the proof is complete. ■

Corollary 6.6. *Suppose that $T \in L(X)$, where X is a Banach space, has the SVEP. If M is a spectral maximal subspace of T then $\sigma_T(x) = \sigma_{T|M}(x)$ for every $x \in M$.*

Proof Combine Theorem 6.5 and Theorem 6.4. ■

The next result shows that if an operator $T \in L(X)$ has the SVEP then every closed local spectral subspace is spectral maximal for T.

Theorem 6.7. *Let $T \in L(X)$, X a Banach space, have the SVEP. Suppose that $\Omega \subseteq \mathbb{C}$ is a closed set for which $X_T(\Omega)$ is closed. Then the analytic subspace $X_T(\Omega)$ is spectral maximal.*

Proof Let Y be a closed T-invariant subspace for which the inclusion $\sigma(T|Y) \subseteq \sigma(T|X_T(\Omega))$ holds. Then by Theorem 2.71 $\sigma(T|Y) \subseteq \Omega$ and therefore, by part (vii) of Theorem 2.6 we obtain $Y \subseteq X_T(\Omega)$. ■

We shall now show that if $T \in L(X)$ has the property (C) then the converse of the preceding risult holds, namely every spectral maximal subspace is a local spectral subspace.

Theorem 6.8. *Suppose that a bounded operator $T \in L(X)$ on a Banach space X has the property (C). Then a closed subspace M is spectral maximal if and only if $M = X_T(\sigma(T|M))$.*

Proof Suppose that M is a spectral maximal subspace of T. By assumption the analytic subspace $X_T(\sigma(T|M))$ is closed, and by Theorem 2.77 the operator T has the SVEP. From Theorem 6.7 it then follows that $X_T(\sigma(T|M))$ is spectral maximal for T. Furthermore, by part (ii) of Theorem 6.2 and Theorem 2.71 we have

$$\sigma(T|X_T(\sigma(T|M)) \subseteq \sigma(T|M) \cap \sigma(T) = \sigma(T|M).$$

Since M is spectral maximal, this entails that $X_T(\sigma(T|M) \subseteq M$. On the other hand, if $x \in M$ by Corollary 6.6 we know that

$$\sigma_T(x) = \sigma_{T|M}(x) \subseteq \sigma(T|M),$$

so $x \in X_T(\sigma(T|M))$. Hence $M = X_T(\sigma(T|M))$.

The converse follows by Theorem 6.7. ∎

2. Decomposable operators on Banach spaces

In the sequel we shall denote by $\mathcal{H}(\mathcal{U}, X)$, where X is a Banach space, the set of all analytic functions $f : \mathcal{U} \to X$ defined on the open set $\mathcal{U} \subseteq \mathbb{C}$. It is known that $\mathcal{H}(\mathcal{U}, X)$ is a Fréchet space with respect to the pointwise vector space operations and the topology of locally uniform convergence. In this topology, a sequence $(f_n) \subset \mathcal{H}(\mathcal{U}, X)$ converges to f if and only if (f_n) converges uniformly to f on every compact set $K \subset \mathcal{U}$.

Given a bounded linear operator T acting from X into Y, X and Y Banach spaces, and an open disc $\mathbb{D}(\lambda_0, \varepsilon)$, let

$$T^\dagger : \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X) \to \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), Y),$$

be the mapping defined by

$$(T^\dagger f)(\lambda) := T(f(\lambda)) \quad \text{for every } \lambda \in \mathbb{D}(\lambda_0, \varepsilon). \tag{157}$$

Evidently $T^\dagger$ is a continuous linear mapping. The operator $T^\dagger$ may be easily represented as follows. Let $f \in \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X)$ be represented by a power series

$$f(\lambda) := \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n x_n, \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon),$$

where $(x_n) \subset X$. Because the series is locally uniform convergent, from the continuity of $T^\dagger$ we obtain

$$(T^\dagger f)(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T(x_n) \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon).$$

The following result shows that the mapping $T^\dagger$ preserves some important properties of T.

Theorem 6.9. *For an arbitrary open disc $\mathbb{D}(\lambda_0, \varepsilon)$ of $\mathbb{C}$ we have:*

(i) *If the bounded operator $T \in L(X, Y)$, X and Y Banach spaces, is surjective then the induced operator $T^\dagger : \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X) \to \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), Y)$ is surjective and open.*

(ii) *If Z is a closed linear subspace of the Banach space X, then the space $\mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X/Z)$ and the quotient space $\mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X)/\mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), Z)$ are topologically isomorphic.*

Proof (i) If $g \in \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X)$ has the representation

$$g(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n y_n, \quad \lambda \in \mathbb{D}(\lambda_0, \varepsilon),$$

the radius of convergence of this series is at least ε. By the open mapping theorem T is open, thus there is a constant $\delta > 0$ such that for every $y \in Y$

we can find $x \in X$ for which $Tx = y$ and $\|x\| \le \delta\|y\|$. Choose $x_n \in X$ such that $Tx_n = y_n$ and $\|x_n\| \le \|y_n\|$. From the estimate

$$\limsup_{n\to\infty} \|x_n\|^{1/n} \le \limsup_{n\to\infty} \delta^{1/n}\|y_n\|^{1/n} = \limsup_{n\to\infty} \|y_n\|^{1/n} \le \varepsilon,$$

we infer that the series

$$f(\lambda) := \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n x_n$$

converges for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$, so $f \in \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), X)$. Clearly, $T^\dagger(f) = g$, hence $T^\dagger$ is surjective, and consequently by the open mapping theorem is open.

(ii) If $T : X \to X/Z$ is the canonical quotient map then $\ker T^\dagger = \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon), Z)$, so $T^\dagger$ is a topological isomorphism. ■

Theorem 6.10. *Suppose that the operator $T \in L(X)$ on the Banach space X has the SVEP and that Ω is a closed subset of $\mathbb{C}$ for which $X_T(\Omega)$ is closed. If $\widetilde{T}$ is the operator induced by T on the quotient $X/X_T(\Omega)$, then $\widetilde{T}$ has the SVEP and*

$$\sigma_T(x) = \sigma_{\widetilde{T}}(x) \quad \textit{for all } x \in X_T(\Omega). \tag{158}$$

Proof Let $Q : X \to X/X_T(\Omega)$ denote the canonical quotient mapping and let $g : \mathbb{D} \to X_T(\Omega)$ be an analytic function defined on some open disc $\mathbb{D}$ for which $(\lambda I - \widetilde{T})g(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. By part (i) of Theorem 6.9 we can find an analytic function $f : \mathbb{D} \to X$ such that $g = Q \circ f$ and hence

$$Q(\lambda I - T)f(\lambda) = (\lambda I - \widetilde{T})g(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}.$$

From this we obtain that $(\lambda I - T)f(\lambda) \in X_T(\Omega)$ for all $\lambda \in \mathbb{D}$, and hence by part (iv) of Theorem 2.6 $f(\lambda) \in X_T(\Omega)$ for all $\lambda \in \mathbb{D} \cap \Omega$. By Theorem 2.71 we know that $\sigma(T|X_T(\Omega)) \subseteq \Omega$, so we may define an analytic function $h : \mathbb{D} \setminus \Omega \to X_T(\Omega)$ by the assignment

$$h(\lambda) := (\lambda I - \widetilde{T})^{-1}(\lambda I - T)f(\lambda) \in X_T(\Omega) \quad \text{for all } \lambda \in \mathbb{D} \setminus \Omega.$$

Now, $(\lambda I - T)(f(\lambda) - h(\lambda)) = 0$ for all $\lambda \in \mathbb{D} \setminus \Omega$, so that the SVEP of T implies that

$$f(\lambda) = h(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D} \setminus \Omega.$$

Thus $f(\lambda) \in X_T(\Omega)$ and hence $g(\lambda) = Q(f(\lambda)) = 0$ for every $\lambda \in \mathbb{D}$. This shows that $\widetilde{T}$ has the SVEP.

To prove the identity (6.10), given $x \in X_T(\Omega)$ and $\lambda \in \rho_T(x)$, let us consider an analytic function $f : \mathbb{D} \to X$ such that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{D}$. Since $\widetilde{T}$ has the SVEP, from the identity $(\lambda I - \widetilde{T})Q(f(\lambda)) = 0$ for all $\lambda \in \mathbb{D}$, we deduce that $Q(f(\lambda)) = 0$ and hence $f(\lambda) \in X_T(\Omega)$ for all $\lambda \in \mathbb{D}$. Therefore $\lambda \in \rho_{T|X_T(\Omega)}(x)$, so $\rho_T(x)(x) \subseteq \rho_{T|X_T(\Omega)}(x)$. The opposite inclusion is obvious, so the equality $\sigma_T(x) = \sigma_{T|X_T(\Omega)}(x)$ is proved. ■

Lemma 6.11. *Let $\mathcal{U}$ be a connected open subset of $\mathbb{C}$ and let Y be a closed subspace of the Banach space X. If $f \in \mathcal{H}(\mathcal{U}, X)$ verifies the property $f(\Omega) \subseteq Y$ for some non-empty open set $\Omega \subseteq \mathcal{U}$, then $f(\mathcal{U}) \subseteq Y$. Moreover, $f \in \mathcal{H}(\mathcal{U}, Y)$.*

Proof For every $n \in \mathbb{N}$, let us consider the set

$$\Lambda_n := \{\lambda \in \mathcal{U} : f^{(n)}(\lambda) \in Y\}.$$

Clearly, Λ_n is closed and hence also $\Lambda := \cap_{n\in\mathbb{N}}\Lambda_n$ is closed. Moreover, Λ is also open and non-empty, since it contains Ω. Therefore $\mathcal{U}$ being connected, $\Lambda = \mathcal{U}$ and this implies that $f(\lambda) \in Y$ for every $\lambda \in \mathcal{U}$. ∎

For every $T \in L(X)$, X a Banach space, let $\rho_\infty(T)$ denote the unbounded component of $\rho(T)$. Let $\sigma_\infty(T)$ denote the *full spectrum* of T, $\sigma_\infty(T) := \mathbb{C} \setminus \rho_\infty(T)$. It is obvious that the full spectrum is the union of the spectrum $\sigma(T)$ and all bounded components of the resolvent set.

For each closed T-invariant subspace Y of X let $T^Y : X/Y \to X/Y$ denote the induced quotient operator, defined for every $\widetilde{x} := x + Y \in X/Y$ by

$$T^Y(\widetilde{x}) := \widetilde{Tx} \quad \text{with } x \in \widetilde{x}.$$

Theorem 6.12. *For a bounded operator $T \in L(X)$ on a Banach space X the following statements hold:*

(i) *If Y is a closed T-invariant subspace of X then*

$$\sigma(T^Y) \subseteq \sigma(T) \cup \sigma(T|Y) \subseteq \sigma_\infty(T).$$

Moreover, Y is invariant under $R(\lambda, T)$ and

$$R(\lambda, T|Y) = R(\lambda, T)|Y \quad \text{for every } \lambda \in \rho_\infty(T).$$

(ii) *If Y and Z are closed T-invariant subspaces for which $X = Y + Z$, then $\sigma(T^Z) \subseteq \sigma_\infty(T|Y)$.*

Proof (i) Let $\lambda \in \rho(T) \cap \rho(T|Y)$ and set $\widetilde{x} := x + Y$ for every $x \in X$. Suppose that $(\lambda I^Y - T^Y)\widetilde{x} = 0$. Then $(\lambda I - T)x \in Y$ and hence

$$x = R(\lambda, T)(\lambda I - T)x = R(\lambda, T|Y)(\lambda I - T)x \in Y.$$

Therefore $\widetilde{x} = 0$, so that $\lambda I^Y - T^Y$ is injective.

Let us consider an arbitrary element $\widetilde{x} \in X/Y$. For every $y \in Y$ there exist two uniquely determined elements $u \in Y$ and $v \in X$ such that

$$(\lambda I - T)u = (\lambda I|Y - T|Y)u = y \quad \text{and} \quad (\lambda I - T)v = x.$$

Set $\widetilde{z} := \widetilde{u} + \widetilde{v}$. Clearly $(\lambda I^Y - T^Y)\widetilde{z} = \widetilde{x}$, so the operator $(\lambda I^Y - T^Y)$ is surjective. Therefore the inclusion $\sigma(T^Y) \subseteq \sigma(T) \cup \sigma(T|Y)$ is established.

To show the second inclusion of (i), let us consider $|\lambda| > \|T\|$. It is known that the resolvent $R(\lambda, T)$ admits the representation

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} \quad \text{for every } |\lambda| > \|T\|,$$

see [**159**, Theorem 44.1], from where the inclusion

$$R(\lambda, T)(Y) \subseteq Y \quad \text{for every} \quad |\lambda| > \|T\|$$

immediately follows. Moreover, $\rho_\infty(T)$ is connected and

$$\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subseteq \rho_\infty(T),$$

so by Lemma 6.11 we can conclude that Y is invariant under $R(\lambda, T)$ and hence

$$R(\lambda, T|Y) = R(\lambda, T)|Y \quad \text{ for every } \quad \lambda \in \rho_\infty(T).$$

(ii) Given $x \in X$, let $[x]_Z := x + Z$ denote the relative equivalence class modulo Z. Let $\varPhi : Y \to X/Z$ be the canonical surjection defined by $\varPhi(y) := [y]_Z$ for every $y \in Y$. Clearly $\ker \varPhi = Y \cap Z$, so the canonical injection

$$\widehat{\varPhi} : Y/\ker \varPhi = Y/(Y \cap Z) \to X/Z$$

is an isomorphism. Consider $\Psi := \widehat{\varPhi}^{-1}$, the inverse of $\widehat{\varPhi}$, and let

$$S := (T|Y)^{Y/(Y\cap Z)}$$

be the quotient map induced by $T|Y$ on $Y/(Y \cap Z)$. It is easy to verify that $S\Psi = \Psi T^Z$, and hence $\sigma(T^Z) = \sigma(S)$. From the part (i) we then conclude that

$$\sigma(S) \subseteq \sigma(T|Y) \cup \sigma(T|(Y \cap Z)) \subseteq \sigma_\infty(T|Y),$$

so the proof is complete. ∎

Theorem 6.13. *Let X_1 and X_2 be closed subspaces of X such that $X = X_1 + X_2$ and let $f \in \mathcal{H}(\mathbb{D}_{\lambda_0}, X)$ be analytic on the open disc $\mathbb{D}_{\lambda_0}$ centred at λ_0. Then there exist two analytic functions $f_k \in \mathcal{H}(\mathbb{D}_{\lambda_0}, X_k)$, $k = 1, 2$, such that*

$$f(\lambda) = f_1(\lambda) + f_2(\lambda) \quad \textit{for every } \lambda \in \mathbb{D}_{\lambda_0}.$$

Proof The assertion will be established if we prove that the linear mapping

$$\Phi : \mathcal{H}(\mathbb{D}_{\lambda_0}, X_1) \times \mathcal{H}(\mathbb{D}_{\lambda_0}, X_2) \to \mathcal{H}(\mathbb{D}_{\lambda_0}, X)$$

defined by the assignment

$$\Phi(f_1, f_2)(\lambda) := f_1(\lambda) + f(\lambda_2), \quad f_i \in \mathcal{H}(\mathbb{D}_{\lambda_0}, X_i) \quad \text{and} \quad \lambda \in \mathbb{D}_{\lambda_0},$$

is continuous, open, and onto.

Let Ψ be the mapping defined by

$$\Psi(f_1, f_2)(\lambda) := (f_1(\lambda), f_2(\lambda)), \quad f_i \in \mathcal{H}(\mathbb{D}_{\lambda_0}, X_i) \quad \text{and} \quad \lambda \in \mathbb{D}_{\lambda_0}.$$

Clearly Ψ is a topological linear isomorphism of the product $\mathcal{H}(\mathbb{D}_{\lambda_0}, X_1) \times \mathcal{H}(\mathbb{D}_{\lambda_0}, X_2)$ onto $\mathcal{H}(\mathbb{D}_{\lambda_0}, X_1 \times X_2)$. Define $T : X_1 \times X_2 \to X$ by $T(x_1, x_2) = x_1 + x_2$, with $x_i \in X_i$. Since T is linear and onto, the induced mapping $T^\dagger : \mathcal{H}(\mathbb{D}_{\lambda_0}, X_1 \times X_2) \to \mathcal{H}(\mathbb{D}_{\lambda_0}, X)$ defined as in (157) is continuous, open, and onto. Since $\Phi = T^\dagger \circ \Psi$ then also Ψ is continuous, open and onto. ∎

The previous result, owed to Gleason [**125**], holds also if $f \in \mathcal{H}(\mathcal{U}, X)$ where $\mathcal{U}$ is an arbitrary open set. The proof of this general case is considerably more involved (see, for a proof, Laursen and Neumann [**214**, Proposition 2.1.14]).

We define now an important property, introduced first by Bishop [**70**], which plays a central role in local spectral theory.

Definition 6.14. *An operator $T \in L(X)$, X a Banach space, is said to have Bishop's property (β) if, for every open set $\mathcal{U} \subseteq \mathbb{C}$ and every sequence $(f_n) \subset \mathcal{H}(\mathcal{U}, X)$ for which $(\lambda I - T)f_n(\lambda)$ converges to 0 uniformly on every compact subset of $\mathcal{U}$, then also $f_n \to 0$ in $\mathcal{H}(\mathcal{U}, X)$.*

For every $T \in L(X)$, X a Banach space, and every open set $\mathcal{U} \subseteq \mathbb{C}$, let us consider the operator $T_{\mathcal{U}} : H(\mathcal{U}, X) \to H(\mathcal{U}, X)$ by

$$(T_{\mathcal{U}}f)(\lambda) := (\lambda I - T)f(\lambda) \quad \text{for all } f \in T_{\mathcal{U}} \text{ and } \lambda \in \mathcal{U}. \tag{159}$$

It is easy to verify that $T_{\mathcal{U}}$ is continuous. The property (β) may be characterized as follows.

Theorem 6.15. *If $T \in L(X)$, where X is a Banach space, then T has the property (β) if and only if $T_{\mathcal{U}}$ is injective and has closed range for all open sets $\mathcal{U} \subseteq \mathbb{C}$.*

Proof Suppose that T has the property (β) and let $\mathcal{U}$ be any open subset of $\mathbb{C}$. By considering the constant sequences in $H(\mathcal{U}, X)$ we easily deduce that $T_{\mathcal{U}}$ is injective.

To show that $T_{\mathcal{U}}$ has a closed range suppose that $T_{\mathcal{U}}f_n \to g$ as $n \to \infty$. We show first that f_n is a Cauchy sequence. To do this suppose that f_n is not a Cauchy sequence. Then one can construct a subsequence f_{n_k} such that the sequence defined by $g_k := f_{n_{k+1}} - f_{n_k}$ does not converge to 0 in $H(\mathcal{U}, X)$. On the other hand, $T_{\mathcal{U}}g_k \to 0$ as $k \to \infty$, therefore property (β) entails also that (g_k) converges to 0, a contradiction. Thus f_n is a Cauchy sequence, and since $H(\mathcal{U}, X)$ is a Fréchet space it follows that (f_n) converges to some $f \in H(\mathcal{U}, X)$. The continuity of $T_{\mathcal{U}}$ yields that $T_{\mathcal{U}}f_n \to T_{\mathcal{U}}f$ as $n \to \infty$, so $g = T_{\mathcal{U}}f$. This shows that $T_{\mathcal{U}}$ has closed range.

Conversely, if $T_{\mathcal{U}}$ is injective and has closed range for all open set $\mathcal{U} \subseteq \mathbb{C}$, then, given an open set $\mathcal{U}$, the operator $T_{\mathcal{U}}$ admits a continuous inverse, say $S_{\mathcal{U}}$, on its range. If $T_{\mathcal{U}}f_n \to 0$ in $H(\mathcal{U}, X)$ then $f_n = S_{\mathcal{U}}T_{\mathcal{U}}f_n \to 0$, as $n \to \infty$. This holds for every open set $\mathcal{U} \subseteq \mathbb{C}$, hence T has the property (β). ∎

We now introduce an important class of operators on Banach spaces which admits a rich spectral theory and contains many important classes of operators.

Definition 6.16. *Given a Banach space X, an operator $T \in L(X)$ is said to be* decomposable *if, for any open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2 = X$ and $\sigma(T|Y_k) \subseteq \mathcal{U}_k$ for $k = 1, 2$.*

Theorem 6.17. *Let $T \in L(X)$, X a Banach space, be decomposable. Then T has the property (β).*

Proof Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set and suppose that the sequence $(f_n) \subset \mathcal{H}(\mathcal{U}, X)$ is such that $(\lambda I - T)f_n(\lambda)$ converges to 0 uniformly on every compact subset of $\mathcal{U}$. To establish property (β) it suffices to show that (f_n) converges to 0 uniformly on every closed disc contained in $\mathcal{U}$.

Let $\lambda_0 \in \mathcal{U}$ and let $\varepsilon > 0$ such that the closed disc $\mathbf{D} := \mathbf{D}(\lambda_0, \varepsilon)$ centred at λ_0 and radius ε is contained in $\mathcal{U}$. Choose $\varepsilon < \varepsilon_1 < \varepsilon_2$ such that $\mathbf{D}_2 := \mathbf{D}(\lambda_0, \varepsilon_2) \subset \mathcal{U}$. Consider the open covering $\{\mathbb{D}(\lambda_0, \varepsilon_1), \mathbb{C} \setminus \mathbf{D}\}$ of $\mathbb{C}$. Since T is decomposable there exist two T-invariant closed subspaces Y_1, Y_2 such that

$$X = Y_1 + Y_2, \quad \sigma(T|Y_1) \subset \mathbb{D}(\lambda_0, \varepsilon_1) \quad \text{and} \quad \sigma(T|Y_2) \cap \mathbf{D} = \varnothing.$$

According with Theorem 6.13 let $(f_n^k) \subset \mathcal{H}(\mathcal{U}, Y_k)$, for $k = 1, 2$, be two sequences such that

$$f_n(\lambda) = f_n^1(\lambda) + f_n^2(\lambda) \quad \text{for every } \lambda \in \mathcal{U} \quad \text{and} \quad n \in \mathbb{N}.$$

Let us consider the operator T^{Y_2} induced by T on the quotient space X/Y_2 and let $\pi : X \to X/Y_2$ denote the canonical quotient map. Clearly, $\pi \circ f_n = \pi \circ f_n^1 \in \mathcal{H}(\mathcal{U}, X/Y_2)$ and furthermore, as $n \to \infty$, the sequence

$$(\lambda I^{Y_2} - T^{Y_2})\pi(f_n^1(\lambda)) = \pi((\lambda I^{Y_2} - T)f_n(\lambda)) \tag{160}$$

converges to 0 uniformly on every compact subset of $\mathcal{U}$. It then follows by Theorem 6.12 that for every $\lambda \in \partial\mathbf{D}_2$, $\partial\mathbf{D}_2$ the boundary of $\mathbf{D}_2$, the operator $\lambda I^{Y_2} - T^{Y_2}$ is invertible so that

$$K := \sup_{\lambda \in \partial D_2} \|R(\lambda, T^{Y_2})\| < \infty.$$

By (160) we then obtain the following estimate

$$\|\pi(f_n^1(\lambda))\| = \|R(\lambda, T^{Y_2})(\lambda I^{Y_2} - T^{Y_2})\pi(f_n^1)\| \leq K\|(\lambda I^{Y_2} - T^{Y_2})\pi(f_n^1(\lambda))\|.$$

This estimate entails that the sequence $(\pi \circ f_n^1) \in \mathcal{H}(\mathcal{U}, X/Y_1)$ converges uniformly on the compact set $\partial\mathbf{D}_2$. By the maximum modulus principle the sequence $(\pi \circ f_n^1)$ then converges uniformly on $\mathbf{D}_2$.

Now, by Theorem 6.9 the space $\mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon_2), X/Y_2)$ is topologically isomorphic to the quotient space $\mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon_2), X)/\mathcal{H}(\mathbf{D}_2, Y_2)$. Hence there exists a sequence $(g_n) \subset \mathcal{H}(\mathbb{D}(\lambda_0, \varepsilon_2), Y_2)$ such that $f_n^1 + g_n$ converges a 0 uniformly on the closed disc $\mathbf{D}(\lambda_0, \varepsilon_1)$. From that we deduce that the sequence $(\lambda I - T)(f_n^1(\lambda) + g_n(\lambda))$ converges to 0 uniformly on $\mathbf{D}(\lambda_0, \varepsilon_1)$, and hence

$$(\lambda I - T)[f_n^2(\lambda) - g_n(\lambda)] = (\lambda I - T)[f_n(\lambda) - (\lambda I - T)(f_n^1(\lambda) + g_n(\lambda)] \tag{161}$$

converges to 0 uniformly on $\mathbf{D}(\lambda_0, \varepsilon_1)$. The inclusion $\mathbf{D} \subseteq \rho(T|Y_2)$ then entails that

$$L := \sup_{\lambda \in D} \|R(\lambda, T|Y_2)\| < \infty.$$

Since $f_n^2(\lambda) - g_n(\lambda) \in Y_2$ for all $\lambda \in \mathbf{D} \subset \mathbf{D}(\lambda_0, \varepsilon_1)$, the equality (161) implies that the estimate

$$\begin{aligned} \|f_n^2(\lambda) - g_n(\lambda)\| &= \|R(\lambda, T|Y_2)(\lambda I - T)(f_n^2(\lambda) - g_n(\lambda))\| \\ &\leq L\|(\lambda I - T)(f_n^2(\lambda) - g_n(\lambda))\| \end{aligned}$$

holds for every $\lambda \in \mathbf{D}$. From this we infer that $\|f_n^2(\lambda) - g_n(\lambda)\| \to 0$ uniformly on $\mathbf{D}$. Finally, since $(\lambda I - T)(f_n^1(\lambda) + g_n(\lambda))$ converges to 0 uniformly on $\mathbf{D}(\lambda_0, \varepsilon_1)$ we may conclude that the sequence

$$f_n(\lambda) = (f_n^1(\lambda) + g_n(\lambda)) + (f_n^2(\lambda) - g_n(\lambda))$$

converges to 0 uniformly on $\mathbf{D}$. ∎

Given an operator $T \in L(X)$, X a Banach space, and a non-empty closed set $\Omega \subseteq \mathbb{C}$ let us denote by $Z(T, \Omega)$ the set of all $x \in X$ such that for every compact $K \subset \mathbb{C} \setminus \Omega$ and $\varepsilon > 0$ there exists an open set $\mathcal{U}$ and an analytic function $f : \mathcal{U} \to X$ such that

$$\|x - (\lambda I - T)f(\lambda)\| < \varepsilon \quad \text{for every } \lambda \in K.$$

Clearly, $Z(T, \Omega)$ is a closed T-invariant subspace of X.

It is easily seen that a bounded operator T on a Banach space X has the SVEP if and only if the operator $T\mathcal{U}$, defined on the Fréchet space $H(\mathcal{U}, X)$ as in (159), is injective. From Theorem 6.15 the property (β) then implies the SVEP. The next result shows that actually we have much more.

Theorem 6.18. *Suppose that $T \in L(X)$, X a Banach space, has the property (β). Then T has the property (C). Moreover,*

$$X_T(\Omega) = Z(T, \Omega) \quad \textit{for every closed } \Omega \subseteq \mathbb{C}. \tag{162}$$

Proof Since $Z(T, \Omega)$ is always closed, in order to prove that the operator T has the property (C) it suffices to establish the equality $X_T(\Omega) = Z(T, \Omega)$ for every closed $\Omega \subseteq \mathbb{C}$. Clearly $X_T(\Omega) \subseteq Z(T, \Omega)$, so we need only to prove the opposite inclusion.

Let x be an arbitrary element of $Z(T, \Omega)$. Consider $\lambda_0 \in \mathbb{C} \setminus \Omega$ and let us denote by $\mathcal{U}$ an open neighbourhood of λ_0 such that $\overline{\mathcal{U}}$ is a compact subset of $\mathbb{C} \setminus \Omega$. Since $x \in Z(T, \Omega)$ there exists a sequence of locally analytic X-valued functions (f_n), defined in some neighbourhoods of $\overline{\mathcal{U}}$, such that

$$\|x - (\lambda I - T)f_n(\lambda)\| < \frac{1}{n} \quad \text{for every } \lambda \in \mathcal{U}. \tag{163}$$

For every $n, j = 1, 2, \dots$ we have

$$\|(\lambda I - T)(f_n(\lambda) - f_j(\lambda))\| < \frac{1}{n} + \frac{1}{j},$$

from which we infer that the first member of the inequality above converges to 0 uniformly on $\mathcal{U}$. Since T has the property (β) then $f_n - f_j \to 0$ in $\mathcal{H}(\mathcal{U}, X)$, as $n, j \to \infty$. From this it follows that (f_n) is a Cauchy sequence in $\mathcal{H}(\mathcal{U}, X)$ and converges uniformly on every compact $K \subseteq \mathcal{U}$ to some $f \in$

$\mathcal{H}(\mathcal{U}, X)$. From the inequality (163) we then obtain that $(\lambda I - T)f(\lambda) = x$ for every $\lambda \in \mathcal{U}$. Thus $\lambda_0 \in \rho_T(x)$ and hence $\sigma_T(x) \subseteq \Omega$. This means that $x \in X_T(\Omega)$, so the equality (162) is proved. ∎

By Theorem 2.77 and Theorem 6.18 we then have the following implications:

$$\text{property } (\beta) \Rightarrow \text{property } (C) \Rightarrow \text{SVEP},$$

and the operator in the Example 2.32 shows that the SVEP is strictly weaker than the property (C). Also property (C) is strictly weaker than the property (β). In fact, T.L. Miller and V.G. Miller have given an example of an operator which has the property (C) and does not have the property (β). Another example in the context of unilateral weighted shifts may be found in Laursen and Neumann [**214**, Section 1.6]. Other examples of operators having property (β) are *hyponormal* operators, *M-hyponormal* operators on complex Hilbert spaces and, more generally, subscalar operators on Banach spaces, see Putinar [**268**], Section 6.4 of Eschmeier and Putinar [**113**], or Laursen and Neumann [**214**, p. 144].

Theorem 6.19. *Let $T \in L(X)$, X a Banach space. Then the following statements are equivalent:*

(i) *T is decomposable;*

(ii) *T has the property (C) and the identity $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$ holds for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$;*

(iii) *For every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$ there exist two spectral maximal subspaces Y_1, Y_2 such that*

$$X = Y_1 + Y_2 \quad \text{and} \quad \sigma(T|Y_k) \subseteq \mathcal{U}_k \quad \text{for } k = 1, 2.$$

Proof (i) ⇒ (ii) By Theorem 6.17 and Theorem 6.18 we need only to prove that the decomposition $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$ holds for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$.

Let Y_1 and Y_2 be two closed T-invariant subspaces for which $X = Y_1 + Y_2$ and $\sigma(T|Y_k) \subseteq \mathcal{U}_k \subseteq \overline{\mathcal{U}_k}$. By part (vi) of Theorem 2.6 we have, for $k = 1, 2$, $Y_k \subseteq X_T(\overline{\mathcal{U}_k})$, and hence

$$X = Y_1 + Y_2 \subseteq X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2}) \subseteq X,$$

from which we conclude that $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$.

(ii) ⇒ (iii) Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$. Let $\{\mathcal{V}_1, \mathcal{V}_2\}$ be another open covering of $\mathbb{C}$ such that $\overline{\mathcal{V}_1} \subseteq \mathcal{U}_1$ and $\overline{\mathcal{V}_2} \subseteq \mathcal{U}_2$. If we set $Y_k := X_T(\overline{\mathcal{V}_k})$, $k = 1, 2$, then $X = Y_1 + Y_2$. By the property (C) the local spectral subspaces Y_k are closed and, by Theorem 6.7, are also spectral maximal for T. Moreover, by Theorem 2.71

$$\sigma(T|Y_k) \subseteq \overline{\mathcal{V}_k} \cap \sigma(T) \subseteq \overline{\mathcal{V}_k} \subseteq \mathcal{U}_k, \qquad k = 1, 2.$$

Since $X = Y_1 + Y_2$ it follows that T has the required decomposition.

(iii)⇒ (i) Obvious. ∎

Another important decomposition property is given by the following one.

Definition 6.20. *An operator $T \in L(X)$, X a Banach space, is said to have the* decomposition property (δ) *if for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ we have $X = \mathcal{X}_T(\overline{\mathcal{U}_1}) + \mathcal{X}_T(\overline{\mathcal{U}_2})$.*

If T has the SVEP then $\mathcal{X}(\overline{\mathcal{U}_k}) = X_T(\overline{\mathcal{U}_k})$, $k = 1, 2$, so the property (δ) is equivalent to saying that the decomposition $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$ holds for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$.

The following result shows that the decomposability of an operator may be described as the conjuction of the two weaker conditions (β) and (δ) or of the two weaker properties (C) and (δ).

Theorem 6.21. *For an operator $T \in L(X)$, X a Banach space, the following assertions are equivalent:*

(i) *T is decomposable;*

(ii) *T has both properties (β) and (δ);*

(iii) *T has both the properties (C) and (δ).*

Proof (i) $\Rightarrow$ (ii) If T is decomposable then by Theorem 6.17 T has the property (β) and therefore has the SVEP. Moreover, from the definition of decomposability, for any arbitrary open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ we have $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$, thus T has also property (δ).

(ii) $\Rightarrow$ (iii) It is obvious by Theorem 6.18.

(iii) $\Rightarrow$ (i) The property (C) entails that T has the SVEP, by Theorem 2.77. The property (δ)then implies that the decomposition $X = X_T(\overline{\mathcal{U}_1}) + X_T(\overline{\mathcal{U}_2})$ holds for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$, and hence by Theorem 6.19, T is decomposable. ∎

One of the deepest results of local spectral theory is that the properties (β) and (δ) are duals of each other, in the sense that an operator $T \in L(X)$, X a Banach space, has one of the properties (β) or (δ) precisely when the dual operator $T^\star$ has the other. This basic result has been established very recently by Albrecht and Eschmeier [**42**]. Moreover, the work of Albrecht and Eschmeier gives two important characterizations of properties (β) and (δ): the property (β) characterizes the restrictions of decomposable operators to closed invariant subspaces, whilst the property (δ) characterizes the quotients of decomposable operators by closed invariant subspaces, see also Chapter 2 of Laursen and Neumann [**214**]. The proof of this complete duality is beyond the scope of this book since it is based on the construction of analytic functional models, for arbitrary operators defined on complex Banach spaces, in terms of certain multiplication operators defined on vector valued Sobolev type of spaces. A detailed discussion of this duality theory, together some interesting applications to the invariant problem, may be found in Chapter 2 of the monograph of Laursen and Neumann [**214**], see also the Laursen's lectures in [**17**]. Furthermore, in section 1.6 of [**214**] one can also find enlighting examples of operators which have only someone of the properties (C), (β), and (δ), but not the others.

Combining the mentioned duality theory on the properties (β) and (δ) and Theorem 6.21 we obtain the following result:

Theorem 6.22. *Let $T \in L(X)$ be a bounded operator on a Banach space X. The following statements hold:*

(i) *If T has the property (δ) then T^* has the SVEP;*

(ii) *If T is decomposable then both T and T^* have the SVEP;*

(iii) *T is decomposable if and only if T^* is decomposable.* ∎

We can now extend to certain spectra of decomposable operators on Banach spaces some classical results valid for normal operators on Hilbert spaces.

Corollary 6.23. *Suppose that a bounded operator $T \in \mathcal{B}(X)$, where X is a Banach space, is decomposable. Then*

$$\sigma_{\mathrm{es}}(T) = \sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{lf}}(T) = \sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T).$$

Proof It is immediate from Corollary 3.53 and Theorem 6.22. ∎

3. Super-decomposable operators

In this section we shall introduce a class of decomposable operators for which it is possible to give a very useful description of the spectral maximal subspaces .

Definition 6.24. *An operator $T \in L(X)$, X a Banach space, is said to be* super-decomposable *if for any open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$ there exists some operator $R \in L(X)$ such that*

$$RT = TR, \ \sigma(T|\overline{R(X)}) \subseteq \mathcal{U}_1 \quad \textit{and} \quad \sigma(T|\overline{(I-R)(X)}) \subseteq \mathcal{U}_2.$$

Note that the condition $TR = RT$ in the definition above ensures that $\overline{R(X)}$ and $\overline{(I-R)(X)}$ are T-invariant subspaces of X. Trivially, taking $Y_1 := \overline{R(X)}$ and $Y_2 := \overline{(I-R)(X)}$ we have $X = Y_1 + Y_2$, so every super-decomposable is decomposable. Generally the converse is not true. Indeed, every super-decomposable operator T is *strongly decomposable*, namely T has the property that every restriction of it to a local spectral subspace $X_T(\Omega)$ is decomposable [**214**, Proposition 1.4.2]. An example of a decomposable operator on a Hilbert space which is not strongly decomposable has been given by Albrecht [**37**], see also Vasilescu [**309**] or Section 1.4 of Laursen and Neumann [**214**].

Definition 6.25. *Let X be a Banach space and $\mathcal{B}$ a closed subalgebra of $L(X)$ containing the identity operator I. $\mathcal{B}$ is said to be* normal *with respect* *to a given operator $T \in \mathcal{B}$ if for every pair of spectral maximal subspaces Y, Z satisfying $\sigma(T|Y) \cap \sigma(T|Z) = \varnothing$ there exists some $R \in \mathcal{B}$ such that*

$$RT = TR, \quad R|Y = 0 \quad \textit{and} \quad (I - R)|Z = 0.$$

The following result establishes that every decomposable operator is super-decomposable precisely when the algebra $L(X)$ is normal with respect to T.

Theorem 6.26. *For a bounded operator $T \in L(X)$, X a Banach space, the following statements are equivalent:*

(i) *T is super-decomposable;*

(ii) *For every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ there exist T-invariant closed subspaces Y_1, Y_2 and an operator $R \in L(X)$ commuting with T such that*

$$R(X) \subseteq Y_1, \quad (I - R)(X) \subseteq Y_2 \quad \textit{and} \quad \sigma(T|Y_k) \subseteq \mathcal{U}_k \textit{ for } k = 1, 2; \tag{164}$$

(iii) *T is decomposable and $L(X)$ is normal with respect to T.*

Proof (i) $\Rightarrow$ (ii) Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$ and choose two open sets $\mathcal{W}_1, \mathcal{W}_2$ in $\mathbb{C}$ such that

$$\overline{\mathcal{W}_1} \subseteq \mathcal{U}_1, \quad \overline{\mathcal{W}_2} \subseteq \mathcal{U}_2 \quad \text{and} \quad \mathcal{W}_1 \cup \mathcal{W}_2 = \mathbb{C}.$$

According the definition of super-decomposability let $R \in L(X)$ denote an operator commuting with T such that

$$\sigma(T|\overline{R(X)}) \subseteq \mathcal{W}_1 \quad \text{and} \quad \sigma(T|\overline{(I - R)(X)}) \subseteq \mathcal{W}_2. \tag{165}$$

Put $Y_k := X_T(\overline{\mathcal{W}_k})$, $k = 1, 2$. From the inclusions (165) and part (vi) of Theorem 2.6, it follows that

$$R(X) \subseteq X_T(\mathcal{W}_1) \subseteq X_T(\overline{\mathcal{W}_1}) = Y_1$$

and

$$(I - R)(X) \subseteq X_T(\mathcal{W}_2) \subseteq X_T(\overline{\mathcal{W}_2}) = Y_2.$$

The local spectral subspaces Y_k are closed since T is decomposable, and hence by Theorem 6.18 has the property (C). Moreover, by Theorem 2.71 we have $\sigma(T|Y_k) \subseteq \overline{\mathcal{W}_k} \subseteq \mathcal{U}_k$.

(ii)$\Rightarrow$(iii) Clearly, T is decomposable. Let Y, Z be two spectral maximal subspaces such that $\sigma(T|Y) \cap \sigma(T|Z) = \varnothing$. Consider the open covering $\{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{C}$, where $\mathcal{U} := \mathbb{C} \setminus \sigma(T|Y)$ and $\mathcal{V} := \mathbb{C} \setminus \sigma(T|Z)$. Let $R \in L(X)$ be a commuting operator with T and let Y_1, Y_2 be two closed T-invariant subspaces for which the inclusions (164) hold. Clearly $R(Y) \subseteq Y_1$. Furthermore, being T decomposable, by Theorem 6.21 T has the property (C), and hence by Theorem 6.8 $Y = X_T(\sigma(T|Y))$. Consequently from part (i) of Theorem 2.6 the subspace Y is hyper-invariant for T, in particular $R(Y) \subseteq Y$. Clearly

$$\begin{aligned} R(Y) \subseteq Y \cap Y_1 &\subseteq X_T(\sigma(T|Y)) \cap X_T(\sigma(T|Y_1)) \\ &\subseteq X_T(\sigma(T|Y)) \cap X_T(\mathbb{C} \setminus \sigma(T|Y)) = X_T(\varnothing) = \{0\}, \end{aligned}$$

so that $R|Y = 0$. A similar argument shows that $(I - R)|Z = 0$.

(iii) $\Rightarrow$ (i) Given an arbitrary open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$, choose open sets $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2$ of $\mathbb{C}$ such that

$$\mathcal{V}_1 \subseteq \overline{\mathcal{V}_1} \subseteq \mathcal{V}_2 \subseteq \overline{\mathcal{V}_2} \subseteq \mathcal{U}_1,$$

and

$$\mathcal{W}_1 \subseteq \overline{\mathcal{W}_1} \subseteq \mathcal{W}_2 \subseteq \overline{\mathcal{W}_2} \subseteq \mathcal{U}_2.$$

Clearly $\Omega_1 := \mathbb{C} \setminus \mathcal{V}_1$ and $\Omega_2 := \mathbb{C} \setminus \mathcal{W}_1$ are closed and disjoint. Again, by Theorem 6.8, the local spectral subspaces $Y := X_T(\Omega_1)$ and $Z := X_T(\Omega_2)$ are spectral maximal. By Theorem 2.71 we also have,

$$\sigma(T|Y) \cap \sigma(T|Z) \subseteq \Omega_1 \cap \Omega_2 = \varnothing.$$

We know, since $L(X)$ is normal with respect to T, that there exists an operator $R \in L(X)$ commuting with T such that $R|Y = 0$ and $(I - R)|Z = 0$. We show now that $\sigma(T\ \overline{R(X)}) \subseteq \mathcal{V}_2$. Obviously $\{\mathcal{V}_2, \mathbb{C} \setminus \overline{\mathcal{V}_1})\}$ being an open covering of $\mathbb{C}$, by Theorem 6.19 we have the decomposition $X = X_T(\overline{\mathcal{V}_2}) + X_T(\overline{\mathbb{C} \setminus \mathcal{V}_1})$. The inclusions $\mathbb{C} \setminus \overline{\mathcal{V}_1} \subseteq \mathbb{C} \setminus \mathcal{V}_1 \subseteq \Omega_1$ then entail that $X = X_T(\overline{\mathcal{V}_2}) + X_T(\Omega_1)$. Since $R|Y = R|X_T(\Omega_1) = 0$, from the last decomposition it readily follows that

$$R(X) = R(X_T(\overline{\mathcal{V}_2})) \subseteq X_T(\overline{\mathcal{V}_2}).$$

Let $\lambda \in \mathbb{C} \setminus \mathcal{V}_2$ and denote by S the inverse operator of $(\lambda I - T)|X_T(\overline{\mathcal{V}_2})$. Given an arbitrary element $x \in R(X)$ choose $y \in X_T(\overline{\mathcal{V}_2})$ such that $x = Ry$. Then

$$Sx = SRy = SR(\lambda I - T)Sy = S(\lambda I - T)RSy = RSy \in R(X).$$

This shows that $S(R(X)) \subseteq R(X)$ and hence that $S(\overline{(R(X)}) \subseteq \overline{R(X)}$, so the restriction $S|\overline{R(X)}$ is the inverse of $(\lambda I - T)|\overline{R(X)}$. Therefore the inclusions $\sigma(T|\overline{(R(X)}) \subseteq \overline{\mathcal{V}_2} \subseteq \mathcal{U}_1$ hold.

A similar reasoning establishes that $\sigma(T|\overline{((I - R)(X)})$ is contained in $\mathcal{U}_2$, so that T is super-decomposable. ∎

For a subalgebra $\mathcal{B}$ of $L(X)$, where X is a Banach space, let $Z(B)$ be the center of $\mathcal{B}$

$$Z(\mathcal{B}) := \{T \in B : TS = ST \text{ for all } S \in \mathcal{B}\}.$$

Lemma 6.27. *Let $\mathcal{B}$ be a closed unital subalgebra of $L(X)$ and Ω any closed subset of $\mathbb{C}$. If $T \in Z(\mathcal{B})$ is decomposable and $S \in \mathcal{B}$ then the following assertions are equivalent:*

(i) $S(X) \subseteq X_T(\Omega)$;

(ii) $X_T(\mathbb{C} \setminus \Omega)) \subseteq \ker S$.

Proof Suppose that $S(X) \subseteq X_T(\Omega)$ and let $x \in X_T(\mathbb{C} \setminus \Omega)$. From the inclusions $\sigma_T(Sx) \subseteq \sigma_T(x)$ we obtain

$$Sx \in X_T(\Omega) \cap X_T(\mathbb{C} \setminus \Omega)) = X_T(\varnothing).$$

The decomposability T entails that T has the SVEP, so by Theorem 2.8 $X_T(\varnothing) = \{0\}$. Therefore $X_T(\mathbb{C} \setminus \Omega)) \subseteq \ker S$.

Conversely, suppose that $X_T(\mathbb{C} \setminus \Omega) \subseteq \ker S$. For every $\varepsilon > 0$ let us consider the following set

$$\Omega_\varepsilon := \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \Omega) \leq \varepsilon\}.$$

Let $\mathcal{U}_1$ be an open set such that $\Omega \subseteq \mathcal{U}_1 \subseteq \Omega_\varepsilon$. Since T is decomposable we know by Theorem 6.19 that there exist two maximal spectral subspaces Y_1, Y_2 such that

$$X = Y_1 + Y_2 \ , \ \sigma(T|Y_1) \subseteq \mathcal{U}_1 \subseteq \Omega_\varepsilon \quad \text{and} \quad \sigma(T|Y_2) \subseteq \mathbb{C} \setminus \Omega_\varepsilon.$$

For an arbitrary element $x \in X$ let $x = y_1 + y_2$, $y_1 \in Y_1$ and $y_2 \in Y_2$, be the corresponding decomposition. The subspaces Y_1, Y_2 are spectral maximal, so by Corollary 6.6 we have

$$\sigma_T(y_1) = \sigma_{T|Y_1}(y_1) \subseteq \sigma(T|Y_1) \subseteq \Omega_\varepsilon$$

and

$$\sigma_T(y_2) = \sigma_{T|Y_2}(y_2) \subseteq \sigma(T|Y_2) \subseteq \mathbb{C} \setminus \Omega_\varepsilon.$$

From this it follows that $y_2 \in X_T(\mathbb{C} \setminus \Omega)$, and hence from the assumption also that $Sy_2 = 0$. Consequently $Sx = Sy_1 + Sy_2 = Sy_1$ and therefore

$$\sigma_T(Sx) = \sigma_T(Sy_1) \subseteq \sigma_T(y_1) \subseteq \Omega_\varepsilon.$$

Therefore $Sx \in X_T(\Omega_\varepsilon)$. Since $\varepsilon > 0$ is arbitrary we then conclude that $Sx \in X_T(\Omega)$, hence the inclusion (i) is proved. ∎

For every $T \in Z(\mathcal{B})$, where $\mathcal{B}$ is any closed unital subalgebra of $L(X)$, let $\widetilde{T} : \mathcal{B} \to \mathcal{B}$ denote the corresponding multiplication operator defined by

$$\widetilde{T}(S) := TS \quad \text{for all } S \in \mathcal{B}.$$

Lemma 6.28. *Let $T \in Z(B)$, where $\mathcal{B}$ is a closed unital subalgebra of $L(X)$ containing the operator identity. Then the following assertions hold:*

(i) $\sigma_T(Sx) \subseteq \sigma_{\widetilde{T}}(S) \cap \sigma_T(x)$ *for every $S \in \mathcal{B}$ and $x \in X$;*

(ii) *If T has the SVEP then $\widetilde{T}$ has the SVEP.*

Proof (i) Let $S \in \mathcal{B}$, $\lambda \in \rho_{\widetilde{T}}(S)$, and $f : \mathbb{D}(\lambda, \varepsilon) \to \mathcal{B}$ an analytic function on the open disc $\mathbb{D}(\lambda, \varepsilon)$ for which the equation $(\mu \widetilde{I} - \mu T) f(\mu) = S$ holds for every $\mu \in \mathbb{D}(\lambda, \varepsilon)$. From the equation $(\mu I - T) f(\mu) x = Sx$ it follows that $\lambda \in \rho_T(Sx)$, and consequently $\sigma_T(Sx) \subseteq \sigma_{\widetilde{T}}(S)$ for every $S \in \mathcal{B}$ and $x \in X$. The inclusion $\sigma_T(Sx) \subseteq \sigma_T(x)$ holds, once it is observed that S and T commutes.

(ii) Clear. ∎

Theorem 6.29. *Let $\mathcal{B}$ be any unital closed subalgebra of $L(X)$ and suppose that $T \in Z(\mathcal{B})$ has the property (C) on the Banach space X. If $\mathcal{B}$ is normal with respect to T then the equality*

$$\mathcal{B}_{\widetilde{T}}(\Omega) = \{S \in \mathcal{B} : S(X) \subseteq X_T(\Omega)\}, \tag{166}$$

holds for every closed subset $\Omega \subseteq \mathbb{C}$.

Proof Suppose that $S \in \mathcal{B}_{\widetilde{T}}(\Omega)$. Then $\sigma_T(Sx) \subseteq \sigma_{\widetilde{T}}(S) \subseteq \Omega$ for every $x \in X$. Hence $S(X) \subseteq X_T(\Omega)$. Conversely, assume that $\mathcal{B}$ is normal with respect to T and let $S \in \mathcal{B}$ such that $S(X) \subseteq X_T(\Omega)$. Fixed $\lambda \notin \Omega$ and choose $0 < \eta < \operatorname{dist}(\lambda, \Omega)$. Let us consider the two sets

$$\Omega_1 := \{\mu \in \mathbb{C} : \operatorname{Im} \mu \leq \operatorname{Im} \lambda - \eta\}.$$

and

$$\Omega_2 := \{\mu \in \mathbb{C} : \operatorname{Im} \mu \geq \operatorname{Im} \lambda + \eta\}$$

Put $Y_k := X_T(\Omega_k)$, $k = 1, 2$. By assumption T has the property (C), so by Theorem 6.8 the two subspaces Y_k are spectral maximal. Furthermore, from Theorem 2.71 we know that

$$\sigma(T|Y_1) \cap \sigma(T|Y_2) \subseteq \Omega_1 \cap \Omega_2 = \varnothing.$$

Since $\mathcal{B}$ is normal, there exists an operator $R \in \mathcal{B}$ commuting with T such that $R|Y_1 = R|X_T(\Omega_1) = 0$ and $(I - R)|Y_2 = (I - R)|X_T(\Omega_2) = 0$. From the inclusions

$$X_T(\mathbb{C} \setminus \overline{(\mathbb{C} \setminus \Omega_1)}) \subseteq X_T(\Omega_1) \subseteq \ker R,$$

and from Lemma 6.27 it then follows that

$$R(X_T(\Omega)) \subseteq R(X) \subseteq X_T(\overline{\mathbb{C} \setminus \Omega_1}).$$

Analogously, using the same argument we easily obtain that

$$(I - R)(X_T(\Omega)) \subseteq (I - R)(X) \subseteq X_T(\overline{\mathbb{C} \setminus \Omega_2}).$$

Hence

$$RS(X) \subseteq R(X_T(\Omega)) \subseteq X_T(\Omega) \cap X_T(\overline{\mathbb{C} \setminus \Omega_1}) = X_T(\Omega \cap \overline{\mathbb{C} \setminus \Omega_1})$$

and

$$(I - R)S(X) \subseteq (I - R)(X_T(\Omega)) \subseteq X_T(\Omega) \cap X_T(\overline{\mathbb{C} \setminus \Omega_2}) = X_T(\Omega \cap \overline{\mathbb{C} \setminus \Omega_2}).$$

Set $X_1 := X_T(\Omega \cap \overline{\mathbb{C} \setminus \Omega_2})$, $X_2 := X_T(\Omega \cap \overline{\mathbb{C} \setminus \Omega_1})$ and consider the two analytic functions

$$\varphi(\mu) := R(\mu, T|X_1)(I - R)S \quad \text{for all } \mu \notin \Omega \cap \overline{\mathbb{C} \setminus \Omega_2}$$

and

$$\psi(\mu) := R(\mu, T|X_2)RS \quad \text{for all } \mu \notin \Omega \cap \overline{\mathbb{C} \setminus \Omega_1}.$$

Let $|\mu| > \|T\|$ and denote by $f : \mathbb{D}_\mu \to B$ an analytic function, defined on an open disc $\mathbb{D}_\mu$ centered at μ, such that

$$(\mu \widetilde{I} - \widetilde{T})f(\mu) = (\mu I - T)f(\mu) = S.$$

For every $x \in X$ we have

$$(\mu I - T)(I - R)f(\mu)x = (I - R)(\mu I - T)f(\mu)x = (I - R)Sx.$$

From these equalities we infer that

$$(\mu I - T)[\varphi(\mu) - (I - R)f(\mu)]x = 0,$$

and therefore $\varphi(\mu) = (I-R)f(\mu) \in \mathcal{B}$. Now, since λ belongs to the unbounded connected component of set $\mathbb{C} \setminus (\Omega \cap \overline{\mathbb{C} \setminus \Omega_2})$ we have $\varphi(\lambda) \in \mathcal{B}$. From the equalities

$$(\lambda\widetilde{I} - \widetilde{T})\varphi(\lambda) = (\lambda I - T)\varphi(\lambda) = (I-R)S,$$

we conclude that $\lambda \in \rho_{\widetilde{T}}((I-R)S)$ and hence $(I-R)S \in \mathcal{B}_{\widetilde{T}}(\Omega)$.

Analogously, $RS \in \mathcal{B}_{\widetilde{T}}(\Omega)$, thus $S = (I-R)S + RS \in \mathcal{B}_{\widetilde{T}}(\Omega)$, so the proof is complete. ∎

Theorem 6.30. *Let $\mathcal{B}$ be a closed unital subalgebra of $L(X)$. For every $T \in Z(\mathcal{B})$ the following assertions are equivalent:*

(i) *For every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ there exist T-invariant closed subspaces Y_1, Y_2 and an operator $R \in \mathcal{B}$ commuting with T such that*

$$R(X) \subseteq Y_1, (I-R)(X) \subseteq Y_2 \quad \textit{and} \quad \sigma(T|Y_k) \subset \mathcal{U}_k,$$

for $k = 1, 2$;

(ii) *T is decomposable and $\mathcal{B}$ is normal with respect to T;*

(iii) *T is super-decomposable and $\mathcal{B}$ is normal with respect to T;*

(iv) *$\widetilde{T} \in L(\mathcal{B})$ is decomposable;*

(v) *$\widetilde{T} \in L(\mathcal{B})$ is super-decomposable.*

Proof The equivalence (i)⇔(ii) may be proved by proceeding exactly as in the proof of Theorem 6.26. Moreover, the statement (i) implies that (ii) is equivalent to the assertion (iii).

(iii) ⇒ (iv) Since T is super-decomposable, hence decomposable, and $\mathcal{B}$ is normal with respect to T, the formula (166) of Theorem 6.29 implies that $\widetilde{T}$ has the property (C). In order to prove that $\widetilde{T} : \mathcal{B} \to \mathcal{B}$ is decomposable it suffices to show, by Theorem 6.19, that for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ we have $\mathcal{B} = \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_1}) + \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_2})$. Let $S \in \mathcal{B}$ be arbitrary. By the equivalent condition (i) there exist T-invariant closed subspaces Y_1, Y_2 and an operator $R \in \mathcal{B}$ commuting with T such that $R(X) \subseteq Y_1, (I-R)(X) \subseteq Y_2$ and $\sigma(T|Y_k) \subset \mathcal{U}_k$ for $k = 1, 2$. Hence

$$R(X) \subseteq X_T(\sigma(T|Y_1)) \subseteq X_T(\overline{\mathcal{U}_1})$$

and

$$(I-R)(X) \subseteq X_T(\sigma(T|Y_2)) \subseteq X_T(\overline{\mathcal{U}_2}).$$

Thus by Theorem 6.29 $R \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_1})$ and $I - R \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_2})$. From the inclusions $RS(X) \subseteq R(X) \subseteq X_T(\overline{\mathcal{U}_1})$ we obtain that $RS \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_1})$, and analogously $(I-R)S \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{U}_2})$. At this point the equality $S = RS + (I-R)S$ shows that the desired decomposition holds.

(iv) ⇒ (iii) Assume that $\widetilde{T}$ is decomposable. Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an arbitrary open covering of $\mathbb{C}$ and choose two open sets $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathbb{C}$ such that $\overline{\mathcal{W}_k} \subseteq \mathcal{U}_k$, for $k = 1, 2$ and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathbb{C}$. $\widetilde{T}$ being decomposable, by Theorem 6.19

we have $\mathcal{B} = \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{W}_1}) + \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{W}_2})$. Since $I \in \mathcal{B}$ there exist $S_1 \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{W}_1})$, $S_2 \in \mathcal{B}_{\widetilde{T}}(\overline{\mathcal{W}_2})$ such that $I = S_1 + S_2$. Hence

$$\sigma_{\widetilde{T}}(S_1) \subseteq \overline{\mathcal{W}_1} \subseteq \mathcal{U}_1, \quad \text{and} \quad \sigma_{\widetilde{T}}(S_2) \subseteq \overline{\mathcal{W}_2} \subseteq \mathcal{U}_2.$$

Now, by Theorem 6.19, $\widetilde{T}$ has the property (C), thus the subspaces $Y_k := X_T(\overline{\mathcal{W}_k})$, $k = 1,2$, are closed and, by Theorem 2.71, $\sigma(T|Y_k) \subseteq \mathcal{U}_k$. Moreover, from the inclusions

$$\sigma_T(S_k x) \subseteq \sigma_T(S_k) \subseteq \overline{\mathcal{W}_k}, \quad k = 1,2,$$

we deduce that $S_1 x \in Y_1$ and $S_2 x = (I - S_1)x \in Y_2$. The equality $x = S_1 x + (I - S_1)x$ then shows that $X = Y_1 + Y_2$, with $\sigma(T|Y_k) \subseteq \mathcal{U}_k$ for $k = 1,2$, so T is decomposable.

It remains only to prove that $\mathcal{B}$ is normal with respect to T. Let Y, Z be two spectral maximal subspaces for which $\sigma(T|Y) \cap \sigma(T|Z) = \varnothing$. Choose two closed subsets Ω_1 and Ω_2 of $\mathbb{C}$ such that

$$\Omega_1 \cap \sigma(T|Z) = \varnothing, \quad \Omega_2 \cap \sigma(T|Y) = \varnothing, \quad \text{int } \Omega_1 \cup \text{int } \Omega_2 = \mathbb{C}.$$

From the decomposability of $\widetilde{T}$ we know that $\mathcal{B} = \mathcal{B}_{\widetilde{T}}(\overline{\Omega}_1) + \mathcal{B}_{\widetilde{T}}(\overline{\Omega}_2)$, so there exist $R_1 \in \mathcal{B}_{\widetilde{T}}(\overline{\Omega_1})$ and $R_2 \in \mathcal{B}_{\widetilde{T}}(\overline{\Omega}_2)$ such that $I = R_1 + R_2$. By Lemma 6.28 and Corollary 6.6 we infer that for every $x \in X$ we have

$$\sigma_T(R_1 x) \subseteq \sigma_{\widetilde{T}}(R_1) \cap \sigma_T(x) \subseteq \overline{\Omega}_1 \cap \sigma(T|Z) = \varnothing.$$

Hence $R_1|Z = 0$ and analogously $R_2 = (I - R_1)|Y = 0$. This shows that $\mathcal{B}$ is normal with respect to T.

The implication (v) $\Rightarrow$ (iv) is obvious. To complete the proof it remains to prove the implication (i)$\Rightarrow$(v).

Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an arbitrary covering of $\mathbb{C}$ and choose two open sets $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathbb{C}$ such that $\overline{\mathcal{W}_k} \subseteq \mathcal{U}_k$, for $k = 1,2$ and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathbb{C}$. According to the assumption (i) let $R \in \mathcal{B}$ be a commuting operator with T such that $R(X) \subseteq Y_1, (I - R)(X) \subseteq Y_2$ and $\sigma(T|Y_k) \subseteq \mathcal{W}_k$ for $k = 1,2$. Consider the two operators $R_1 := R,\ R_2 := I - R \in \mathcal{B}$. Obviously R_1, R_2 commute with T and $R_1 + R_2 = I$. The corresponding left multiplication operators $\widetilde{R_1}, \widetilde{R_2} \in L(\mathcal{B})$ satisfy $\widetilde{R_1} + \widetilde{R_2} = I_{\mathcal{B}}$, where $I_{\mathcal{B}}$ is the identity operator on $\mathcal{B}$. Moreover, from the inclusions $\sigma(T|\overline{R_k(X)}) \subseteq \overline{W_k}$ we obtain

$$R_k(X) \subseteq X_T(\sigma(T|\overline{R_K(X)}) \subseteq X_T(\overline{W_k})$$

for $k = 1,2$. Since T is decomposable on $\mathcal{B}$, by Theorem 6.29 the spectral maximal subspaces for $\widetilde{T} : \mathcal{B} \to \mathcal{B}$ are given by

$$\mathcal{B}_{\widetilde{T}}(\Omega) = \{S \in \mathcal{B} : S(X) \subseteq X_T(\Omega)\},$$

for all closed subsets $\Omega \subseteq \mathbb{C}$. Now put $X_k := \mathcal{B}_{\widetilde{T}}(\overline{W_k})$ for $k = 1,2$. Clearly the subspaces X_k are closed and $\widetilde{T}$-invariant. Moreover,

$$\sigma(\widetilde{T}|X_k) \subseteq (\overline{W_k}) \subseteq U_k$$

and $\widetilde{R_k}(\mathcal{B}) = X_k$ for $k = 1, 2$. Hence $\widetilde{R}(\mathcal{B}) \subseteq X_1$ and $\widetilde{I} - \widetilde{R})(\mathcal{B}) \subseteq X_2$, so by Theorem 6.26 $\widetilde{T}$ is super-decomposable. ∎

An important example of super-decomposable operator is given by every operator which has totally disconnected spectrum.

Theorem 6.31. *Suppose that $T \in L(X)$, X a Banach space, has a totally disconnected spectrum. Then T is super-decomposable.*

Proof Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$. Since $\sigma(T)$ is totally disconnected there exists a spectral subset Ω of $\sigma(T)$ such that $\Omega \subseteq \mathcal{U}_1$ and $\mathbb{C} \setminus \Omega \subseteq \mathcal{U}_2$. Let P denote the spectral projection associated with Ω and let $Y_1 := P(X)$, $Y_2 := (I - P)(X) = \ker P$. Then P commutes with T and Y_1, Y_2 are closed T-invariant subspaces. Moreover,

$$\sigma(T|Y_1) = \Omega \subseteq \mathcal{U}_1, \quad \sigma(T|Y_2) = \mathbb{C} \setminus \Omega \subseteq \mathcal{U}_2,$$

so by part (ii) of Theorem 6.26 T is super-decomposable. ∎

By Theorem 6.31 every Riesz operator, or more generally, every operator having a discrete spectrum, is super-decomposable. The class of all super-decomposable operators contains some other classes larger than that of operators having a totally disconnected spectrum. For instance, the class of all generalized scalar operators or, more generally, the so called class of all $\mathcal{A}$-spectral operators, see [**214**], or also [**83**]. Later we shall show that several multipliers on commutative semi-simple Banach algebras are super-decomposable.

Recall that a linear subspace M of a vector space is said to T *divisible* if $(\lambda I - T)(M) = M$ for every $\lambda \in \mathbb{C}$. By definition the algebraic subspace $E_T(\varnothing)$, introduced in Chapter 2, is the largest T-divisible subspace.

The following result describes in a simple way the local spectral subspaces of a super-decomposable operator T in the case that $E_T(\varnothing) = \{0\}$.

Theorem 6.32. *Let $T \in L(X)$ be super-decomposable and suppose that $\{0\}$ is the only T-divisible linear subspace of X. Then $X_T(\Omega) = E_T(\Omega)$ for all closed $\Omega \subseteq \mathbb{C}$.*

Proof We need only to prove that $E_T(\Omega) \subseteq X_T(\Omega)$, because the opposite inclusion is satisfied for every operator $T \in L(X)$. It suffices to show that $E_T(\Omega) \subseteq X_T(\overline{\mathcal{W}})$ holds for every open set $\mathcal{W} \supseteq \Omega$, since $X_T(\cdot)$ preserves countable intersections by Theorem 2.6 part (v).

Choose an open set $\mathcal{U}$ such that $\Omega \subseteq \mathcal{U} \subset \overline{\mathcal{U}} \subseteq \mathcal{W}$ and let $R \in L(X)$ be an operator commuting with T such that

$$\sigma(T|\overline{R(X)}) \subseteq \mathbb{C} \setminus \overline{\mathcal{U}} \subseteq \mathbb{C} \setminus \Omega \quad \text{and} \quad \sigma(T|\overline{(I-R)(X)}) \subseteq \mathcal{W}.$$

We have $(I - R)(X) \subseteq X_T(\overline{\mathcal{W}})$, so the inclusion $E_T(\Omega) \subseteq X_T(\overline{\mathcal{W}})$ will follow if we verify that $R(E_T(\Omega)) = \{0\}$.

To see that let Z denote the largest linear space of $\overline{R(X)}$ such that $(\lambda I - T)(Z) = Z$ for all $\lambda \in \mathbb{C} \setminus \Omega$. From the inclusion $\sigma(T|\overline{R(X)}) \subseteq \mathbb{C} \setminus \Omega$ it follows that Z is T-divisible, and by assumption this forces Z to be trivial. On the other hand, we have $R(E_T(\Omega)) = (\lambda I - T)R(E_T(\Omega))$ for all $\lambda \in \mathbb{C}\setminus\Omega$, and therefore $R(E_T(\Omega)) \subseteq Z = \{0\}$, so the proof is complete. ∎

Note that super-decomposable operators need not have any non-trivial T-divisible linear subspaces. For instance, if $X := C[0,1]$ and T is the quasi-nilpotent operator considered in Example 2.35, then T is super-decomposable and hence super-decomposable. Moreover, as is easy to verify, the non-trivial subspace

$$Y := \{f \in C^\infty[0,1] : f^{(n)}(0) = 0 \text{ for all } n = 0, 1, \cdots\}$$

is T-divisible for X.

We conclude this section with a result which will be useful in the sequel.

Theorem 6.33. *Let X,Y be Banach spaces. Suppose that $T \in L(X)$, $S \in L(Y)$, $B \in L(X,Y)$ injective and $BT = SB$. If T has the SVEP and the property (δ) then $\sigma(T) \subseteq \sigma(S)$.*

Proof Assume that T has the SVEP and the property (δ). Combining Theorem 2.45 and Theorem 2.43 we know that

$$\sigma(T) = \sigma_{\rm su}(T) = \bigcup_{x\in X} \sigma_T(x)$$

so, in order to establish the inclusion $\sigma(T) \subseteq \sigma(S)$, it suffices to prove that, for every $x \in X$, the local spectrum $\sigma_T(x)$ is contained in any arbitrary open neighborhood $\mathcal{V}$ of $\sigma(S)$.

To prove this let us consider a closed neighborhood Ω of $\sigma(S)$ such that $\sigma(S) \subseteq \Omega \subseteq \mathcal{V}$ and put $\mathcal{U} := \mathbb{C} \setminus \Omega$. Obviously $\{\mathcal{U}, \mathcal{V}\}$ is an open covering of $\mathbb{C}$, so the property (δ) implies, since T has the SVEP, that every $x \in X$ may be decomposed as $x = u + v$, with $\sigma_T(u) \subseteq \mathcal{U}$ and $\sigma_T(\mathrm{v}) \subseteq \mathcal{V}$. Next we want show that $u = 0$. To see this let Γ denote a closed curve in the interior of Ω that surrounds $\sigma(S)$. Since T has the SVEP we can find an analytic function $f : \mathbb{C}\setminus\sigma_T(u) \to X$ such that $(\lambda I - T)f(\lambda) = u$ for all $\lambda \in \mathbb{C}\setminus\sigma_T(u)$. We have

$$\begin{aligned} Bx &= \frac{1}{2\pi i}\int_\Gamma R(\lambda,S)Bx\, d\lambda = \frac{1}{2\pi i}\int_\Gamma R(\lambda,S)B(\lambda I - T)f(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma R(\lambda,S)(\lambda I - S)Bf(\lambda)\, d\lambda = \frac{1}{2\pi i}\int_\Gamma Bf(\lambda)\, d\lambda = 0. \end{aligned}$$

By the injectivity of B we conclude that $u = 0$, and from this it follows that $\sigma_T(x) = \sigma_T(\mathrm{v}) \subseteq \mathcal{V}$, as desired. ∎

4. Decomposable right shift operators

In this section we shall consider the problem of decomposability of weighted right shift operators on $\ell^p(\mathbb{N})$, where $1 \leq p < \infty$. We first establish an useful characterization of Riesz operators among the class of decomposable operators, and successively we shall explore some situations for which a bounded operator $T \in L(X)$ on a Banach space X has a closed analytic core $K(T)$.

Theorem 6.34. *If X is a Banach space and $T \in L(X)$ then the following conditions are equivalent:*

(i) *T is a Riesz operator;*

(ii) *T is decomposable and all local spectral subspaces $X_T(\Omega)$, where Ω is closed set for which $0 \notin \Omega$, are finite-dimensional.*

Proof (i) $\Rightarrow$ (ii) We know that every Riesz operator is decomposable and hence has the SVEP. Let $\Omega \subseteq \mathbb{C} \setminus \{0\}$ be closed. By Theorem 2.6 we have $X_T(\Omega) = X_T(\Omega \cap \sigma(T))$, so we may suppose that Ω is a subset of $\sigma(T)$. Clearly Ω is a finite subset of $\sigma(T) \setminus \{0\}$, say $\{\lambda_1, \cdots, \lambda_n\}$, since the spectrum $\sigma(T)$ of a Riesz operator is a finite set or a countable set of eigenvalues which clusters at 0. By Theorem 2.6 we also have

$$X_T(\Omega) = \bigoplus_{i=1}^{n} X_T(\{\lambda_i\}) = \bigoplus_{i=1}^{n} H_0(\lambda_i I - T),$$

where the last equality follows from Theorem 2.20. Now, $\lambda_i I - T \in \Phi(X)$ for all $i = 1, \ldots, n$ and every λ_i is isolated in $\sigma(T)$, so that by Theorem 3.77 the subspaces $H_0(\lambda_i I - T)$ are finite-dimensional and hence also $X_T(\Omega)$ is finite-dimensional.

(ii) $\Rightarrow$ (i) First we show that each non-zero spectral point is an isolated point of $\sigma(T)$. Since T is decomposable, for an arbitrary $\varepsilon > 0$ we have

$$X = X_T(\mathbf{D}(0, \varepsilon))) + X_T(\mathbb{C} \setminus \mathbb{D}(0, \varepsilon/2))$$

and

$$\sigma(T|X_T(\mathbf{D}(0, \varepsilon)) \subseteq \mathbf{D}(0, \varepsilon).$$

Let $\lambda \in \sigma(T)$ be such that $|\lambda| > \varepsilon$. Then $\lambda I - T$ is bijective on $X_T(\mathbf{D}(0, \varepsilon))$, so that

$$\sigma(T) \cap (\mathbb{C} \setminus \mathbf{D}(0, \varepsilon)) \subseteq \sigma(X_T(\mathbb{C} \setminus \mathbb{D}(0, \varepsilon/2)).$$

From assumption we have that $X_T(\mathbb{C} \setminus \mathbb{D}(0, \varepsilon/2))$ is finite-dimensional, hence $\sigma(T) \cap (\mathbb{C} \setminus \mathbf{D}(0, \varepsilon))$ is a finite set. Since ε is arbitrary it then follows that every non-zero spectral point λ is isolated in $\sigma(T)$.

Now, by decomposability T has the SVEP, and hence $H_o(\lambda I - T) = X_T(\{\lambda\}$ by Theorem 2.19. By assumption it then follows that $H_o(\lambda I - T)$ is finite-dimensional, and hence by Theorem 3.77 and Theorem 3.111 we conclude that T is a Riesz operator. ∎

Recall that if 0 is an isolated point of $\sigma(T)$ then by Theorem 3.74, $H_0(T) = P_0(X)$ and $K(T) = \ker P_0$, where P_0 denotes the spectral projection associated with 0. Therefore if 0 is an isolated point of $\sigma(T)$, $K(T)$ is closed. The converse of this implication in general does not hold, for instance the unilateral left shift operator T on $X := \ell^p(\mathbb{N})$, for arbitrary $1 \leq p < \infty$, has closed analytical core $K(T) = X$, since it is surjective, whilst 0 certainly is a cluster point of $\sigma(T)$.

Definition 6.35. *Given a bounded operator $T \in L(X)$ on a Banach space X, $\lambda \in \mathbb{C}$, is said to be a* support point *for T if for every closed disc $\mathbf{D}(\lambda, \varepsilon)$ the glocal subspace $\mathcal{X}_T(\mathbf{D}(\lambda, \varepsilon))$ is not $\{0\}$. The set of all support points for T will be denoted by $\sigma_{\mathrm{supp}}(T)$.*

In the following theorem we collect some basic properties of $\sigma_{\mathrm{supp}}(T)$. Recall that T is said to have fat local spectra if $\sigma_T(x) = \sigma(T)$ for all $0 \neq x \in X$.

Theorem 6.36. *For every bounded operator $T \in L(X)$ on a Banach space X, the following assertions hold:*

(i) *$\sigma_{\mathrm{supp}}(T)$ is a closed subset of $\mathbb{C}$ and $\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{supp}}(T) \subseteq \sigma_{\mathrm{ap}}(T)$;*

(ii) *If T has the property (δ) then $\sigma_{\mathrm{supp}}(T) = \sigma_{\mathrm{ap}}(T) = \sigma(T)$;*

(iii) *If T has local fat spectra, $\sigma_{\mathrm{supp}}(T)$ is non-empty if and only if $\sigma_{\mathrm{supp}}(T)$ is a singleton.*

Proof (i) It is easy to see that $\sigma_{\mathrm{supp}}(T)$ is closed, whilst the inclusion $\sigma_{\mathrm{p}}(T) \subseteq \sigma_{\mathrm{supp}}(T)$ follows from the inclusions

$$\{0\} \neq \ker\,(\lambda I - T) \subseteq H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\}) \subseteq \mathcal{X}_T(\mathbf{D}(\lambda, \varepsilon))$$

for every closed disc $\mathbf{D}(\lambda, \varepsilon)$ centred at $\lambda \in \sigma_{\mathrm{p}}(T)$, see Theorem 2.20. Moreover, by Theorem 2.46 we know that $\mathcal{X}_T(\Omega) = \{0\}$ for all closed sets $\Omega \subseteq \mathbb{C}$ for which $\Omega \cap \sigma_{\mathrm{ap}}(T) = \varnothing$. Hence given any $\lambda \in \sigma_{\mathrm{supp}}(T)$ we have $\mathbf{D}(\lambda, \varepsilon) \cap \sigma_{\mathrm{ap}}(T) \neq \varnothing$ for every $\varepsilon > 0$, and therefore $\lambda \in \sigma_{\mathrm{ap}}(T)$ since $\sigma_{\mathrm{ap}}(T)$ is closed.

(ii) Let $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{supp}}(T)$, and choose an $\varepsilon > 0$ such that $\mathcal{X}_T(\mathbf{D}(\lambda, \varepsilon)) = \{0\}$. Evidently the open disc $U := \mathbb{D}(\lambda, \varepsilon)$ and $V := \mathbb{C} \setminus \mathbf{D}(\lambda, \varepsilon/2)$ form an open cover of $\mathbb{C}$, so the property (δ) implies that $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V}) = \mathcal{X}_T(\overline{V})$. From $\lambda \notin \overline{V}$ we obtain that $\lambda \notin \sigma_{\mathrm{su}}(T)$. Since $\lambda \notin \sigma_{\mathrm{p}}(T)$ by part (i), we then conclude that $\lambda \notin \sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{su}}(T)$ as claimed.

(iii) Immediate. ∎

Lemma 6.37. *Suppose that $T \in L(X)$, X a Banach space, has the SVEP. Suppose that there exists a sequence of compact sets $\Omega_n \subseteq \mathbb{C}$ with the property $0 \notin \Omega_n$ and $X_T(\Omega_n) \neq \{0\}$ for all $n \in \mathbb{N}$. If*

$$\gamma_n := \sup_{\lambda \in \Omega_n} |\lambda| \to 0 \quad \text{as } n \to \infty,$$

then $K(T)$ is not closed.

Proof Assume that $K(T)$ is closed. By part (ii) of Theorem 1.21 the restriction $T|K(T)$ is surjective on the Banach space $K(T)$ and has the SVEP since every restriction inherits SVEP. From Corollary 2.24 it then follows that $\lambda I - T$ is injective, so $\lambda \in \rho(T)$. Because the resolvent set is open there exists an open disc $\mathbb{D}(0, \varepsilon)$ such that $\mathbb{D}(0, \varepsilon) \cap \sigma(T|K(T)) = \varnothing$. Choose n such that $\Omega_n \subseteq \mathbb{D}(0, \varepsilon)$ and observe that

$$X_T(\Omega_n) \subseteq X_T(\mathbb{C} \setminus \{0\} = K(T) \quad \text{for all } n \in \mathbb{N},$$

and

$$K(T) = X_{T|K(T)}(\sigma(T|K(T)) \subseteq X_T(\sigma(T|K(T)).$$

From this, and taking into account Theorem 2.6, we obtain that

$$X_T(\Omega_n) \subseteq X_T(\Omega_n) \cap X_T(\sigma(T|K(T)) = X_T(\Omega_n \cap \sigma(T|K(T))) = X_T(\varnothing),$$

and hence by Theorem 2.8 since T has the SVEP we conclude that $X_T(\Omega_n) = \{0\}$, a contradiction. ■

The preceding result reveals that for every operator $T \in L(X)$ with the SVEP the space $K(\lambda I - T)$ fails to be closed whenever λ is a cluster point of $\sigma_{\mathrm{p}}(T)$, since $\ker(\lambda I - T) \subseteq X_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Note that the SVEP is crucial here, as shown by the example of the left shift T on $\ell^p(\mathbb{N})$ for any $1 \leq p < \infty$. In fact, as already observed, T does not have the SVEP, $\sigma_{\mathrm{p}}(T)$ is the open unit disc, whilst $K(T)$ is closed (since coincides with $\ell^p(\mathbb{N})$, being T onto), and $\lambda = 0$ is certainly a cluster point of $\sigma_{\mathrm{p}}(T)$.

Theorem 6.38. *Suppose that $T \in L(X)$ has the SVEP. If $K(T)$ is closed then 0 is not a cluster point of $\sigma_{\mathrm{supp}}(T)$. Moreover, if $\sigma(T) = \sigma_{\mathrm{supp}}(T)$ then $K(T)$ is closed if and only if 0 is not a cluster point of $\sigma(T)$. In particular, this equivalence holds if T has the property (δ).*

Proof Suppose that 0 is a cluster point of $\sigma_{\mathrm{supp}}(T)$. Choose a sequence of point $\lambda_n \in \sigma_{\mathrm{supp}}(T) \setminus \{0\}$ such that $\lambda_n \to 0$ as $n \to \infty$, and consider for each $n \in \mathbb{N}$ the closed disc $\Omega_n := \mathbf{D}(\lambda_n, \varepsilon_n)$, where $0 < \varepsilon_n < |\lambda_n|$. By Lemma 6.37 then $K(T)$ is not closed, which proves the first assertion. The equivalence and the last assertion are then clear from Theorem 3.74 and part (ii) of Theorem 6.36. ■

Corollary 6.39. *If $T \in L(X)$ is a non-invertible decomposable operator then the following assertions are equivalent:*

(i) *$K(T)$ is closed;*

(ii) *$K(T^*)$ is closed;*

(iii) *0 is an isolated point of $\sigma(T)$.*

Proof We know by Theorem 6.22 that T^* is decomposable, and hence has both the SVEP and the property (δ), so the assertions are equivalent by Theorem 6.38 and Theorem 6.36. ■

Note that the equivalence (i) $\Leftrightarrow$ (iii) does remain valid for operators having the SVEP and with the property (δ).

Corollary 6.40. *Suppose that $T \in L(X)$ is a decomposable operator for which $\sigma(T)$ contains no isolated points. Then for each $\lambda \in \mathbb{C}$ the following statements are equivalent:*

(i) *$K(\lambda I - T)$ is closed;*

(ii) *$K(\lambda I - T) = X$, or equivalently $\lambda I - T$ is surjective;*

(iii) *$\lambda I - T$ is invertible.*

Proof Immediate by Corollary 6.39. ∎

In the opposite direction we obtain the following result for Riesz operators.

Corollary 6.41. *Suppose that $T \in L(X)$ is Riesz operator. Then the following statements are equivalent:*

(i) *$K(T)$ is finite-dimensional;*

(ii) *$K(T)$ is closed;*

(iii) *$K(\lambda I - T)$ is closed for all $\lambda \in \mathbb{C}$;*

(iv) *$\sigma(T)$ is finite;*

(v) *$K(T^*)$ if finite-dimensional;*

(vi) *$K(T^*)$ is closed.*

Proof Obviously (i) $\Rightarrow$ (ii). If T is a Riesz operator every $\lambda \in \sigma(T) \setminus \{0\}$ is isolated in $\sigma(T)$, so $K(\lambda I - T)$ is closed since it is the kernel of the spectral projection associated with $\{\lambda\}$. Therefore (ii) and (iii) are equivalent. For a Riesz operator T the spectrum $\sigma(T)$ is finite if and only if 0 is not a cluster point of $\sigma(T)$, so from Theorem 6.36 and Theorem 6.38 we deduce that (ii) $\Leftrightarrow$ (iv). Finally, assume that the spectrum $\sigma(T)$ is finite. By the characterization of Riesz operators established in Theorem 6.34 we obtain that $K(T) = X_T(\sigma(T) \cap (\mathbb{C} \setminus \{0\})$ is finite-dimensional. Hence (iv) $\Rightarrow$ (i), so the statements (i)-(iv) are equivalent.

The equivalences (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) are obvious, since $\sigma(T) = \sigma(T^*)$, and if T is Riesz then also T^* is Riesz by Corollary 3.114. ∎

We now consider operators for which $K(T) = \{0\}$. Recall that this condition is satisfied by every weighted right shift on $\ell^p(\mathbb{N})$. If (ω_n) is the weight sequence of T, from Theorem 2.88 we know that if the quantity

$$c(T) = \lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n}$$

is strictly greater than 0 then T^* does not have the SVEP. Therefore by Theorem 6.22 T cannot be decomposable, and in particular does not have the property (δ). This shows that in many concrete cases shifts fail to be decomposable. The next result shows that for a weighted right shift T both property (δ) and decomposability are actually equivalent to T being quasi-nilpotent.

Theorem 6.42. *Let $T \in L(X)$, where X is a Banach space, be such that $K(T) = \{0\}$. Then the following statements are equivalent:*

(i) *T is decomposable;*

(ii) *T has the property (δ);*

(iii) *0 is an isolated point of $\sigma(T)$;*

(iv) *T is quasi-nilpotent;*

(v) *T is a Riesz operator;*

(vi) *$\lambda I - T$ has finite descent of all $\lambda \neq 0$.*

Proof Clearly (i) $\Rightarrow$ (ii). Assume (ii). Since $\ker(\lambda I - T) \subseteq K(T) = \{0\}$ for all $\lambda \neq 0$ it follows that $\ker(\lambda I - T) \subseteq K(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$, so T has the SVEP by part (iii) of Theorem 2.22. Clearly the condition $K(T) = \{0\}$ entails that T is not surjective, so $0 \in \sigma(T)$, and by Theorem 6.38 0 is an isolated point of $\sigma(T)$. Therefore (ii) $\Rightarrow$ (iii).

To show that (iii) $\Rightarrow$ (iv) suppose that 0 is isolated in $\sigma(T)$ and denote by P_0 the spectral projection associated with $\{0\}$. Then by Theorem 3.74 $\{0\} = K(T) = \ker P_0$ and hence $H_0(T) = P_0(X) = X$, so by Theorem 1.68 T is quasi-nilpotent. Since (iv) implies (i) trivially, the assertions (i)–(iv) are equivalent.

Clearly (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vi) by Theorem 3.111, so it remains only to prove the implication (vi) $\Rightarrow$ (iv). Suppose that T is not quasi-nilpotent and $\lambda \in \sigma(T) \setminus \{0\}$. The argument used in the first part of the proof shows that $\ker(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$. This forces $q(\lambda I - T)$ to be infinite, otherwise we would obtain $p(\lambda I - T) = q(\lambda I - T) = 0$, by Theorem 3.3 and hence $\lambda \in \rho(T)$; a contradiction. This shows that (vi) $\Rightarrow$ (iv), so the proof is complete. ∎

By Corollary 2.57 for every non-invertible isometry T, T^* fails to have the SVEP at every λ greater than $r(T) = 1$. Therefore by Theorem 6.22 every non-invertible isometry T does not have the property (δ), and in particular T is not decomposable. However, every isometry T has the property (β) and the decomposability, as well as the property (δ), are equivalent to T being invertible. We refer to Section 1.6 of [**214**] for a proof of these results.

We now introduce a property intermediate between the SVEP and the Dunford property (C).

Definition 6.43. *A bounded operator $T \in L(X)$, X a Banach space, is said to have the* property (Q) *if $H_0(\lambda I - T)$ is closed for every $\lambda \in \mathbf{C}$.*

Clearly a quasi-nilpotent operator has the property (Q), since $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$ and $H_0(T) = X$. More generally, if the spectrum $\sigma(T)$ is finite then T has the property (Q). In fact, if $\lambda \in \sigma(T)$ is isolated then $H_0(\lambda I - T)$ coincides with the range of the spectral projection associated with the singleton set $\{\lambda\}$, see Theorem 3.74. Another important class of operators having the property (Q) is given, by Theorem 4.33, by the multipliers of semi-simple Banach algebras.

Clearly, since the property (C) entails the SVEP, from Theorem 2.19 and Theorem 2.20 we infer that if T has the property (C) then $H_0(\lambda I - T) = X_T(\{\lambda\})$ is closed for every $\lambda \in \mathbf{C}$, so that the following implications hold:

$$\text{property } (C) \Rightarrow \text{property } (Q) \Rightarrow \text{SVEP}. \tag{167}$$

Note that neither of the implications (167) may be reversed in general. A first counter example of an operator which has the SVEP but not the property (Q) is given by the operator T defined in Example 2.32. An example of an operator which shows that the first implication is not reversed in general, may be found amongst the convolution operators T_μ of group algebras $L^1(G)$, since these operators have the property (Q), see Theorem 4.33, whilst it may not have the property (C), as we shall see in the next section.

The next result shows that in many concrete cases the property (Q) and the decomposability of a right shift T on $\ell^p(\mathbb{N})$ are equivalent to T being quasi-nilpotent.

Theorem 6.44. *Suppose that infinitely many weights ω_n are zero. Then for the corresponding right shift T on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, the following statements are equivalent:*

(i) *T is quasi-nilpoltent;*

(ii) *T is decomposable;*

(iii) *T has the property (δ);*

(iv) *T has the property (β);*

(v) *T has the property (C);*

(vi) *T has the property (Q);*

(vii) *$H_0(T)$ is closed .*

Proof The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) have been proved in Theorem 6.42. The implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are satisfied by every bounded operator, by Theorem 6.21 and Theorem 6.18. Moreover, (v) $\Rightarrow$ (vi), as already observed.

(vi) $\Rightarrow$ (i) Suppose that $H_0(T)$ is closed. Since $Te_n = \omega_n e_{n+1}$ for all $n \in \mathbb{N}$, if $\omega_n = 0$ then $e_n \in \ker T \subseteq H_0(T)$. Suppose that $\omega_n \neq 0$ and let k be the smallest integer such that $\omega_{n+k} = 0$. It is easy to check that

$$T^{k+2} e_n = \omega_n \omega_{n+1} \cdots \omega_{n+k} \; e_{n+k+1} = 0,$$

so $e_n \in \ker T^{k+2} \subseteq H_0(T)$. This shows that $H_0(T) = \ell^p(\mathbb{N})$ and hence T is quasi-nilpotent.

Therefore (vi) $\Rightarrow$ (i) and consequently the assertions (i)–(vi) are equivalent. The implication (vi) $\Rightarrow$ (vii) is obvious, whilst (vii) $\Rightarrow$ (vi) follows from part (iv) of Theorem 2.82, so the proof is complete. ∎

Example 6.45. A simple example of an operator having the SVEP but without the property (Q) is provided by the following right shift on $\ell^p(\mathbb{N})$. Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the bounded sequence of positive real numbers defined by

$$\omega_n := \begin{cases} 0 & \text{if } n \text{ is a square of an integer,} \\ 1 & \text{otherwise.} \end{cases}$$

If T is the corresponding right shift on $\ell^p(\mathbb{N})$, with $1 \le p < \infty$, it is easily seen that $\|T^n\| = 1$ for all $n \in \mathbb{N}$, so that T is not quasi-nilpotent. This excludes by Theorem 6.42 that $H_0(T)$ is closed.

5. Decomposable multipliers

In this section we shall give several characterizations of decomposable multipliers of semi-simple Banach algebra. We begin first with some preliminary remarks about the maximal ideal space of the unitization of a commutative complex Banach algebra A without unit.

Let $A_e := A \oplus \mathbb{C}e$ denote the standard unitization of A endowed with the usual structure of a commutative Banach algebra. Then we can identify each $m \in \Delta(A)$ with its canonical extension $m^\star$ from A to A_e, and if $m_\infty : A_e \to \mathbb{C}$ denotes the canonical extension of the zero functional defined by

$$m_\infty(x + \lambda u) := \lambda \quad \text{for all } x \in A \quad \text{and} \quad \lambda \in \mathbb{C},$$

it is easily seen that $\Delta(A_e) = \Delta(A) \cup \{m_\infty\}$. Observe that if A is semi-simple then also A_e is semi-simple and that this construction is still valid when A possesses a unit element, which, however, will be different from the unit element of the unitization A_e.

Lemma 6.46. *Let A be a commutative complex Banach algebra without identity and suppose that for an element $a \in A$ the Gelfand transform $\widehat{a}$ is hk-continuous. Then $\widehat{a}$ is also hk-continuous on $\Delta(A_e)$.*

Proof Let $E \subseteq \Delta(A_e)$ be an arbitrary set. It follows immediately from the definition of hulls and kernels that

$$h_1(k_1(E)) \subseteq h_A(k_A(\Delta(A) \cap E)) \cup \{m_\infty\}, \tag{168}$$

where h_1, k_1 refer to the hull and kernel operations with respect to A_e. Given an arbitrary non-empty closed subset F of $\mathbb{C}$, let us denote by

$$E := \{m^\star \in \Delta(A_e) : m^\star(a) \in F\}$$

its pre-image under $\widehat{a} : \Delta(A_e) \to \mathbb{C}$. Clearly, in order to show that $\widehat{a}$ is hk-continuous on $\Delta(A_e)$ it suffices to prove that E is hk-closed in $\Delta(A_e)$. Since by assumption $\widehat{a}$ is hk-continuous on $\Delta(A)$, the set $\Delta(A) \cap F$ is hk-closed in $\Delta(A)$.

We distinguish now the two cases $0 \in F$ and $0 \notin F$.

If $0 \in F$ then $m_\infty \in E$ and hence

$$h_1(k_1(E)) \subseteq h_A(k_A(\Delta(A) \cap E)) \cup \{m_\infty\} = (\Delta(A) \cap E)) \cup \{m_\infty\} = E,$$

so E is hk-closed in $\Delta(A_e)$.

In the remaining case $0 \notin F$ we have $m_\infty \notin E$, so $E \subseteq \Delta(A)$. From that it follows that E is a hull in $\Delta(A)$, and if $\varepsilon := \inf\{|\lambda| : \lambda \in F\}$ we also have

$$|\widehat{a}(m)| \geq \varepsilon \quad \text{for every } m \in E.$$

By Theorem 4.22 part (ii) there exists an element $b \in A$ such that

$$\widehat{a}(m)\widehat{b}(m) = 1 \quad \text{for every } m \in E.$$

Since the element $v := e - ab \in A_e$ satisfies

$$m_\infty(v) = m_\infty(e) = 1 \quad \text{and} \quad \widehat{v}(m) = 0 \quad \text{for every } m \in E,$$

we conclude that $m_\infty \notin h_1(k_1(E))$. From the inclusion (168) we then obtain

$$h_1(k_1(E)) \subseteq h_A(k_A(\Delta(A) \cap E)) = h_A(k_A(E)) = E,$$

thus, also in the case $0 \notin F$, the set E is hk-closed in $\Delta(A_e)$. ∎

Theorem 6.47. *Let A be a commutative semi-simple Banach algebra, X a Banach space, and $\Phi : A \to L(X)$ an algebraic homomorphism. Then for every $a \in A$ for which the Gelfand transform $\widehat{a}$ is hk-continuous on $\Delta(A)$ the corresponding operator $T := \Phi(a) : \Delta(A) \to L(X)$ is super-decomposable.*

Proof The proof is divided in two parts. In the first part we shall consider the case A has a unit u for which $\Phi(u) = I$, whilst in the second part we shall consider the case where A either has no unit at all or the unit $u \in A$ does not satisfy the condition $\Phi(u) = I$.

First case: Suppose that A has a unit u and $\Phi(u) = I$. In this case $\Delta(A)$ is compact in the hk-topology. Given any arbitrary open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ let us consider a pair of open subsets $G, H \subset \mathbb{C}$ such that $\overline{G} \subseteq \mathcal{U}_1$, $\overline{H} \subseteq \mathcal{U}_2$, and $G \cup H = \mathbb{C}$. Since $\mathbb{C} \setminus G$ and $\mathbb{C} \setminus H$ are two disjoint closed sets and the Gelfand transform $\widehat{a}$ is hk-continuous, it follows that the pre-images $\widehat{a}^{-1}(\mathbb{C} \setminus G)$ and $\widehat{a}^{-1}(\mathbb{C} \setminus H)$ are disjoint hulls in the compact space $\Delta(A)$. By [**279**, Corollary 3.6.10], there exists then an element $z \in A$ such that

$$\widehat{z} \equiv 0 \text{ on } \widehat{a}^{-1}(\mathbb{C} \setminus G) \quad \text{and} \quad \widehat{z} \equiv 1 \text{ on } \widehat{a}^{-1}(\mathbb{C} \setminus H).$$

Let us consider the operator $R := \Phi(z) \in L(X)$. We claim that R satisfies the conditions for the super-decomposability of T with respect to the covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$. Obviously T and R commute. In order to show the inclusion $\sigma(T|\overline{R(X)} \subseteq \overline{U_1}$ let $\lambda \in \mathbb{C} \setminus \overline{U_1}$ be arbitrarily given and let $\delta := \operatorname{dist}(\lambda, \overline{G})$ be the distance from λ to $\overline{G}$. Clearly $\delta > 0$ and

$$|(\widehat{a} - \lambda\widehat{u})(m)| \geq \delta \quad \text{for all } m \in h_A(\widehat{a}^{-1}(\mathbb{C} \setminus \overline{G})).$$

By part (ii) of Theorem 4.22 there exists an element $s \in A$ such that

$$(\widehat{a} - \lambda\widehat{u})\widehat{s} \equiv 1 \quad \text{on} \quad \widehat{a}^{-1}(\overline{G}).$$

Since $\widehat{z} \equiv 0$ on $\widehat{a}^{-1}(\mathbb{C} \setminus G)$ it follows that

$$(\widehat{a} - \lambda\widehat{u})\widehat{s}\widehat{z}(m) = \widehat{z}(m) \quad \text{for every } m \in \Delta(A).$$

From the semi-simplicity of A it follows that $(a - \lambda u)sz = z$.

Now let us consider $S := \Phi(s) \in L(X)$ and apply the homomorphism Φ to the equation $(a - \lambda u)sz = z$. We obtain that

$$((T - \lambda I)SR)x = (S(T - \lambda I)R)x = Rx$$

holds for all $x \in X$ and therefore $(T - \lambda I)S = S(T - \lambda I) = I$ on $\overline{R(X)}$. Since the subspace $\overline{R(X)}$ is invariant under S it follows that λ belongs to the resolvent of the restriction of T on $\overline{R(X)}$, which proves the inclusion $\sigma(T|\overline{R(X)}) \subseteq \overline{U_1}$.

To prove the remaining inclusion $\sigma(T|\overline{(I-R)(X)}) \subseteq \overline{U_2}$ we use a similar argument. Take $\mu \in \mathbb{C} \setminus \overline{U_2}$ and let $\varepsilon := \operatorname{dist}(\mu, \overline{H})$. Clearly $\varepsilon > 0$ and

$$|(\widehat{a} - \mu\widehat{u})(m)| \geq \varepsilon \quad \text{for all } m \in h_A(\widehat{a}^{-1}(\mathbb{C} \setminus \overline{H})).$$

Again, by part (ii) of Theorem 4.22, there exists an element $v \in A$ such that

$$(\widehat{a} - \mu\widehat{u})\widehat{v} \equiv 1 \quad \text{on} \quad \widehat{a}^{-1}(\overline{H}).$$

From $\widehat{z} = 1$ on $\widehat{a}^{-1}(\mathbb{C}\setminus\overline{H})$ we obtain $(a-\mu u)v(u-z) = u-z$, so if $V := \Phi(v)$ the equality

$$[(T-\mu I)V(I-R)]x = (I-R)x$$

holds for every $x \in X$ and hence

$$(\mu I - T)V = V(\mu I - T) = I \quad \text{on} \quad \overline{(I-R)(X)}.$$

As before, this implies the invertibility of $\mu I - T$ on $\overline{(I-R)(X)}$ and therefore $\sigma(T|\overline{(I-R)(X)}) \subseteq \overline{U_2}$.

Second case: If A has no unit or if the unit $u \in A$ does not satisfy the equality $\Phi(u) = I$, the statement of the theorem can be reduced to the case $\Phi(u) = I$ by means of the following construction. In either case let us consider the standard unitization A_e and the canonical extension $\Phi^\star$ of the homomorphism Φ from A to the algebra A_e, defined by

$$\Phi^\star(x + \lambda e) := \Phi(x) + \lambda \quad \text{for all } x \in A \text{ and } \lambda \in \mathbb{C}.$$

Observe that this construction works even if A has an identity element, which, as observed before, ceases to be the identity of the extension A_e.

In order to prove that $T = \Phi(a) = \Phi^\star(a)$ is super-decomposable we need to prove, by the first part of the proof, that $\widehat{a}$ is hk-continuous on the extended maximal ideal space $\Delta(A_e)$. We consider two cases. If A has an identity element then $\Delta(A)$ is obviously hk-closed in $\Delta(A_e)$ and from the hk-continuity of $\widehat{a}$ on $\Delta(A)$ there easily follows the hk-continuity of $\widehat{a}$ on $\Delta(A_e)$.

In the remaining case the hk-continuity of $\widehat{a} : \Delta(A_e) \to \mathbb{C}$ follows from Lemma 6.46, so the proof is complete. ∎

We shall now consider the problem of decomposability for a multiplier T defined on a semi-simple commutative Banach algebra. Not surprisingly the continuity of the transform $\widehat{T}$ on $\Delta(M(A))$ here has a certain importance, as well as the continuity of the Helgason–Wang function $\varphi_T = \widehat{T}|\Delta(A)$, where both the maximal ideal spaces $\Delta(A)$ and $\Delta(M(A))$ are endowed with the hk-topology.

We need first to introduce a definition obtained by borrowing a term from harmonic analysis:

Definition 6.48. *If A is a commutative semi-simple Banach algebra, a multiplier $T \in M(A)$ is said to have a* natural spectrum *if* $\sigma(T) = \overline{\widehat{T}(\Delta(A))}$.

Observe that by Theorem 7.79 if $T \in M_{00}(A)$ and A is not unital then T has a natural spectrum. It is easy to see that if A has an unit then every $T \in M(A)$ has natural spectrum. Indeed, in this case $A = M(A)$, and hence by Theorem 7.79

$$\sigma(T) = \widehat{T}(\Delta(M(A)) = \widehat{T}(\Delta(A)).$$

On the other hand, if A has no unit $\Delta(A)$ need not be dense in $\Delta(M(A))$ with respect to the Gelfand topology, so there exist multipliers which does not natural spectrum.

The next result shows that the property (δ), and a *fortiori* the decomposability, in the case of a multiplier implies the hk-continuity of φ_T and that T has a natural spectrum. First recall that since every multiplier $T \in M(A)$ of a semi-prime Banach algebra A has the SVEP, by Theorem 2.19 the local spectral subspace and the glocal spectral subspace of T associated with a closed set $\Omega \subseteq \mathbb{C}$ coincide. The condition (δ) for a multiplier $T \in M(A)$, where A is a semi-prime Banach algebra, may then be formulated saying that for every open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$ we have $A = A_T(\overline{\mathcal{U}_1}) + A_T(\overline{\mathcal{U}_2})$.

Theorem 6.49. *Suppose that A is a semi-prime commutative Banach algebra and $T \in M(A)$ has the property (δ). Then the Helgason–Wang transform $\varphi_T = \widehat{T}|\Delta(A)$ is hk-continuous on $\Delta(A)$. If A is semi-simple then T has a natural spectrum.*

Proof Suppose that $\varphi_T = \widehat{T}|\Delta(A)$ is not hk-continuous on $\Delta(A)$. Then there exists a closed subset F of $\mathbb{C}$ such that

$$E := \widehat{T}^{-1}(F) = \{m \in \Delta(A) : \widehat{T}(m) \in F\}$$

is not hk-closed in $\Delta(A)$, so $h_A(k_A(E)) \neq E$. Let $m_0 \in h_A(k_A(E)) \setminus E$ and choose an element $x \in A$ such that $m_0(x) = 1$. Clearly $\lambda := \widehat{T}(m_0) \notin F$. Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$ such that $\overline{\mathcal{U}_1} \subseteq \mathbb{C} \setminus \{\lambda\}$ and $\overline{\mathcal{U}_2} \subseteq \mathbb{C} \setminus F$. Since T has the property (δ) then $A = A_T(\overline{\mathcal{U}_1}) + A_T(\overline{\mathcal{U}_2})$, so there exist y and $z \in A$ such that

$$x = y + z, \quad \sigma_T(y) \subseteq \overline{\mathcal{U}_1}, \quad \sigma_T(z) \subseteq \overline{\mathcal{U}_2}.$$

Since $\lambda \notin \sigma_T(y)$ there exists some element $u \in A$ such that $y = (\lambda I - T)u$. From this it follows that

$$m_0(y) = \widehat{y}(m_0) = [\lambda - \widehat{T}(m_0)]\widehat{u}(m_0) = 0.$$

Hence $1 = m_0(x) = m_0(y) + m_0(z) = m_0(z)$.

On the other hand, for any $m \in E$ we have $\mu := \widehat{T}(m) \in F$, so $\mu \notin \overline{\mathcal{U}_2}$. Since $z \in A_T(\overline{\mathcal{U}_2})$ we obtain that $z = (\mu I - T)v$ for some $v \in A$, and consequently

$$m(z) = \widehat{z}(m) = [\mu - \widehat{T}(m)]\widehat{v}(m) = 0.$$

Since $m(z) = 0$ for all $m \in E$ and $m_0(z) = 1$ we infer that $m_0 \notin h_A(k_A(E))$, a contradiction. Hence $\widehat{T}$ is hk-continuous on $\Delta(A)$.

Assume now that T has the property (δ) and that A is semi-simple.

To prove the equality $\sigma(T) = \overline{\widehat{T}(\Delta(A))}$, denote, as usual, by $C_b(\Delta(A))$ the Banach algebra of all Gelfand continuous bounded complex-valued functions on $\Delta(A)$ endowed with the supremum norm. Let $B : A \to C_b(\Delta(A))$ denote the Gelfand transform defined by $Bx := \widehat{x}$ for all $x \in A$, and let $S : C_b(\Delta(A)) \to C_b(\Delta(A))$ denote the operator given by multiplication by the function $\varphi_T = \widehat{T}$. Clearly $BT = TS$. Since B is injective by the semi-simplicity of A and since T has the property (δ), by Lemma 6.33 we have $\sigma(T) \subseteq \sigma(S)$. On the other hand, the inclusions $\sigma(S) \subseteq \overline{\widehat{T}(\Delta(A))} \subseteq \sigma(T)$ are trivial. Hence $\sigma(T) = \overline{\widehat{T}(\Delta(A))}$. ∎

Remark 6.50. Observe that in general the converse of the previous Theorem does not hold, the hk-continuity of $\widehat{T}|\Delta(A) = \varphi_T$ does not imply that T has the property (δ), as the following example shows.

Let $A := L^1(G)$, G a non discrete locally compact Abelian group, and consider the convolution operator T_μ, where $\mu \in \mathcal{M}(G)$. Since $L^1(G)$ is regular the Gelfand topology and the hk-topology coincide on $\Delta(L^1(G)) = \widehat{G}$, the dual group of G. Thus for any $\mu \in \mathcal{M}(G)$ the transform $\varphi_{T_\mu} = \widehat{\mu}|\widehat{G}$ is hk-continuous. On the other hand, as we shall see in Chapter 8, there exists a measure μ such that the spectrum $\sigma(T_\mu) = \sigma(\mu) \neq \widehat{\mu}(\widehat{G})$, hence by Theorem 6.49 the corresponding convolution operator T_μ cannot have the property (δ), and, in particular, is not decomposable.

The next theorem relates, for an arbitrary $T \in M(A)$, the decomposability of T to the decomposability of the corresponding multiplication operator $L_T : M(A) \to M(A)$.

Theorem 6.51. *Let A be a semi-simple commutative Banach algebra and $T \in M(A)$. Then $\widehat{T}$ is hk-continuous on $\Delta(M(A))$ if and only if the multiplication operator $L_T : M(A) \to M(A)$ is super-decomposable. Moreover, these equivalent conditions imply that T is decomposable on A.*

Proof The hk-continuity of $\widehat{T}$ on $\Delta(M(A))$ implies, by Theorem 6.47 applied to left regular representation $\Phi : T \in M(A) \to L_T \in L(M(A))$, that L_T is super-decomposable on $M(A)$. Conversely, if $L_T : M(A) \to M(A)$ is super-decomposable then L_T has the property (δ) so that, by Theorem 6.49, $\widehat{L_T}|\Delta(M(A)) = \widehat{T}$ is hk-continuous on $\Delta(M(A))$. The last assertion follows by Theorem 6.30, on taking $\mathcal{B} = M(A)$. ∎

Corollary 6.52. *Let A be a commutative semi-simple Banach algebra. If $M(A)$ is regular then every $T \in M(A)$ is decomposable on A.*

Proof (i) If $M(A)$ is regular then the Gelfand topology and the hk-topology on $\Delta(M(A))$ coincide, so every $\widehat{T}$ is hk-continuous on $\Delta(M(A))$. ∎

Recall that given a complex commutative Banach algebra A and $x \in A$, the support supp $\widehat{x}$ is defined as the closure in $\Delta(A)$ of the set $\{m \in \Delta(A) :$

$\widehat{x}(m) \neq 0\}$. It is not difficult to see that if A is semi-simple then for any $T \in M(A)$ we have

$$\widehat{T}(\text{supp } \widehat{x}) \subseteq \sigma_T(x) \quad \text{for all } x \in A. \tag{169}$$

Indeed, if $\lambda \in \rho_T(x)$ and $\lambda = \widehat{T}(m)$ for some $m \in \Delta(A)$, then $(\lambda I - T)u = x$ for some $u \in A$ and hence $0 = [\lambda I - \widehat{T}(m)]\widehat{u}(m) = \widehat{x}(m)$. From this we obtain

$$\widehat{T}(\{x^{-1}\{0\}) \subseteq \sigma_T(x)$$

and hence $\widehat{T}(\text{supp } \widehat{x}) \subseteq \sigma_T(x)$, by the continuity of the Gelfand transform $\widehat{T}$ on $\Delta(A)$.

The following definition is a local spectral version of the notion of natural spectrum.

Definition 6.53. *Given a semi-simple commutative Banach algebra A, $T \in M(A)$ is said to have* natural local spectra *if $\sigma_T(x) = \overline{\widehat{T}(\text{supp } \widehat{x})}$ holds for every $x \in A$.*

For any closed subset $\Omega \subseteq \mathbb{C}$ let $Z_T(\Omega)$ denote the set

$$Z_T(\Omega) := \{x \in A : \widehat{T}(\text{supp } \widehat{x}) \subseteq \Omega\}.$$

Clearly, $Z_T(\Omega)$ is a closed ideal of A. It is easily seen that also the local spectral subspace $A_T(\Omega)$ is an ideal, not necessarily closed, of A. In the next theorem we shall describe some relations between these two ideals.

Theorem 6.54. *Let A be a commutative semi-simple Banach algebra and $T \in M(A)$. Then the following assertions hold:*

(i) *$A_T(\Omega) \subseteq Z_T(\Omega)$ for all closed subsets $\Omega \subseteq \mathbb{C}$;*

(ii) *T has natural local spectra if and only if equality $A_T(\Omega) = Z_T(\Omega)$ holds for all closed subsets $\Omega \subseteq \mathbb{C}$, and this is the case if and only if $\sigma(T|Z_T(\Omega)) \subseteq \Omega$ for all closed sets Ω of $\mathbb{C}$;*

(iii) *If T has natural local spectra then T has the property (C) and the equalities*

$$A_T(\Omega) = Z_T(\Omega) = \bigcap_{\lambda \notin \Omega} (\lambda I - T)(A) = E_T(\Omega),$$

hold for all closed sets Ω of $\mathbb{C}$, where, as usual, $E_T(\Omega)$ denotes the algebraic spectral subspace associated with Ω;

(iv) *If the restriction $T|Z_T(\Omega)$, Ω a closed subset of $\mathbb{C}$, has the property (δ) then T has natural local spectra.*

Proof (i) The inclusion $A_T(\Omega) \subseteq Z_T(\Omega)$, for all closed subsets $\Omega \subseteq \mathbb{C}$, follows easily from the identity (169).

(ii) If $T \in M(A)$ has natural local spectra then obviously $A_T(\Omega) = Z_T(\Omega)$ for all closed sets $\Omega \subseteq \mathbb{C}$. Conversely, suppose that $Z_T(\Omega) = A_T(\Omega)$ for all closed sets $\Omega \subseteq \mathbb{C}$, and consider for every $x \in A$ the closed sets of the form $\Omega := \overline{\widehat{T}(\text{supp } \widehat{x})}$. Then $x \in Z_T(\Omega) = A_T(\Omega)$ so that $\sigma_T(x) \subseteq \Omega$, which shows that T has natural local spectra.

Assume now that T has natural local spectra, or, equivalently, that $Z_T(\Omega) = A_T(\Omega)$. Then $A_T(\Omega)$ is closed, and since T has the SVEP from Theorem 2.71 it follows that $X_T\sigma(T|Z_T(\Omega)) \subseteq \Omega$. Conversely, since $Z_T(\Omega)$ is a closed T-invariant subspace of X, the inclusion $\sigma(T|Z_T(\Omega)) \subseteq \Omega$ implies by part (vi) of Theorem 2.6 that $Z_T(\Omega) \subseteq A_T(\Omega)$. By part (i) we then conclude that $Z_T(\Omega) = A_T(\Omega)$.

(iii) T has the property (C) since by part (ii) $A_T(\Omega) = Z_T(\Omega)$ is closed for all closed subsets $\Omega \subseteq \mathbb{C}$. Clearly

$$A_T(\Omega) \subseteq \bigcap_{\lambda \notin \Omega} (\lambda I - T)(A) \subseteq Z_T(\Omega),$$

and by Corollary 2.70 $E_T(\Omega) = \bigcap_{\lambda \notin \Omega}(\lambda I - T)(A)$. If T has natural spectra, or, equivalently, if $Z_T(\Omega) = A_T(\Omega)$ then all these subspaces coincide.

(iv) Observe first that by Theorem 5.1 the closed ideal $Z_T(\Omega)$ is semi-simple and $T|Z_T(\Omega)$ is a multiplier. From Theorem 6.49, since by assumption $T|Z_T(\Omega)$ has the property (δ), we have that

$$\sigma(T|Z_T(\Omega) = \overline{\widehat{T}(\Delta(Z_T(\Omega))}. \tag{170}$$

Now, from Theorem 4.21 we know that $\Delta(Z_T(\Omega)) = \Delta(A) \setminus h_A(Z_T(\Omega))$. Moreover, $\widehat{T}^{-1}(\mathbb{C} \setminus \Omega) \subseteq h_A(Z_T(\Omega))$, so that from the equality (170) it follows that

$$\sigma(T|Z_T(\Omega) \subseteq \overline{\widehat{T}(\widehat{T}^{-1}(\Omega))} \subseteq \Omega \quad \text{for all closed subsets } \Omega \subseteq \mathbb{C}.$$

By part (iii) we then conclude that T has natural local spectra. ∎

Lemma 6.55. *Let A be a semi-simple commutative Banach algebra and J be a closed ideal of A for which $J = \mathrm{span}\,(AJ)$. Then J is also an ideal of $M(A)$. Moreover, if $T \in M(A)$ has the property (δ) then $T|J : J \to J$ has also the property (δ).*

Proof We have

$$M(A)J = M(A)\ \mathrm{span}\,(AJ) \subseteq \mathrm{span}\,(AJ) = J,$$

so J is a closed T-invariant subspace for any $T \in M(A)$. Suppose that T has the property (δ). Let $x \in A$ arbitrary and choose an open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{C}$. Let $x := \sum_{k=1}^n a_k x_k$, where $a_k \in A$ and $x_k \in J$, $k = 1, 2, ...n$. Using the property (δ), for every a_k we can find $u_k, v_k \in A$ such that

$$\sigma_T(u_k) \subseteq \mathcal{U}_1 \quad \sigma_T(v_k) \subseteq \mathcal{U}_2 \quad \text{and} \quad a_k = u_k + v_k$$

for all $k = 1, 2, ...n$. Let $f_k : \mathbb{C} \setminus \mathcal{U}_1 \to A$ denote an analytic function for which $u_k = (\lambda I - T)f_k(\lambda)$ holds for every $\lambda \in \mathbb{C} \setminus \mathcal{U}_1$. Then

$$u_k x_k = (\lambda I - T)f_k(\lambda)x_k \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathcal{U}_1,$$

so that the J-valued analytic functions $g_k(\lambda) := f_k(\lambda)x_k$ on $\mathbb{C} \setminus \mathcal{U}_1$ verify $u_k x_k = (\lambda I - T)g_k(\lambda)$ and hence

$$\sigma_{T|J}(u_k x_k) \subseteq \mathcal{U}_1 \quad \text{for all } k = 1, 2, \cdots, n.$$

A similar argument shows that $\sigma_{T|J}(v_k x_k) \subseteq \mathcal{U}_2$ for all $k = 1, 2, \cdots, n$. Therefore

$$\sigma_{T|J}\left(\sum_{k=1}^{n} u_k x_k\right) \subseteq \mathcal{U}_1 \quad \text{and} \quad \sigma_{T|J}\left(\sum_{k=1}^{n} v_k x_k\right) \subseteq \mathcal{U}_2.$$

Since $x = \sum_{k=1}^{n} u_k x_k + \sum_{k=1}^{n} v_k x_k$ this shows that the restriction $T|J$ has the property (δ), as claimed. ■

Theorem 6.56. *Let A be a semi-simple commutative Banach algebra and suppose that for every closed ideal J of A we have $J = \text{span}\,(AJ)$. Then $T \in M(A)$ is decomposable if and only if T has the property (δ).*

Proof By Theorem 6.21 we need only to show that (δ) implies the decomposability of T. Suppose that T has the property (δ) and let $\Omega \subseteq \mathbb{C}$ denote an arbitrary closed set. By Lemma 6.55 the restriction $T|Z_T(\Omega)$ has the property the (δ), and therefore by part (iv) of Theorem 6.54 T has natural local spectra, from which we conclude by part (iii) of Theorem 6.54 that T has the property (C). By Theorem 6.21 it then follows that T is decomposable. ■

Corollary 6.57. *Let A be a semi-simple commutative Banach algebra with a bounded approximate identity. Then $T \in M(A)$ is decomposable if and only if T has the property (δ).*

Proof We only need to prove that (δ) implies decomposability. This follows from Theorem 6.56 and from the Cohen factorization. ■

Note that the result of Corollary 6.57 applies to the group algebra $L^1(G)$, with G a locally compact Abelian group.

We shall now address the problem of decomposability for multipliers which belong to the ideals $M_0(A)$ and $M_{00}(A)$. Recall that by definition

$$M_0(A) := \{T \in M(A) : \varphi_T = \widehat{T}|\Delta(A) \text{ vanishes at infinity in } \Delta(A)\},$$

whilst

$$M_{00}(A) := \{T \in M(A) : \widehat{T} \equiv 0 \text{ on } h_{M(A)}(A)\},$$

and

$$A \subseteq M_{00}(A) \subseteq M_0(A) \subseteq M(A).$$

Moreover, if A is semi-simple then $M(A)$, $M_0(A)$, and $M_{00}(A)$ are also semi-simple Banach algebras, see Theorem 4.25.

The following fundamental result shows that for these operators the decomposability is equivalent to the formally weaker property (δ).

Theorem 6.58. *Let A be a semi-simple commutative Banach algebra and $T \in M_0(A)$. Then T is decomposable if and only if T has the property (δ).*

Proof By Theorem 6.21 we only need to prove that the property (δ) implies property (C). According to part (iv) and part (iii) of Theorem 6.54 it suffices to show that $T|Z_T(\Omega)$ has the property (δ) whenever Ω is a closed subset of $\mathbb{C}$.

Suppose first that $0 \notin \Omega$. Since $T \in M_0(A)$ the set $\widehat{T}^{-1}(\Omega)$ is a compact subset of $\Delta(A)$ in the Gelfand topology. By Theorem 6.49 the set $\widehat{T}^{-1}(\Omega)$ is also hk-closed in $\Delta(A)$, and hence by [**279**, Theorem 3.6.9] there exists an element $a \in A$ such that $\widehat{a} \equiv 1$ on $\widehat{T}^{-1}(\Omega)$. Since A is semi-simple it then follows that $ax = x$ for every $x \in Z_T(\Omega)$. Using Lemma 6.55 we then conclude that $T|Z_T(\Omega)$ has the property (δ).

Next, suppose that 0 belongs to the interior int Ω of Ω. Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$. We may assume, with no loss of generality, that $0 \in \mathcal{N} :=$ int $\Omega \cap \mathcal{U}_1$. Let us consider an open subset $\mathcal{W} \subseteq \mathbb{C}$ such that $0 \notin \overline{\mathcal{W}}$ and so $\mathcal{N} \cup \mathcal{W} = \mathbb{C}$. For every $x \in Z_T(\Omega)$, by using the property (δ) we may choose $u, v \in A$ such that

$$a = u + v, \quad \sigma_T(u) \subseteq \mathcal{N}, \quad \sigma_T(\mathrm{v}) \subseteq \mathcal{W}.$$

From the inclusion $\mathcal{N} \subseteq \Omega$ it is clear that $u \in Z_T(\Omega)$ and hence $\sigma_{T|Z(\Omega)}(u) \subseteq \mathcal{U}_1$. Moreover, by the first part of this proof we have

$$v \in A_T(\overline{\mathcal{W}}) \cap Z_T(\Omega) = Z_T(\overline{\mathcal{W}}) \cap Z_T(\Omega) = Z_T(\overline{\mathcal{W}} \cap \Omega).$$

We already know that $T|Z_T(\overline{\mathcal{W}} \cap \Omega)$ has the property (δ), and since $\{\mathcal{U}_1, \mathcal{U}_2\}$ is an open covering of $\mathbb{C}$ we can find $z_1, z_2 \in Z_T(\overline{\mathcal{W}} \cap \Omega)$ such that

$$v = z_1 + z_2 \quad \text{and} \quad \sigma_{T|Z_T(\overline{\mathcal{W}} \cap \Omega)}(z_k) \subseteq \mathcal{U}_k, \; k = 1, 2.$$

From this it follows that $\sigma_{T|Z_T(\Omega)}(z_k) \subseteq \mathcal{U}_k$ for $k = 1, 2$. From the equality $x = (u + z_1) + z_2$ we then conclude that $T|Z_T(\Omega)$ has the property (δ) also when $0 \in$ int Ω.

To conclude the proof let us consider the case where $0 \in \Omega$. Choose a sequence of closed subsets Ω_n of $\mathbb{C}$ for which 0 belongs to int Ω_n and $\Omega = \cap_{n \in \mathbb{N}} \Omega_n$. By the preceding part of the proof we know that $T|Z_T(\Omega_n)$ has the property (δ), and hence by part (iv) and part (ii) of Theorem 6.54 $Z_T(\Omega_n) = A_T(\Omega_n)$ for all $n \in \mathbb{N}$. From this we obtain that

$$Z_T(\Omega) = \bigcap_{n \in \mathbb{N}} Z_T(\Omega_n) = \bigcap_{n \in \mathbb{N}} A_T(\Omega_n) = A_T(\Omega).$$

The equality $Z_T(\Omega) = A_T(\Omega)$ entails, again by part (ii) of Theorem 6.54, that T has natural local spectra and hence has the property (C), so the proof is complete. ∎

Lemma 6.59. *Let A be a semi-simple commutative Banach algebra. If $T \in M_0(A)$ is decomposable then $M(A)$ and $M_0(A)$ are normal with respect to T.*

Proof It suffices to prove the normality of the algebra $M_0(A)$. Let $T \in M_0(A)$ and consider a pair of spectral maximal spaces Y and Z for T for which $\sigma(T|Y) \cap \sigma(T|Z) = \varnothing$ is satisfied. Let us consider the following two sets:

$$E := \widehat{T}^{-1}(\sigma(T|Y)) = \{m \in \Delta(A) : \widehat{T}(m) \in \sigma(T|Y)\}$$

and

$$F := \widehat{T}^{-1}(\sigma(T|Z)) = \{m \in \Delta(A) : \widehat{T}(m) \in \sigma(T|Z)\}.$$

Being T decomposable on A we then have by Theorem 6.51 that $\widehat{T}|\Delta(A)$ is hk-continuous, so the two sets E, F are hulls in $\Delta(A)$. Moreover, since by assumption $\sigma(T|Y)$ and $\sigma(T|Z)$ are disjoint, we may assume that $0 \notin \sigma(T|Z)$. If we put

$$\varepsilon := \inf_{\lambda \in \sigma(T|Z)} |\lambda|,$$

we then obtain

$$|\widehat{T}(m)| \geq \varepsilon > 0 \quad \text{for all } m \in F.$$

Since $T \in M_0(A)$ we infer that F is compact in $\Delta(A)$ with respect to the Gelfand topology. The two sets E and F are disjoint, and hence by part (iii) of Theorem 4.22 there exists an element $e \in A$ such that

$$\widehat{e} \equiv 0 \text{ on } E \quad \text{and} \quad \widehat{e} \equiv 1 \text{ on } F.$$

Now let us consider the multiplication operator $R := L_e \in M_0(A)$. Let $y \in Y$ be arbitrarily given and take $m \in \Delta(A) \backslash E$. Then $\lambda = \widehat{T}(m) \notin \sigma(T|Y)$ and hence there exists some $w \in Y$ such that $y = (\lambda I - T)w$, which implies

$$\widehat{y}(m) = [\lambda - \widehat{T}(m)]\widehat{w}(m) = 0.$$

Hence $\widehat{y} \equiv 0$ on $\Delta(A) \setminus E$ and consequently $\widehat{e}\,\widehat{y} \equiv 0$ on $\Delta(A)$. Since A is semi-simple it then follows that $Rv = 0$, and therefore $R|Y \equiv 0$.

A similar argument shows that $R|Z \equiv I$, so $M_0(A)$ is normal with respect to T. ∎

For multipliers which belong to the ideals $M_0(A)$ and $M_{00}(A)$ the property of being decomposable is equivalent to many of the conditions previously introduced.

Theorem 6.60. *Let A be a semi-simple commutative Banach algebra. If $T \in M_0(A)$ then the following assertions are equivalent:*

(i) *T is super-decomposable on A;*

(ii) *T is decomposable on A;*

(iii) *T has the property (δ);*

(iv) *$L_T : M(A) \to M(A)$ is super-decomposable;*

(v) *$L_T : M(A) \to M(A)$ is decomposable;*

(vi) $\widehat{T}$ *is hk-continuous on* $\Delta(M(A))$;

(vii) $T \in M_{00}(A)$ *and* $\widehat{T}$ *is hk-continuous on* $\Delta(A)$.

Moreover, if $T \in M_{00}(A)$ *the statements above are equivalent to:*

(viii) $\widehat{T}$ *is hk-continuous on* $\Delta(A)$.

Proof The statements (i), (ii), (iv), and (v) are equivalent by Theorem 6.30, since by Lemma 6.59 $M_0(A)$ is normal with respect to T. The equivalence (ii) $\Leftrightarrow$ (iii) has been established in Theorem 6.58. The equivalence (iv) $\Leftrightarrow$ (vi) has been proved in Theorem 6.51. Hence the statements (i)–(iv) are equivalent.

(vi)$\Rightarrow$(vii) Let $T \in M_0(A)$ and suppose that $\widehat{T}$ is hk-continuous on $\Delta(M(A))$. Then the restriction $\widehat{T}|\Delta(A)$ is hk-continuous, so we need only to show that $\widehat{T}$ vanishes on the set

$$h_{M(A)}(A) = \{\varphi \in \Delta(M(A)) : \varphi|\Delta(A) \equiv 0\}.$$

Let $\varepsilon > 0$ be arbitrarily given. From our assumption the set

$$E := \{m \in \Delta(A) : |\widehat{T}(m)| \geq \varepsilon\}$$

is a Gelfand compact hull in $\Delta(A)$. Therefore, see [**279**, Theorem 3.6.7], the kernel $k_A(E)$ is a regular ideal in A, and consequently by Lemma 3.1.15 of [**279**]) there exists an element $v \in A$ such that $\widehat{v} \equiv 1$ on E. For every $m \in E$ we have

$$|\widehat{(Tv)}(m)| = |\widehat{T}(m)\widehat{v}(m)| \geq \varepsilon \quad \text{for all } m \in E,$$

so by part (ii) of Theorem 4.22 there exists some $z \in A$ such that $\widehat{T}\widehat{v}\widehat{z} \equiv 1$ on E. Clearly, for the multiplier $S := I - L_{T(vz)}$, where $L_{T(vz)}$ is the multiplication operator by $T(vz)$, we have $\widehat{S} \equiv 0$ on E and $\widehat{S} \equiv 1$ on $h_{M(A)}(A)$. From this it follows that $h_{M(A)}k_{M(A)}(E) \cap h_{M(A)}(A) = \varnothing$.

Now assume that there exists a functional $\phi_0 \in h_{M(A)}(A)$ such that $|\widehat{T}(\phi_0)| > \varepsilon$. Since by assumption $\widehat{T}$ is hk-continuous on $\Delta(M(A))$ there exists a hk- open neighborhood $\mathcal{U}$ of ϕ_0 in $\Delta(M(A))$ such that

$$|\widehat{T}(\phi)| > \varepsilon \quad \text{for all } \varphi \in \mathcal{U}.$$

Since by Theorem 4.24 $\Delta(A)$ is hk-dense in $\Delta(M(A))$ it follows that the intersection $\mathcal{U} \cap \mathcal{V} \cap \Delta(A)$ is non-empty for all hk-open neighborhoods $\mathcal{V}$ of ϕ_0 in $\Delta(M(A))$, and hence $\phi_0 \in h_{M(A)}k_{M(A)}(E)$. But this contradicts that $\phi_0 \in h_{M(A)}(A)$ and $h_{M(A)}k_{M(A)}(E) \cap h_{M(A)}(A) = \varnothing$. Therefore for every $\varepsilon > 0$ we have

$$|\widehat{T}(\varphi)| \leq \varepsilon \quad \text{for all } \varphi \in h_{M(A)}(A)$$

which shows that $T \in M_{00}(A)$.

(vii) $\Rightarrow$ (vi) Let us consider an arbitrary closed non-empty subset F of $\mathbb{C}$ and let

$$E := \widehat{T}^{-1}(F) = \{\varphi \in \Delta(M(A)) : \widehat{T}(\varphi) \in F\}.$$

To prove the hk-continuity of $\widehat{T} : \Delta(M(A)) \to \mathbb{C}$ it suffices to show that E is hk-closed, namely $E = h_{M(A)}k_{M(A)}(E)$.

We distinguish two cases: Suppose first that $0 \in F$. Since $\widehat{T} \equiv 0$ on $h_{M(A)}(A)$ we have

$$\Delta(M(A)) \setminus E = \{m \in \Delta(A) : \widehat{T}(m) \in \mathbb{C} \setminus F\} \subseteq \Delta(A).$$

We know by Theorem 4.18 that $\Delta(A)$ is hk-open in $\Delta(M(A))$, and since by assumption $\widehat{T}|\Delta(A)$ is hk-continuous this implies $\Delta(M(A)) \setminus E$ is hk-open in $\Delta(M(A))$. Thus if $0 \in F$, E is hk-closed in $\Delta(M(A))$.

Consider the remaining case $0 \notin F$. Then $\varepsilon := \inf\{|\lambda| : \lambda \in F\} > 0$. By assumption the Helgason–Wang function $\widehat{T}|\Delta(A)$ is hk-continuous and $\widehat{T}|\Delta(A)$ vanishes at infinity, so the set

$$\Gamma := \{m \in \Delta(A) : |\widehat{T}(m)| \geq \varepsilon\}$$

is a Gelfand compact hull in $\Delta(A)$. By using the same argument as in the proof of the implication (vi)$\Rightarrow$(vii), it is possible to find two elements $v, z \in A$ such that

$$\widehat{v} \equiv 1 \text{ on } \Gamma \quad \text{and} \quad \widehat{T}\widehat{v}\widehat{z} \equiv 1 \text{ on } \Gamma.$$

Trivially, since $E \subseteq \Gamma$, we then have $\widehat{T}\widehat{v}\widehat{z} \equiv 1$ on E. This implies that the multiplier $S := I - L_{T(vz)}$ satisfies both the identities

$$\widehat{S} \equiv 0 \text{ on } E \quad \text{and} \quad \widehat{S} \equiv 1 \text{ on } h_{M(A)}(A).$$

From this it follows that

$$h_{M(A)}k_{M(A)}(E) \cap h_{M(A)}(A) = \varnothing$$

and hence $h_{M(A)}k_{M(A)}(E) \subseteq \Delta(A)$. Since by assumption E is a hull in $\Delta(A)$, the inclusion $h_{M(A)}k_{M(A)}(E) \subseteq \Delta(A)$ then implies that $h_{M(A)}k_{M(A)}(E) = E$, so E is hk-closed in $\Delta(M(A))$. ∎

Corollary 6.61. *Let A be a commutative semi-simple regular Banach algebra and suppose that $T \in M_0(A)$. Then T is decomposable on A if and only if $T \in M_{00}(A)$ and this happens if and only if T has natural local spectra.*

Proof This is an obvious consequence of the equivalence (ii) $\Rightarrow$ (vii) of Theorem 6.60, since on $\Delta(A)$ the Gelfand topology and the hk-topology coincide. ∎

The next result is an obvious consequence of Theorem 6.60, once it is observed that $A \subseteq M_{00}(A) \subseteq M_0(A)$.

Corollary 6.62. *Let A be a semi-simple commutative Banach algebra. Then for each $a \in A$ the following statements are equivalent:*

(i) *$\widehat{a}$ is hk-continuous on $\Delta(A)$;*

(ii) *The multiplication operator L_a is super-decomposable on A;*

(iii) *The multiplication operator* L_a *is decomposable on* A*;*

(iv) *The multiplication operator* L_a *has the property* (δ) *on* A*.* ∎

The last result leads immediately to a very nice characterization of regularity for a commutative semi-simple Banach algebra in terms of decomposability of the multiplication operators L_a.

Corollary 6.63. *Let* A *be a semi-simple commutative Banach algebra. Then the following statements are equivalent:*

(i) A *is regular;*

(ii) *For each* $a \in A$, $\widehat{a}$ *is hk-continuous on* $\Delta(A)$*;*

(iii) *For each* $a \in A$, L_a *is super-decomposable;*

(iv) *For each* $a \in A$, L_a *is decomposable.* ∎

Given an arbitrary semi-simple commutative Banach algebra A let us consider the following subset of A

$$\operatorname{Dec} A := \{a \in A : \text{ the multiplication operator } L_a \text{ is decomposable}\}.$$

We now show that $\operatorname{Dec} A$ is a closed subalgebra of A. In the sequel we shall denote by $r(a)$ the spectral radius of $a \in A$.

Theorem 6.64. *If* A *is a semi-simple commutative Banach algebra the following assertions hold:*

(i) $\operatorname{Dec} A$ *is a closed subalgebra of* A*;*

(ii) *Let* $(a_n) \subset A$ *be a sequence and suppose that for some* $a \in A$ *we have* $\lim_{n\to\infty} r(a - a_n) = 0$*. Then* $a \in \operatorname{Dec} A$*;*

(iii) *If* $a \in \operatorname{Dec} A$ *and if* J *is an arbitrary Banach algebra contained in* A *as an ideal, then the restriction* $L_a|J$ *is super-decomposable on* J*;*

(iv) *If* $a \in \operatorname{Dec} A$ *and if* $C \supseteq A$ *is a not necessarily commutative semi-simple Banach algebra containing* A *as a subalgebra, then the canonical extension* $S_a : C \to C$ *defined by* $S_a(x) := ax$ *for all* $x \in C$ *is super-decomposable.*

Proof (i) and (ii) easily follows from Corollary 6.62. To prove (iii) let us consider an arbitrary open covering $\{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{C}$. Let us consider the left regular representation $\Phi : A \to L(A)$ defined by $\Phi(a) := L_a$ for all $a \in A$. Arguing as in the proof of Theorem 6.47, one can find an element $z \in A \oplus \mathbb{C}e$, $A \oplus \mathbb{C}e$ the unization of A, with the following two properties: for each $\lambda \in \mathbb{C} \setminus \mathcal{U}$ there exists some element $a_\lambda \in A \oplus \mathbb{C}e$ such that $(a - \lambda e)a_\lambda z = z$, and for every $\mu \in \mathbb{C} \setminus \mathcal{V}$ there exists some element $b_\mu \in A \oplus \mathbb{C}e$ such that $(a - \mu e)b_\mu(e - z) = e - z$. As in the proof of Theorem 6.47, for the operator $\Phi(z) := L_z$ the following inclusions hold:

$$\sigma(L_a|\overline{L_z(J)} \subseteq \mathcal{U} \quad \text{and} \quad \sigma(L_a|\overline{(I - L_z)(J)} \subseteq \mathcal{V}.$$

Obviously $J = \overline{L_z(J)} + \overline{(I - L_z)(J)}$, so $L_a|J$ is decomposable.

The proof of the assertion (iv) is obtained by using the same argument above. ∎

Theorem 6.65. *Every semi-simple commutative Banach algebra A contains a greatest regular closed subalgebra. Moreover, this subalgebra is also closed in the spectral radius norm and is contained in* $\operatorname{Dec} A$.

Proof Let B denote the union of all regular Banach subalgebras M of A and let C denote the closure in the spectral radius norm of the (possibly trivial) subalgebra generated by B. We claim that C is regular. By Corollary 6.63 it suffices to prove that the multiplication operator $L_a|C : C \to C$ is decomposable for all $a \in C$.

Given an element $a \in B$ we have $a \in M$ for some regular subalgebra M of A and, since on $\Delta(M)$ the Gelfand topology and the hk-topology coincide, the Gelfand transform $\widehat{a} : \Delta(M) \to \mathbb{C}$ is hk-continuous. Let $\Phi : M \to L(C)$ denote the mapping defined by $\Phi(x) := L_x|C$ for all $x \in M$. By Theorem 6.47 $L_x|C : C \to C$ is decomposable, and by using a similar argument it follows that L_x is also decomposable as operator acting on A. From that we conclude that C is regular and B is contained in $\operatorname{Dec} A$. Moreover, since by Theorem 6.64, $\operatorname{Dec} A$ is a closed subalgebra with respect to the spectral radius norm, we also have $C \subseteq \operatorname{Dec} A$, so the proof is complete. ∎

Let $\operatorname{Reg} A$ denote the greatest regular closed subalgebra which is contained in A. By Theorem 6.65 $\operatorname{Reg} A$ is closed in the spectral radius norm on A and

$$\operatorname{Reg} A \subseteq \operatorname{Dec} A \subseteq A.$$

It is easy to see that if $\operatorname{Dec} A$ is an ideal in A then $\operatorname{Reg} A = \operatorname{Dec} A$. Clearly, if A is regular by Corollary 6.63, then $A = \operatorname{Reg} A = \operatorname{Dec} A$. It is an open problem whether the equality $\operatorname{Reg} A = \operatorname{Dec} A$ hold for any semi-simple commutative Banach algebra A.

Let $M_{\mathrm{d}}(A)$ the set defined by

$$M_{\mathrm{d}}(A) := \{T \in M(A) : T \text{ is decomposable on } A\}.$$

Note that the sums, products, and restrictions of decomposable operators may be far from being decomposable. It is also an open problem whether the sum and the product of two commuting decomposable operators are again decomposable. Anyway, it is possible to show that the set $M_\delta(A)$ of all multipliers $T \in M(A)$ such that T has the property (δ) is a closed full commutative subalgebra of $L(A)$. We shall not give the proof of this result, which is rather involved and requires some technicalities and concepts which have not been introduced in this book. Anyway, the interested reader can find this proof in Laursen and Neumann [**214**, Theorem 4.2.12].

Theorem 6.66. *Let A be a semi-simple commutative Banach algebra. Then we have:*

(i) $M_{\rm d}(A)$ *is a closed full subalgebra of* $M(A)$ *if* A *has a bounded approximate identity;*

(ii) Reg $M(A) \subseteq$ Dec $M(A) \subseteq M_{\rm d}(A)$*;*

(iii) Reg $M_0(A) \subseteq$ Dec $M_0(A) = M_{\rm d}(A) \cap M_0(A) \subseteq M_{00}(A)$*;*

(iv) $M_{\rm d}(A) \cap M_0(A)$ *is a closed subalgebra of* $M_{00}(A)$*;*

(iv) Reg $M_0(A) =$ Dec $M_0(A) = M_{\rm d}(A) \cap M_0(A) = M_{00}(A)$ *if* A *is regular.*

Proof (i) By Theorem 6.57 $M_{\rm d}(A) = M_\delta(A)$. The assertions (ii) and (iii) are immediate consequences of Theorem 6.47, Theorem 6.51, and Theorem 6.60, whilst (iv) follows from Theorem 6.60.

(v) Assume that A is regular. Since $\Delta(A) = \Delta((M_{00}(A))$, also the subalgebra $M_{00}(A)$ is regular and is therefore contained in Reg $M_0(A)$. Obviously this implies that the inclusions of (iii) are identities. ∎

The next corollary improves the result of Theorem 4.25.

Corollary 6.67. *Let* A *be a semi-simple commutative Banach algebra. Then the following statements are equivalent:*

(i) A *is regular and* $\Delta(A) = \Delta(M_0(A))$*;*

(ii) $M_0(A)$ *is regular;*

(iii) A *is regular and* $M_0(A) = M_{00}(A)$*.*

Proof (i)⇒(ii) If A is regular and $\Delta(A) = \Delta(M_0(A))$ then $M_0(A)$ is obviously regular.

(ii)⇒ (iii) If $M_0(A)$ is regular then by the inclusions (iii) in Theorem 6.66 we have $M_0(A) = M_{\rm d}(A) \cap M_0(A) = M_{00}(A)$. In particular, it follows that for each $a \in A$ the multiplication operator $L_a \in M_0(A)$ is decomposable on A. By Corollary 6.63 we conclude that A is regular.

(iii) ⇒(i) From the equality $M_0(A) = M_{00}(A)$ it follows that $\Delta(A) = \Delta(M_{00}(A)) = \Delta(M_0(A))$. ∎

Theorem 6.68. *Let* A *be a semi-simple commutative Banach algebra and* J *a closed ideal of* A *invariant under every* $T \in M(A)$*. Consider a* $T \in M(A)$ *such that* $\widehat{T}$ *is hk-continuous on* $\Delta(M(A))$*. Then the restriction* $S := T|J \in M(J)$ *is decomposable and has a natural spectrum. Moreover, the spectral maximal spaces of* S *are given for all closed subsets* $\Omega \subseteq \mathbb{C}$ *by*

$$J_S(\Omega) = \bigcap_{\lambda \notin \Omega} (\lambda I - T)(J) = Z_S(\Omega) = \{u \in J : \text{ supp } \widehat{u} \subseteq \widehat{T}^{-1}(\Omega)\},$$

where $\Delta(J)$ *is canonically embedded in* $\Delta(A)$ *and* supp $\widehat{u}$ *denotes the support of* $\widehat{u}$ *with respect to the Gelfand topology on* $\Delta(J)$*.*

Proof Let us consider the homomorphism $\Phi : M(A) \to L(J)$ defined by $\Phi(R) := R|J$ for all $R \in M(A)$. Then by Theorem 6.47, since $\widehat{T}$ is *hk*-continuous on $\Delta(M(A))$, the operator $\Phi(T)$ is decomposable, and hence by Theorem 6.49 has a natural spectrum. Since J is an ideal in $M(A)$, by Theorem 4.21 $\Delta(J)$ can be canonically embedded in $\Delta(M(A))$ and $\Delta(M(A)) = \Delta(J) \cap h_{M(A)}(J)$. For each closed subset $\Omega \subseteq \mathbb{C}$ the inclusions

$$J_S(\Omega) \subseteq \bigcap_{\lambda \notin \Omega} (\lambda I - T)(J) \subseteq Z_S(T)$$

have been observed in the proof of part (iii) of Theorem 6.54. To prove the sets coincide let U denote an arbitrary open neighborhood of Ω and choose an open subset V of $\mathbb{C}$ such that $U \cup V = \mathbb{C}$ and $\Omega \cap \overline{V} = \varnothing$. Since $\widehat{T}$ is hull- kernel continuous on $\Delta(M(A))$, there exists a multiplier $R \in M(A)$ such that

$$\sigma(S|\overline{R(J)}) \subseteq U \quad \text{and} \quad \sigma(S|\overline{(I-R)(J)}) \subseteq V$$

(see the proof of Theorem 6.47). By spectral maximality this implies

$$R(J) \subseteq J_S(\overline{U}) \quad \text{and} \quad (I-R)(J) \subseteq J_S(\overline{V}).$$

Moreover, if $u \in Z_S(\Omega)$, from the disjointness of Ω and $\overline{V}$ and the semi-simplicity of J we obtain that

$$(I-R)(u) \in Z_S(\Omega) \cap J_S(\overline{V}) \subseteq Z_S(\Omega) \cap Z_S(\overline{V}) = \{0\}.$$

Hence $u = Ru \in J_S(\overline{U})$, so $Z_S(\Omega) \subseteq J_S(\overline{U})$ and therefore $Z_S(\Omega) \subseteq J_S(\Omega)$, as desired. ∎

Note that the last theorem applies, in addition to the trivial case $J = A$, to each ideal J which is the intersection of regular maximal ideals, since by Theorem 4.15 each ideal of this type is invariant under every $T \in M(A)$. Since every closed ideal in a Banach algebra with an approximate identity, not necessarily bounded, is obviously invariant under all multipliers, Theorem 6.68 does apply to arbitrary closed ideals of the group algebra $L^1(G)$, where G is a locally compact Abelian group.

In the next theorem we add another equivalence, in terms of natural local spectra, to those which have already been established in Theorem 6.60.

Theorem 6.69. *Let A be a semi-simple commutative Banach algebra and $T \in M_0(A)$. Then T is decomposable on A if and only if T has natural local spectra and $\widehat{T}$ is hk-continuous on $\Delta(A)$.*

Proof Suppose that $T \in M_0(A)$ is decomposable, or, equivalently, by Theorem 6.60 that $\widehat{T}$ is hk-continuous on $\Delta(M(A))$. Then also the restriction $\widehat{T}|\Delta(A)$ is hk-continuous and T has natural local spectra by Theorem 6.68.

Conversely, suppose that T has natural local spectra and $\widehat{T}|\Delta(A)$ is hk-continuous on $\Delta(A)$. By Theorem 6.54 the assumption on the natural local spectra implies that $A_T(\Omega) = Z_T(\Omega)$ for all closed subsets $\Omega \subseteq \mathbb{C}$. Moreover, from part (iii) of Theorem 6.54 we know that T has the property (C), and hence by Theorem 2.71 we have $\sigma(T|A_T(\Omega)) = \sigma(T|Z_T(\Omega)) \subseteq \Omega$. Therefore to prove the decomposability of T it suffices to show that the decomposition $A = Z_T(\overline{U}) + Z_T(\overline{V})$ holds for an arbitrary open covering $\{U, V\}$ of $\mathbb{C}$. Without loss of generality we may assume that $0 \in V$.

Now, since V is open there exists an open disc $\mathbb{D}(0, \varepsilon) \subseteq V$. By assumption $T \in M_0(A)$, thus $\varphi_T = \widehat{T}|\Delta(A)$ vanishes at infinity and hence $T^{-1}(\mathbb{C} \setminus V)$ is compact in $\Delta(A)$ with respect to the Gelfand topology. Now, since $\widehat{T}|A$ is hk-continuous on $\Delta(A)$, the two sets $\Delta(A) \setminus T^{-1}(U)$ and $\Delta(A) \setminus T^{-1}(V)$ are disjoint hulls in $\Delta(A)$. By Theorem 4.22 it then follows that there exists an element $e \in A$ such that

$$\widehat{e} \equiv 0 \text{ on } \Delta(A) \setminus T^{-1}(U) \quad \text{and} \quad \widehat{e} \equiv 1 \text{ on } \Delta(A) \setminus T^{-1}(V).$$

This implies that for every $x \in A$ we have

$$\text{supp } \widehat{ex} \subseteq T^{-1}(\overline{U}) \quad \text{and} \quad \text{supp } (\widehat{x - ex}) \subseteq T^{-1}(\overline{V}),$$

so that $ex \in Z_T(\overline{U})$ and $x - ex \in Z_T(\overline{U})$. Therefore every $x \in A$ admits the decomposition

$$x = y + z \quad \text{with} \quad y := xe \in Z_T(\overline{U}), \quad z := x - xe \in Z_T(\overline{V}),$$

so T is decomposable on A. ∎

It has been already observed that given a closed set $\Omega \subseteq \mathbb{C}$ the two ideals $Z_T(\Omega)$ and the ideal $A_T(\Omega)$ generally do not coincide. An example for which this situation occurs is provided by a convolution operator T of $A = L^1(G)$, G a locally compact Abelian group, without a natural spectrum. An example of a such operator may be found in Section 4.10 of [**214**]. Obviously for this operator the two ideals $A_T(\Omega)$ and $Z_T(\Omega)$ cannot coincide for all closed subsets Ω of $\mathbb{C}$, otherwise by Theorem 6.54 T would have a natural spectrum.

We want now show that under rather mild conditions on A, $A_T(\Omega)$ is dense in $Z_T(\Omega)$. First we need the following preliminary result on regular algebras.

Lemma 6.70. *Let A be a commutative regular semi-simple Banach algebra and $x \in A$ for which* supp $\widehat{x}$ *is compact in $\Delta(A)$. Then*

$$\sigma_T(x) = \widehat{T}(\text{supp } \widehat{x}) \quad \text{for all } T \in M(A).$$

Moreover, for all closed subsets $\Omega \subseteq \mathbb{C}$ we have

$$h_A(A_T(\Omega)) = h_A(Z_T(\Omega)) = \overline{\Delta(A) \setminus \widehat{T}^{-1}(\Omega)}. \tag{171}$$

Proof Let $x \in A$ be such that $\widehat{x}$ has compact support in $\Delta(A)$. Since A is regular there exists by Theorem 4.22 an element $z \in A$ such that $\widehat{z} \equiv 1$ on supp $\widehat{x}$. Let us consider the set:

$$J := \{a \in A : \text{ supp } \widehat{a} \subseteq \text{ supp } \widehat{x}\}.$$

Clearly, J is a closed ideal in A which is invariant under all multipliers in $M(A)$ and therefore is an ideal in $M(A)$. Moreover, the multiplier $S := T|J$ satisfies the inclusion $\sigma_T(x) \subseteq \sigma_S(x) \subseteq \sigma(S)$, and if we let $u := Tz$ then $S = L_u|J$, the restriction of L_u to J. Since A is regular, by Corollary 6.63

and Theorem 6.64 it follows that S is decomposable on J, being the restriction of a decomposable multiplication operator on A. Moreover, because J semi-simple, see part (i) of Theorem 5.1, from Theorem 6.51 we obtain that $\sigma(S) = \overline{\widehat{S}(\Delta(J))}$. From the definition of the ideal J it easily follows that $\Delta(J) \subseteq \text{supp}\,\widehat{x}$, where according Theorem 4.21 $\Delta(J)$ is canonically embedded in $\Delta(A)$. Putting together all these results and taking into account that $\widehat{z} = 1$ on supp $\widehat{x}$ we then obtain

$$\sigma_T(x) \subseteq \sigma(S) \subseteq \widehat{a}(\text{supp}\,\widehat{x}) = (\widehat{T}\widehat{z})(\text{supp}\,\widehat{x}) = \widehat{T}(\text{supp}\,\widehat{x}).$$

Since the reverse inclusion $\widehat{T}(\text{supp}\,\widehat{x}) \subseteq \sigma_T(x)$ holds for all $x \in A$, we conclude that $\sigma_T(x) = \widehat{T}(\text{supp}\,\widehat{x})$.

To show that the equality (171) is satisfied, let us consider an arbitrary closed set $\Omega \subseteq \mathbb{C}$. Clearly, from the definition of a hull we have

$$h_A(A_T(\Omega)) \supseteq h(Z_T(\Omega)) \supseteq \Delta(A) \setminus \widehat{T}^{-1}(\Omega)).$$

To show the opposite inclusion let $m \notin \overline{\Delta(A) \setminus \widehat{T}^{-1}(\Omega)}$. Then m lies in the interior of $\widehat{T}^{-1}(\Omega)$, and since $\Delta(A)$ is locally compact the regularity of A entails that there exists some $x \in A$ such that $\widehat{x}(m) = 1$ and supp $\widehat{x}$ is compactly contained in $\widehat{T}^{-1}(\Omega)$. By the first part of the proof we obtain

$$\sigma_T(x) = \widehat{T}(\text{supp}\,\widehat{x}) \subseteq \Omega,$$

and hence $x \in A_T(\Omega)$. Since $m(x) \neq 0$ it then follows that $m \notin h_A(A_T(\Omega))$, and hence the inclusion $\widehat{T}^{-1}(\Omega) \subseteq \overline{\Delta(A) \setminus \widehat{T}^{-1}(\Omega)}$ is proved, so the identities (171) are satisfied. ∎

Theorem 6.71. *Suppose that A is a commutative semi-simple regular Tauberian commutative Banach algebra and has approximate units. If $T \in M(A)$ then*

$$Z_T(\Omega) = \overline{A_T(\Omega)} \quad \textit{for all closed sets } \Omega \subseteq \mathbb{C}.$$

Proof The inclusion $A_T(\Omega) \subseteq Z_T(\Omega)$ has been observed in part (i) of Theorem 6.54 without any assumption on A. Obviously, since $Z_T(\Omega)$ is closed then $\overline{A_T(\Omega)} \subseteq Z_T(\Omega)$.

To show the reverse inclusion let us consider an element $x \in Z_T(\Omega)$ and take $\varepsilon > 0$. Since A has approximate units we can find an element $u \in A$ such that $\|x - ux\| < \varepsilon$. Furthermore, since A is Tauberian there exists an element $v \in A$ such that supp $\widehat{v}$ is compact and $\|u - v\| < \dfrac{\varepsilon}{1 + \|x\|}$. Clearly $\|x - vx\| < 2\varepsilon$ and $vx \in Z_T(\Omega)$. Moreover, $\widehat{vx}$ has compact support, so from Lemma 6.70 we obtain

$$\sigma_T(vx) = \widehat{T}(\text{supp}\,\widehat{vx}) \subseteq \Omega,$$

and therefore $vx \in A_T(\Omega)$. Since $\|x - vx\| < 2\varepsilon$ and ε is arbitrary we then conclude that $x \in \overline{A_T(\Omega)}$. ∎

The next theorem shows that under certain conditions on A the property of having natural local spectra is equivalent to the Dunford property (C). In particular the next result applies to every convolution operator T_μ on $L^1(G)$, with G a locally compact Abelian group.

Theorem 6.72. *Assume that the commutative semi-simple Banach algebra A is regular and Tauberian. If A has an approximate identity and $T \in M(A)$ then the following conditions are equivalent:*

(i) *T has natural local spectra;*

(ii) *T has the property (C).*

Furthermore, if $T \in M_0(A)$ then the conditions (i) *and* (ii) *are equivalent to each one of the conditions* (i)-(vii) *of Theorem 6.60.*

Proof (i) $\Rightarrow$ (ii) By Theorem 6.71 we know that $Z_T(\Omega) = \overline{A_T(\Omega)}$ for all closed $\Omega \subseteq \mathbb{C}$. Since by assumption T has natural local spectra, by Theorem 6.54 we also have $A_T(\Omega) = Z_T(\Omega)$, from which we conclude that $A_T(\Omega)$ is closed for every closed $\Omega \subseteq \mathbb{C}$.

(ii) $\Rightarrow$ (i) By assumption the local spectral subspace $A_T(\Omega)$ is closed for every closed $\Omega \subseteq \mathbb{C}$, so again by Theorem 6.71 $A_T(\Omega) = \overline{A_T(\Omega)} = Z_T(\Omega)$. From part (ii) of Theorem 6.54 we conclude that T has natural local spectra.

The last assertion is clear: if T is decomposable then T has the property (C), by Theorem 6.19. Conversely, if $T \in M_0(A)$ has natural local spectra, from the regularity of A it follows that $\widehat{T}$ is hk-continuous on $\Delta(A)$ and hence, by Theorem 6.69, T is decomposable. ∎

The next result establishes for a multiplier $T \in M(A)$, two formulas for the local spectrum and the local spectral subspace of the corresponding multiplication operator $L_T : M(A) \to M(A)$.

Theorem 6.73. *Let A be a semi-prime commutative Banach algebra and $T \in M(A)$. For every closed set $\Omega \subseteq \mathbb{C}$ we have*

$$\sigma_{L_T}(S) = \overline{\bigcup_{a \in A} \sigma_T(Sa)} \quad \text{for all } S \in M(A), \tag{172}$$

and

$$M(A)_{L_T}(\Omega) = \{S \in M(A) : S(A) \subseteq A_T(\Omega)\}$$

for every closed subset $\Omega \subseteq \mathbb{C}$.

Proof It is easy to verify that for an arbitrary multiplier $S \in M(A)$, the local spectrum of L_T at S contains the set $\bigcup_{a\in A} \sigma_T(Sa)$ as well as its closure.

Conversely, suppose that λ is an interior point of $\bigcap_{a\in A} \rho_T(Sa)$. Then, for some neighborhood $\mathcal{U}$ of λ there is for every $a \in A$ an analytic function $f_a : \mathcal{U} \to A$ which satisfies the identity

$$(\mu I - T) f_a(\mu) = Sa \quad \text{for all } \mu \in \mathcal{U}.$$

Fix $\mu \in \mathcal{U}$ and define the mapping $W_\mu : A \to A$ by

$$W_\mu(a) := f_a(\mu) \quad \text{for all } \ a \in A.$$

Clearly $(\mu L_I - L_T)W_\mu = S$. Hence it only remains to show that W_μ is a multiplier of A and that W_μ depends analytically on $\mu \in \mathcal{U}$.

To prove that $W_\mu \in M(A)$ let us consider two arbitrarily given elements $a, b \in A$. Let $\eta := aW_\mu(b) - W_\mu(a)b$. Clearly

$$(\mu I - T)(\eta) = aS(b) - S(a)b = 0,$$

and hence $\eta \in \ker\,(\mu I - T) \subseteq A_T(\{\mu\}$. On the other hand, we have

$$\sigma_T(W_\mu(a)) = \sigma_T(f_a(\mu)) = \sigma_T(Sa)$$

and

$$\sigma_T(W_\mu(b)) = \sigma_T(f_b(\mu)) = \sigma_T(Sb),$$

which implies $\eta \in A_T(\sigma_T(Sa) \cap \sigma_T(Sb))$. Since $\mu \notin \sigma_T(Sa) \cup \sigma_T(Sb)$ and T has the SVEP we conclude that $\eta = 0$, and hence

$$aW_\mu(b) = W_\mu(a)b \quad \text{for all } a, b \in A,$$

so $W_\mu \in M(A)$. To show that the operator W_μ depends analytically on $\mu \in \mathcal{U}$ fix $\mu \in \mathcal{U}$ and choose $\delta > 0$ such that the closed disc $\mathbf{D}(\lambda, 2\delta)$ is contained in $\mathcal{U}$. Then for each $a \in A$, if we let

$$M_a := \sup_{z \in \mathbf{D}(\lambda, 2\delta)} \|f_a(z)\|,$$

by means of a standard application of Cauchy's formula to the analytic function f_a we obtain the estimate

$$\left\| \frac{(W_\xi(a) - W_\mu(a))}{\xi - \mu} \right\| = \left\| \frac{(f_a(\xi) - f_a(\mu)}{\xi - \mu} \right\| \leq \frac{M_a}{\delta}$$

for every $\xi \in \mathbb{C}$ with $|\xi - \mu| \leq \delta$. By the uniform boundedness principle there exists a constant $M > 0$ such that $\|W_\xi - W_\mu\| \leq M$ and hence

$$\|W_\xi - W_\mu\| \leq M|\xi - \mu| \quad \text{for all } \xi \in \mathbb{C} \text{ with } \ |\xi - \mu| \leq \delta$$

This shows that the function $\xi \to W_\xi$ is norm continuous at μ. From the classical Morera Theorem we conclude that the function $\mu \to W_\mu$ is analytic on $\mathcal{U}$. Hence $\mathcal{U}$ is contained in the local resolvent of L_T at S, and this completes the proof of (172).

The last assertion follows immediately from the description of the local spectra of L_T. ∎

6. Riesz multipliers

We wish now to find conditions for which a decomposable operator T is a Riesz operator. In order to do this we first need a general result on Banach spaces.

For a given operator $T \in L(X)$ on a Banach space X let

$$Z(T) := \{S \in L(X) : TS = ST\}$$

be the commutant of T. Evidently $Z(T)$ is a closed subalgebra of $L(X)$. If $L_T : Z(T) \to Z(T)$ is the operator of multiplication by T on $Z(T)$ it is easily seen that $\sigma(T) = \sigma(L_T)$.

Lemma 6.74. *If $T \in L(X)$, where X is a Banach space, then T is Riesz if and only if L_T is Riesz.*

Proof Obviously $\lambda \neq 0$ is an isolated point of $\sigma(T)$ if and only if it is an isolated point of $\sigma(L_T)$. In this case the spectral projection $P(\lambda, L_T)$ of L_T associated with the spectral set $\{\lambda\}$ is related to the spectral projection $P(\lambda, T)$ of T associated with $\{\lambda\}$ by the identity

$$P(\lambda, L_T)(S) = SP(\lambda, T) \quad \text{for all } S \in Z(T).$$

Now, if $S \in Z(T)$ then S commutes with $P(\lambda, T)$, so $X = X_1 \oplus X_2$ with $X_1 := P(\lambda, T)(X)$ and $X_2 := \ker P(\lambda, T)$. Write $T = T_1 \oplus T_2$, $S = S_1 \oplus S_2$ and $P(\lambda, T)S = S_1 \oplus W_2$, where $S_1 \in Z(T_1)$. Therefore

$$P(\lambda, L_T)(Z(T)) = \{P(\lambda, T)S : S \in Z(T)\} = \{S_1 \oplus W_2 : S_1 \in Z(T_1)\}.$$

If T is a Riesz operator then $0 \neq \lambda$ is an isolated point of $\sigma(T)$ and dim $(X_1) < \infty$. From this it follows that

$$\dim\, (Z(T_1)) \leq \dim\, L(X_1) < \infty,$$

so dim $P(\lambda, L_T)(Z(T)) < \infty$, and hence by Theorem 3.111, L_T is a Riesz operator on $Z(T)$.

Conversely, let L_T be a Riesz operator on $Z(T)$ and $0 \neq \lambda \in \sigma(T)$. Then λ is an isolated point of $\sigma(L_T)$ and

$$\dim\, P(\lambda, L_T)(Z(T)) = \dim\, (Z(T_1)) < \infty.$$

Since the algebra generated by T_1 is contained in $Z(T_1)$ it must be finite-dimensional, so there exists a non-zero polynomial p such that $p(T_1) = 0$. But $\sigma(T_1) = \{\lambda\}$, so $(\lambda I - T_1)^k = \{0\}$ for some $k \in \mathbb{N}$. If ker $(\lambda I - T_1)$ is infinite-dimensional then it contains an infinite linearly independent set $\{x_n\}$ for which $T_1 x_n = \lambda x_n$ for all $n \in \mathbb{N}$. Moreover, there exists $0 \neq f \in X_1{}^*$ such that $T^* f = \lambda f$. From this it easily follows that the infinite linearly independent set $\{f \otimes x_n\}$ of finite rank-one operators lies in $Z(T_1)$. Hence ker $(\lambda I - T_1)$ is finite-dimensional, so ker $(\lambda I - T_1)^k = X_1$ and therefore T is a Riesz operator. ∎

We shall see now that several versions of the multiplication operator L_T acting on $M(A)$, $M_0(A)$, and $M_{00}(A)$ as a Riesz operator are all equivalent.

Theorem 6.75. *Let A be a semi-simple commutative Banach algebra and $T \in M(A)$. Then the following assertions are equivalent:*

(i) *T is a Riesz operator on A;*

(ii) *The multiplication operator L_T is a Riesz operator on $M(A)$;*

(iii) *$T \in M_0(A)$ and $L_T : M_0(A) \to M_0(A)$ is a Riesz operator;*

(iii) *$T \in M_{00}(A)$ and $L_T : M_{00}(A) \to M_{00}(A)$ is a Riesz operator.*

Proof (i) $\Leftrightarrow$ (ii) If $T \in M(A)$ then $M(A)$ is a closed L_T-invariant subspace of $Z(T)$, and since L_T is a Riesz operator on $Z(T)$ its restriction on $M(A)$ is still a Riesz operator, by part (iii) of Theorem 3.113.

Conversely, if L_T is a Riesz operator on $M(A)$, by Theorem 6.31 L_T is then super-decomposable on $M(A)$, and by Theorem 6.34 the spectral subspace $M(A)_{L_T}(\Omega)$ is finite-dimensional for each closed set $\Omega \subseteq \mathbb{C} \setminus \{0\}$. From the inclusion $A_T(\Omega) \subseteq M(A)_{L_T}(\Omega)$ we then deduce that also $A_T(\Omega)$ is finite-dimensional for each closed set $\Omega \subseteq \mathbb{C} \setminus \{0\}$. Moreover, the super-decomposability of L_T on $M(A)$ yields by Theorem 6.51 that T is decomposable operator, from which, again by Theorem 6.34, we conclude that T is a Riesz operator on A.

(i) $\Rightarrow$ (iii) Suppose that T is a Riesz operator on A or, equivalently, that L_T is a Riesz operator on $M(A)$. Then each $\lambda \in \sigma(T) \setminus \{0\}$ is an isolated point of the spectrum and the ideal $J := (\lambda I - T)(A)$ is finite-codimensional. By Theorem 4.38 and Corollary 4.39

$$h_A(J) = \{m \in \Delta(A) : \widehat{T}(m) = \lambda\} = \widehat{T}^{-1}(\{\lambda\}) \cap \Delta(A)$$

is then a finite set. Let Ω be a closed set of $\mathbb{C}$ such that $0 \notin \Omega$. Evidently $\Omega \cap \sigma(T)$ is a finite set, and we can consider an open subset U of $\mathbb{C}$ such that $\Omega \cap \sigma(T) = U \cap \sigma(T)$. Since $\widehat{T}(\Delta(A)) \subseteq \sigma(T)$, by part (ii) of Theorem 7.79 it then follows that

$$\widehat{T}^{-1}(\Omega) \cap \Delta(A) = \widehat{T}^{-1}(U) \cap \Delta(A),$$

from which we conclude that $\widehat{T}^{-1}(\Omega) \cap \Delta(A)$ is a finite set of isolated points in $\Delta(A)$. This shows, in particular, that $T \in M_0(A)$.

It remains to prove that L_T is a Riesz operator on $M_0(A)$. This is a consequence of part (iii) of Theorem 3.113, since $L_T : M_0(A) \to M_0(A)$ is the restriction of the Riesz operator L_T acting on $M(A)$.

(iii) $\Rightarrow$ (iv) Since L_T is decomposable on $M_0(A)$, by part (iii) of Theorem 6.64 the two operators L_T and T are then decomposable on $M_{00}(A)$ and A, respectively. Moreover, as $T \in M_0(A)$, from Theorem 6.60 we conclude that $T \in M_{00}(A)$.

(iv) $\Rightarrow$ (i) We have $A \subseteq M_{00}(A)$ so part (iii) of Theorem 6.64 entails that T is decomposable operator on A. Since L_T is a Riesz operator on $M_{00}(A)$, from Theorem 6.34 we then obtain that the local spectral subspace $M_{00}(A)_{L_T}(\Omega)$ is finite-dimensional for each closed set $\Omega \subseteq \mathbb{C} \setminus \{0\}$. From the inclusion $A_T(\Omega) \subseteq M(A)_{L_T}(\Omega)$ we conclude that also $A_T(\Omega)$ is finite-dimensional for every closed set $\Omega \subseteq \mathbb{C} \setminus \{0\}$. Since T is a decomposable operator this last fact implies, again by Theorem 6.34, that T is a Riesz operator. ∎

Another useful characterization of Riesz multipliers is given by the next theorem.

Theorem 6.76. *Let A be a semi-simple commutative Banach algebra and let $T \in M(A)$. Then T is a Riesz operator on A precisely when $T \in$*

$M_{00}(A)$, and for each closed subset Ω contained in $\mathbb{C}\setminus\{0\}$ the set $\widehat{T}^{-1}(\Omega)\cap\Delta(A)$ is a finite set of isolated points in $\Delta(A)$.

Proof From Theorem 6.75 we know that if T is a Riesz operator on A then $T\in M_{00}(A)$. Moreover, from the proof of the implication (i) $\Rightarrow$ (iii) we see that for each closed subset Ω contained in $\mathbb{C}\setminus\{0\}$ the set $\widehat{T}^{-1}(\Omega)\cap\Delta(A)$ is finite.

Conversely, suppose that for $T\in M_{00}(A)$ the set $\widehat{T}^{-1}(\Omega)\cap\Delta(A)$ is finite for every closed subset Ω contained in $\mathbb{C}\setminus\{0\}$. To show that T is a Riesz operator it suffices, by Theorem 6.75, to show that the multiplication operator L_T on $M(A)$ is a Riesz operator. Since $\sigma(T)\subseteq\widehat{T}(\Delta(A))\cup\{0\}$ for each $T\in M_{00}(A)$, our assumption entails that $\sigma(T)$ is countable. From the equality $\sigma(T)=\sigma(L_T)$ and from Theorem 6.31 we then obtain that L_T is super-decomposable on $M(A)$.

We show now that the space $M(A)_{L_T}(\Omega)$ is finite-dimensional for every closed subset Ω contained in $\mathbb{C}\setminus\{0\}$. Note first that our assumption implies that $\widehat{T}^{-1}(\Omega)\cap\Delta(A)$ is both compact and open in the Gelfand topology of $\Delta(A)$. By Theorem 4.22 then we can find an element $u\in A$ such that

$$\widehat{u}\equiv 1 \text{ on } \widehat{T}^{-1}(\Omega)\cap\Delta(A) \quad\text{and}\quad \widehat{u}\equiv 0 \text{ on } \Delta(A)\setminus\widehat{T}^{-1}(\Omega).$$

From Theorem 6.73 we know that if $S\in M(A)_{L_T}(\Omega)$ then $S(A)\subseteq A_T(\Omega)\subseteq Z_T(\Omega)$, so that

$$\operatorname{supp}\widehat{Sx}\subseteq\widehat{T}^{-1}(\Omega)\cap\Delta(A)\quad\text{for all } x\in A.$$

From this we obtain that $\widehat{Sx}=\widehat{uSx}$, and since A is semi-simple it then follows that $Sx=uSx=(Su)x$ for all $x\in A$. Hence S is the multiplication operator by Su. From the inclusion $A\subseteq M(A)$ we then obtain that $M(A)_{L_T}(\Omega)\subseteq Z_T(\Omega)$. Finally, since $\widehat{T}^{-1}(\Omega)\cap\Delta(A)$ is finite and the Gelfand transform is injective, $Z_T(\Omega)$ is then finite-dimensional. Therefore the subspace $M(A)_{L_T}(\Omega)$ is finite-dimensional for every closed subset Ω contained in $\mathbb{C}\setminus\{0\}$, and the decomposability of L_T then implies by Theorem 6.34 that L_T is a Riesz operator on $M(A)$. ∎

Let us consider the important case that $T\in M_0(A)$ for a semi-simple commutative Banach algebra having a discrete maximal regular ideal space. We see now that to the equivalent conditions of Theorem 6.60 we can add some other significant conditions.

Theorem 6.77. *Let A be a semi-simple Banach algebra for which $\Delta(A)$ is discrete. If $T\in M_0(A)$ then the following assertions are equivalent:*

(i) *T is decomposable;*

(ii) *T has natural local spectra;*

(iii) *T has natural spectrum;*

(iv) *$\sigma(T)$ is countable;*

(v) *T is a Riesz operator;*

(vi) *Every* $0 \neq \lambda \in \sigma(T)$ *is a simple pole of* $R(\lambda, T)$*;*

(vii) $T \in M_{00}(A)$.

Proof Observe first that the algebra A is regular, since $\Delta(A)$ is discrete, so the Gelfand topology and the hk-topology coincide on $\Delta(A)$. Therefore $\widehat{T}$ is hk-continuous and from Theorem 6.69 it follows that the equivalence (i) $\Leftrightarrow$ (ii) holds. The equivalence (i) $\Leftrightarrow$ (vii) is clear from Corollary 6.61, while the implication (i) $\Rightarrow$ (iii) follows from Theorem 6.49.

To show the implication (iii) $\Rightarrow$ (iv) assume that T has a natural spectrum. Since $T \in M_0(A)$, by Theorem 7.79 we have

$$\sigma(T) = \overline{\widehat{T}(\Delta(A))} = \widehat{T}(\Delta(A) \cup \{0\}.$$

For every $n \in \mathbb{N}$ let $\Omega_n := \{m \in \Delta(A) : |\widehat{T}(m)| \geq 1/n\}$. Clearly Ω_n is compact and countable, hence $\widehat{T}(\Omega_n)$ is a countable subset of $\mathbb{C}$. Since

$$\widehat{T}(\Delta(A)) \setminus \{0\} = \bigcup_{n \in \mathbb{N}} \widehat{T}(\Omega_n)$$

we then conclude that $\widehat{T}(\Delta(A))$ is countable and hence also the spectrum $\sigma(T)$ is countable.

Finally, by Theorem 6.49 we have (iv) $\Rightarrow$ (i) so the statements (i)–(iv) are equivalent. From Theorem 3.111 we have (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv), whilst the implication (vii) $\Rightarrow$ (v) easily follows from Theorem 6.76, since $\Delta(A)$ is discrete. ∎

7. Decomposable convolution operators

In this section we shall apply the general results established in the preceeding sections to convolution operators on group algebras. Consider an arbitrary locally compact Abelian group G and let $\mathcal{M}(G)$ denote the measure algebra of all complex regular Borel measures. We know that the group algebra $A := L^1(G)$ is a commutative semi-simple Banach algebra with a bounded approximate identity and $\Delta(L^1(G) = \widehat{G}$, $\widehat{G}$ the dual group of G. Moreover, the multiplier algebra $M(A)$ may be identified by convolution with the semi-simple Banach algebra $\mathcal{M}(G)$. By Theorem 4.32, for every $\mu \in \mathcal{M}(G)$ the convolution operator T_μ has the SVEP and hence

$$\sigma(\mu) = \sigma(T_\mu) = \sigma_{\rm su}(T_\mu),$$

whilst from Theorem 5.88, $L^1(G)$ being regular and Tauberian we obtain

$$\sigma(\mu) = \sigma_{\rm ap}(T_\mu) = \sigma_{\rm se}(T_\mu).$$

As was observed after Theorem 5.54, if G is compact, we also have

$$\sigma_{\rm r}(T) = \varnothing \quad \text{and} \quad \sigma_{\rm p}(T_\mu) = \widehat{\mu}(\widehat{G}).$$

The ideal $M_0(A)$, if $A = L^1(G)$, becomes the ideal $\mathcal{M}_0(G)$ of all measures $\mu \in \mathcal{M}(G)$ whose Fourier–Stieltjes transforms $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ vanish at infinity. Again, $M_{00}(A)$ is identified with the ideal $\mathcal{M}_{00}(G)$ of all measures $\mu \in \mathcal{M}(G)$

whose Fourier–Stieltjes transforms $\widehat{\mu} : \Delta(\mathcal{M}(G) \to \mathbb{C}$ vanish outside $\widehat{G}$. Finally, identifying, via the Radom–Nykodim theorem, $L^1(G)$ with the ideal $\mathcal{M}_a(G)$ of all measures on G which are absolutely continuous with respect to the Haar measure on G we have

$$L^1(G) = \mathcal{M}_a(G) \subseteq \mathcal{M}_{00}(G) \subseteq \mathcal{M}_o(G) \subseteq \mathcal{M}(G).$$

Compact and Riesz convolution operators are precisely described if the group G is compact. In fact, from Corollary 5.62 we have

$$T_\mu \text{ is compact} \Leftrightarrow \mu \in \mathcal{M}_a(G),$$

whilst from Theorem 6.77

$$T_\mu \text{ is Riesz} \Leftrightarrow \mu \in \mathcal{M}_{00}(G).$$

For an arbitrary measure $\mu \in \mathcal{M}(G)$ the Browder spectrum $\sigma_{\mathrm{b}}(T_\mu)$ and the Weyl spectrum $\sigma_{\mathrm{w}}(T_\mu)$ coincide, since T_μ has the SVEP. If G is compact and $\mu \in \mathcal{M}_0(G)$, to the description of these spectra given in Corollary 5.113 for an arbitrary measure, we may add, by virtue of Theorem 6.77, the following characterizations

$$\begin{aligned}\sigma_{\mathrm{w}}(T_\mu) &= \sigma_{\mathrm{b}}(T_\mu) = \bigcap_{\nu \in \mathrm{Dec}\,\mathcal{M}(G)} \sigma(\mu+\nu) \\ &= \bigcap \{\sigma(\mu+\nu) : \sigma(\nu) \text{ is countable}\}.\end{aligned}$$

Specializing the results of Theorem 7.79, Theorem 6.51, and Theorem 6.68 to the special case $T_\mu : \mathcal{M}(G) \to \mathcal{M}(G)$ we also obtain:

Theorem 6.78. *For every $\mu \in \mathcal{M}(G)$ the convolution operator T_μ on $\mathcal{M}(G)$ is decomposable if and only if $\widehat{\mu}$ is hk-continuous on $\Delta(\mathcal{M}(G))$. In this case, if $A = L^1(G)$ the restriction $T_\mu|A$ is also decomposable and*

$$\sigma_{\mathcal{M}(G)}(\mu) = \sigma(T_\mu) = \sigma(T_\mu|L^1(G)) = \overline{\widehat{\mu}(\widehat{G})}.$$

Moreover, for every closed subset $\Omega \subseteq \mathbb{C}$ the local spectral subspaces of $T_\mu|A$ are given by

$$A_{T_\mu|A}(\Omega) = \bigcup_{\lambda \notin \Omega} [(\lambda\delta_o - \mu) \star L^1(G)] = \{f \in L^1(G) : \mathrm{supp}\ \widehat{x} \subseteq \widehat{\mu}^{-1}(\Omega)\},$$

where δ_0 denotes the Dirac measure concentrated at the identity and the Fourier transforms are taken with respect to the dual group $\widehat{G}$.

Theorem 6.78 provides us an useful tool for giving classes of measures for which the corresponding convolution operators is decomposable on $\mathcal{M}(G)$ or $L^1(G)$. Note that since $L^1(G)$ is regular then by Theorem 6.66 $\mathcal{M}_{00}(G)$ is a closed regular algebra of $\mathcal{M}(G)$, and hence, again by Theorem 6.66, the following inclusions always hold

$$\mathcal{M}_{00}(G) \subseteq \mathrm{Reg}\ (\mathcal{M}(G)) \subseteq \mathrm{Dec}\,(\mathcal{M}(G)).$$

Suppose now that H is a closed subgroup of G. Then $L_1(H)$ may be canonically identified with the space of all measures on G which are concentrated on H and absolutely continuous with respect the Haar measure on H. From this it follows that $L_1(H)$ is a closed regular subalgebra of $\mathcal{M}(G)$, thus

$$L_1(H) \subseteq \text{Reg } (\mathcal{M}(G)) \subseteq \text{ Dec}\,(\mathcal{M}(G)) \subseteq M_{\rm d}(\mathcal{M}(G)),$$

where $M_{\rm d}(\mathcal{M}(G))$ denotes the set of all decomposable convolutions operators.Observe that if H is a non-trivial subgroup of G then all measures in $L_1(H)$ are singular with respect to the Haar measure on G. Hence Reg $(\mathcal{M}(G))$ and therefore Dec$\,(\mathcal{M}(G))$ may contain singular measures. Also the subalgebra $\mathcal{M}_{\rm dis}(G)$ of all discrete measures on G is contained in Dec$\,(\mathcal{M}(G))$. Indeed, $\mathcal{M}_{\rm dis}(G)$ may be canonically identified with the regular closed subalgebra $L_1(G_{\rm disc})$ of $\mathcal{M}(G)$, where $G_{\rm dis}$ is G provided with the discrete topology. Since Dec$\,(\mathcal{M}(G))$ is a subalgebra then

$$\mathcal{M}_{00}(G) + L_1(H) + \mathcal{M}_{\rm dis}(G) \subseteq \text{ Dec}\,(\mathcal{M}(G)).$$

By Theorem 6.78 we then have:

Corollary 6.79. *For every $\mu \in \mathcal{M}_{00}(G) + L_1(H) + \mathcal{M}_{\rm dis}(G)$, the operator T_μ is decomposable both on $\mathcal{M}(G)$ and $L^1(G)$ with $\sigma(T_\mu) = \sigma(T_\mu|L^1(G)) = \overline{\widehat{\mu}(\widehat{G})}$.*

Observe that Corollary 6.79 applies to all measures whose continuous part is absolutely continuous. Hence Corollary 6.79 extends classical results established by Beurling [**67**] in the case $G = \mathbb{R}$ and of Hartman [**155**] for the circle group $G = \mathbb{T}$. Although Corollary 6.79 identifies some classes of measures $\mu \in \mathcal{M}(G)$ for which the corresponding convolution operator T_μ is decomposable, a full characterization of all these measures is still missing. Now we turn to the subalgebra $\mathcal{M}_o(G)$. The following result is an immediate consequence of Corollary 6.60, Corollary 6.61 and Theorem 6.72.

Theorem 6.80. *Let G be a locally compact Abelian group. For every $\mu \in \mathcal{M}_o(G)$ the following statements are equivalent:*

(i) $T_\mu : \mathcal{M}(G) \to \mathcal{M}(G)$ *is decomposable;*

(ii) $T_\mu : \mathcal{M}(G) \to \mathcal{M}(G)$ *is super-decomposable;*

(iii) $T_\mu : L^1(G) \to L^1(G)$ *is decomposable;*

(iv) $T_\mu : L^1(G) \to L^1(G)$ *is super-decomposable;*

(v) $\widehat{\mu}$ *is hk-continuous on* $\Delta(\mathcal{M}(G))$*;*

(vi) $\mu \in \mathcal{M}_{00}(G)$*;*

(vii) $T_\mu : L^1(G) \to L^1(G)$ *has local natural spectra;*

(viii) $T_\mu : L^1(G) \to L^1(G)$ *has the property* (δ)*;*

(ix) $T_\mu : L^1(G) \to L^1(G)$ *has the property* (C)*.* ∎

If G is compact, by virtue of Theorem 6.77 the list of equivalent conditions of Theorem 6.80 may be extended as follows:

Theorem 6.81. *Let G be a compact Abelian group. For every $\mu \in \mathcal{M}_o(G)$ the conditions* (i)–(ix) *are equivalent to each one of the following statement:*

(i) *μ has natural spectrum, i.e., $\sigma(\mu) = \overline{\widehat{\mu}(\widehat{G})}$;*

(ii) *$\sigma(\mu)$ is countable;*

(iii) *T_μ is a Riesz operator;*

(iv) *Every $0 \neq \lambda \in \sigma(T_\mu)$ is a simple pole of $R(\lambda, T_\mu)$.* ∎

We know that if the group G is discrete then the algebra $L^1(G)$ has an identity. In this case the measure algebra $\mathcal{M}(G)$ may be identified with $L^1(G)$, so that all convolution operators T_μ are decomposable by Corollary 6.63. By Theorem 6.49 T_μ then has natural spectrum whenever G is discrete.

From Theorem 6.81 we see that for every measure $\mu \in \mathcal{M}_o(G)$ with natural spectrum the corresponding convolution operator T_μ is decomposable. For non-compact groups this is no longer true in general. In fact, a non-discrete locally compact Abelian group G is not compact if and only if there exists a measure $\mu \in \mathcal{M}_0(G)$ having a natural spectrum without natural local spectra, see Theorem 4.11.9 of [**214**]. Other examples of non-decomposable convolution operators T_μ, with $\mu \in \mathcal{M}_o(G)$, are provided by non-zero measures $\mu \in \mathcal{M}_o(G)$, G a locally compact Abelian group, such that μ^n is singular with respect to the Haar measure for all $n \in \mathbb{N}$, see Theorem 4.11.12 of Laursen and Neumann [**214**].

7.1. Comments. The concept of spectral maximal space has been explicitly introduced for the first time by Foiaş in [**120**]. The work of Foiaş has some ideas and results in common with those of the article [**70**] of Bishop. The spectral maximal space was introduced in order to obtain spectral decompositions for a bounded operator in a Banach space, with respect to subsets of the spectrum. This led to a class of operators which have sufficiently many spectral maximal spaces: the decomposable operators. Hence the original definition of decomposable operator, given by Foiaş in [**120**], involved the concept of spectral maximal space, and this is, for instance, the approach to this theory in the books by Colojoară and Foiaş [**83**] and by Erdelyi and Lange [**105**]. That the notion of decomposable operators can be given, in easier way, in terms of closed invariant subspaces was clarified in 1979 by Albrecht [**38**]. This is also the approach to this theory of the modern text of Laursen and Neumann [**214**], see also the monograph by Vasilescu [**309**]. Other equivalent conditions for the decomposability may be found in Jafarian and Vasilescu [**173**], Radjabalipour [**269**], Lange [**195**], Lange and Wang [**196**]. The decomposability of an operator may be also characterized by means of the concept of spectral capacity, see for instance Theorem 1.2.23 of Laursen and Neumann [**214**].

All the second section of our book is modeled after Albrecht [**38**]. The property (β), which was introduced by Bishop [**70**], plays a remarkable role in local spectral theory. The fact that the decomposability of an operator $T \in L(X)$, X a Banach space, is the conjuction of the weaker conditions (β) and (δ) has been observed in Albrecht et al. [**41**]. That the properties (β) and (δ) are the dual of each other has been proved by Albrecht, and Eschmeier [**42**]. The remarkable characterization of property (β) is that $T \in L(X)$ has the property (β) if and only if the restriction of T to every closed invariant subspace is decomposable, as well as the dual result that the property (δ) characterizes the quotients of decomposable operators by closed invariant subspaces, is owed to Albrecht and Eschmeier [**42**]. It should be noted that recently Eschmeier [**111**] has proved that if $T \in L(X)$ satisfies the SVEP then T has the property (β) on the Fredholm resolvent $\rho f(T)$. Here T is said to have the property (β) on an open set $\mathcal{U}$ if the mapping defined as in (159) is injective with closed range for each open subset $\mathcal{V}$. The proof of this result uses tools from the theory of analytic sheaves.

The class of super-decomposable operators was introduced by Laursen and Neumann in [**209**], from which almost all the material of the third section is taken. Further elements of the theory of super-decomposable operators may be found in Albrecht et al. [**41**], Erdely and Wang [**106**], Rădulescu and Vasilescu [**270**]. That an operator $T \in L(X)$ is super-decomposable if and only if T is decomposable and $L(X)$ is normal with respect to T was proved by Apostol [**47**]. Theorem 6.33 is owed to Eschmeier [**109**].

The local spectral characterization of Riesz operators established in Theorem 6.34 is taken from Aiena and Laursen [**28**] while the remaining part of the section on decomposable right shift operators is modeled after T. L. Miller, V. G. Miller, and Neumann [**238**].

Theorem 6.47 is owed to Neumann [**243**]. The problem of decomposability for a multiplier has been first considered by Colojoară and Foiaş in [**83**]. For a general locally compact Abelian group they considered the multiplication operator given by a fixed element of a regular semi-simple commutative Banach algebra of which the group algebra $L^1(G)$ is an example. They showed that any such multiplication operator is decomposable. Colojoară and Foiaş also posed the problem of describing all the measures $\mu \in \mathcal{M}(G)$ for which the corresponding convolution operator T_μ is decomposable.

In 1982 Albrecht [**36**] and Eschemeier [**109**] independently showed that if G is non-discrete there always exists a non decomposable convolution operator. Moreover, they showed that the convolution operator corresponding to any measure on G whose continuous part is absolutely continuous with respect to the Haar measure on G is decomposable on $L^1(G)$. Some other results in this direction have been proved by Laursen and Neumann [**210**], but untill now a measure-theoretic characterization of decomposable convolution operators T_μ on group algebras is still missing.

The part concerning the hk-topology on $\Delta(A)$ contains standard material from Banach algebra theory, except for Theorem 6.47 which is taken from Neumann [**243**]. The part concerning the property (δ) in the framework of multiplier theory is modeled after Laursen and Neumann [**211**], but some ideas, in particular those concerning the result of Theorem 6.49, traced back to Albrecht [**40**] and Eschmeier [**109**]. The investigation of Zafran [**330**] of the spectra of the multipliers of convolution operators on $L^p(G)$ dates back to 1973. He considered the problem of characterizing those measures $\mu \in \mathcal{M}(G)$, G a locally compact Abelian group, which induce convolution operators T_μ on $L^1(G)$ having a natural spectrum. He also found an example of a measure $\mu \in \mathcal{M}(G)$ which has a non-natural spectrum.

All the material of the section concerning the relationships between the decomposability of a multiplier $T \in M(A)$, the hk-continuity of its Gelfand transform on $\Delta(A)$ and $\Delta(M(A))$, and the property of having a natural spectrum, is entirely based on the paper of Laursen and Neumann[**206**].

That every multiplication operator on a regular semi-simple commutative Banach algebra is decomposable was first noted by Colojoară and Foiaş in [**83**]. Later it was observed by Frunza [**118**] that the decomposability of all multiplication operators characterizes the regularity of a semi-simple commutative Banach algebra. To Neumann [**243**] is owed the improvement of these results, given here in Corollary 6.62, which showed that the decomposability of a multiplication operator $L_a : A \to A$, $a \in A$, on an arbitrary semi-simple commutative Banach algebra A is equivalent to the hk-continuity of the Gelfand transform $\widehat{a} : \Delta(A) \to \mathbb{C}$. In this paper there are also proved the related results contained in Corollary 6.63, Theorem 6.64. The existence of a greatest regular subalgebra was discovered by Albrecht [**38**] in the semi-simple case. The proof here given in Theorem 6.65 is that given in Neumann [**243**].

The concept of local natural spectra was introduced by Eschmeier, Laursen, and Neumann [**112**]. In this paper one can find all the material developed in the section of multipliers with natural local spectra, except for Theorem 6.68, which was proved in Laursen and Neumann [**210**]. The result on the section of Riesz multipliers are modeled after Aiena [**5**], Aiena and Laursen [**28**], Laursen and Neumann [**209**] and Eschmeier, Laursen, and Neumann [**112**]. The characterization of Riesz multipliers given in Theorem 6.76 was established by Neumann [**246**].

The last section concerning the decomposability of a convolution operator is taken from Laursen and Neumann [**210**] and extends results of Albrecht [**40**], Laursen and Neumann [**209**]. The characterization of Riesz convolution operators given in Theorem 6.81 was first obtained in Aiena [**5**]. All these results are strongly influenced by the work of Zafran [**330**] which introduced the concept of natural spectrum for convolution operators.

CHAPTER 7

Perturbation classes of operators

In this chapter we concentrate our attention on some perturbation classes of operators which occur in Fredholm theory. In this theory we find two fundamentally different classes of operators: semi-groups, such as the class of Fredholm operators, the classes of upper and lower semi-Fredholm operators, and ideals of operators, such as the classes of finite-dimensional, compact operators.

A perturbation class associated with one of these semi-groups is a class of operators T for which the sum of T with an operator of the semi-group is still an element of the semi-group. A paradigm of a perturbation class is, for instance, the class of all compact operators: on adding a compact operator to a Fredholm operator we obtain a Fredholm operator. For this reason the perturbation classes of operators are often called classes of *admissible* perturbations. In the first section of this chapter we shall see that the class of inessential operators is the class of *all* admissible perturbations, since it is the biggest perturbation class of the semi-group of all Fredholm operators, as well as the semi-groups of left Atkinson or right Atkinson operators. These results will be established in the general framework of operators acting between two different Banach spaces. Moreover, we shall see that the class of inessential operators $I(X, Y)$ presents also an elegant duality theory. In fact every operator $T \in I(X, Y)$ may be characterized either in terms of nullity α or, alternatively, in terms of the deficiency β.

In the second section we shall introduce two classes of operators $\Omega_+(X)$ and $\Omega_-(X)$ which are in a sense the dual of each other, and we shall see that the ideal $I(X)$ is the uniquely determined maximal ideal of $\Omega_+(X)$ operators, or, dually, of $\Omega_-(X)$ operators. Moreover, every Riesz operator belongs to each of the classes $\Omega_+(X)$ and $\Omega_-(X)$, and the Riesz operators are precisely $\Omega_+(X)$ operators, dually $\Omega_-(X)$ operators, for which the spectrum is finite or a sequence with 0 as unique cluster point.

The third section addresses the study of two important classes of operators, the class $SS(X, Y)$ of all strictly singular operators and the class $CS(X, Y)$ of all strictly cosingular operators. Both these classes of operators are contained in $I(X, Y)$. If the sets $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ are non-empty these two classes are contained, respectively, in the perturbation class $P\Phi_+(X, Y)$ of upper semi-Fredholm and in the perturbation class $P\Phi_-(X, Y)$ of lower semi-Fredholm operators. For many classical Banach spaces these inclusions are actually equalities, but we shall also give a recent example of González [**131**] which shows that these inclusions are, in general, proper. This counterexample, which solves an old open problem in operator theory, is constructed by considering the Fredholm theory of a very special class of Banach spaces: the class of indecomposable Banach spaces. An indecomposable Banach space is a Banach space which cannot be split into the direct sum of two infinite-dimensional closed subspaces. The existence of infinite-dimensional indecomposable Banach spaces has been a long standing open problem and was raised by Banach in the early 1930s. This problem has been positively solved by Gowers and Maurey, who constructed in [**137**] an example of reflexive hereditarily indecomposable Banach space. We also give several examples of Banach spaces X and Y for which the equalities $SS(X,Y) = P\Phi_+(X,Y)$ and $SC(X,Y) = P\Phi_-(X,Y)$ hold.

The fourth section is devoted to the improjective operators $\mathrm{Imp}(X,Y)$ between Banach spaces. This class of operators contains $I(X,Y)$ for all Banach spaces, and it has been for several years an open problem whether the two classes coincide. We shall give the recent counter example given by Aiena and González [**21**], which shows that if Z is an indecomposable Banach space Z which is neither hereditarily indecomposable nor quotient hereditarily indecomposable, then $I(Z)$ is properly contained in Imp(Z). Again Gowers and Maurey [**137**] provide an example of a such Banach space. We also see that for the most classical Banach spaces the improjective operators coincide with the inessential operators, giving in such a way for these space an intrinsic characterization of inessential operators. In the last section we shall briefly discuss two notions of incomparability of Banach spaces which originate from the class of all inessential and improjective operators. In particular, from the theory of indecomposable Banach spaces we shall see disprove that these two notions of incomparability do not coincide.

1. Inessential operators between Banach spaces

In Chapter 5 we have defined the inessential ideal $I(X)$ (or Riesz ideal) of operators on a complex Banach space X, as

$$I(X) := \pi^{-1}(\mathrm{rad}\ (L(X)/F(X)).$$

where $\pi : L(X) \to L(X)/F(X)$ denotes the canonical quotient mapping. By the Atkinson characterization of Fredholm operators, $T \in L(X)$ is a Fredholm operator if and only if T is invertible modulo the ideal $F(X)$ of all finite-dimensional operators, or, equivalently, modulo the ideal $K(X)$ of all compact operators, see Section 5 of Chapter 5. From the characterization (133) of the radical of an algebra we then obtain

$$\begin{aligned} (173) \qquad I(X) &= \{T \in L(X) : I - ST \in \Phi(X) \text{ for all } S \in L(X)\} \\ &= \{T \in L(X) : I - TS \in \Phi(X) \text{ for all } S \in L(X)\}. \end{aligned}$$

Taking in (173) $S = \lambda I$, with $\lambda \neq 0$, we obtain that $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$, so every inessential operator is a Riesz operator.

Note that by the results of Section 5 of Chapter 5 the set of Fredholm operators $\Phi(X)$ coincides with the class of all operators $T \in L(X)$ invertible modulo every ideal of operators $J(X)$ such that $F(X) \subseteq J(X) \subseteq I(X)$.

Recall that if X and Y are Banach spaces the class of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) := \{T \in L(X,Y) : \alpha(T) := \dim \ker T < \infty,\ T(X) \text{ closed}\},$$

whilst the class of lower semi-Fredholm operators is defined as

$$\Phi_-(X,Y) := \{T \in L(X,Y) : \beta(T) := \operatorname{codim} T(X) < \infty\}.$$

Observe that in the case $X = Y$ the class $\Phi(X)$ is non-empty since the identity trivially is a Fredholm operator. This is a substantial difference from the case in which X and Y are different. In fact, if $T \in \Phi(X,Y)$ for some infinite-dimensional Banach spaces X and Y then there exist two subspaces M and N such that $X = \ker\, T \oplus M$ and $Y = T(X) \oplus N$, with M and $T(X)$ closed infinite-dimensional subspaces of X and Y, respectively. The restriction of T to M clearly has a bounded inverse, so the existence of a Fredholm operator from X into a different Banach space Y implies the existence of isomorphisms between some closed infinite-dimensional subspaces of X and Y. For this reason, for certain Banach spaces X, Y no bounded Fredholm operator from X to Y exists, i.e., $\Phi(X,Y) = \varnothing$. In this chapter we shall give several examples of pairs of Banach spaces X, Y for which $\Phi(X,Y) = \varnothing$.

Throughout this chapter, given a closed subspace M of X we shall denote by J_M the canonical embedding of M into X, whilst the canonical quotient map of X onto X/M will be denoted by Q_M. The identity operator on a Banach space X will be denoted by I_X.

The study of the following two classes of operators was initiated by Atkinson [**51**].

Definition 7.1. *If X and Y are Banach spaces then $T \in L(X,Y)$ is said to be* left Atkinson *if there exists $S \in L(Y,X)$ such that $I_X - ST \in K(X)$. The operator $T \in L(X,Y)$ is said to be* right Atkinson *if there exists $S \in L(Y,X)$ such that $I_Y - TS \in K(X)$. The class of left Atkinson operators and right Atkinson operators will be denoted by $\Phi_{\mathrm{l}}(X,Y)$ and $\Phi_{\mathrm{r}}(X,Y)$, respectively.*

Note that in the definitions above the ideal of compact operators may be replaced by the ideal of finite-dimensional operators. The concept of relatively regular operators introduced in Chapter 5 may be easily extended to operators acting between different Banach spaces. Given $T \in L(X,Y)$, T is said to be *relatively regular* if there exists $S \in L(Y,X)$ such that $TST = T$.

Theorem 7.2. *If $T \in L(X,Y)$ the following assertions hold:*

(i) *T is relatively regular if and only if* $\ker T$ *is complemented in X and $T(X)$ is complemented in Y;*

(ii) *If for some $U \in L(Y,X)$ the operator $TUT - T$ is relatively regular then T is relatively regular.*

Proof (i) In the proof of Theorem 3.88 it is shown that if $T = TST$ then TS is a projection of Y onto $T(X)$, whilst $I_X - ST$ projects X onto $\ker T$. Conversely, suppose that $X = \ker T \oplus M$ and $Y = T(X) \oplus N$. Let T_0 be the restriction of T to M. Clearly T_0 is a bijective map of M onto $T(X)$. Since $T(X)$ is a Banach space it follows that ${T_0}^{-1}$ is continuous, by the open mapping theorem. Let Q be the (bounded) projection of Y onto $T(X)$. It is easily seen that $S := {T_0}^{-1}Q$ satisfies $TST = T$.

(ii) Suppose that R satisfies the equality

$$(TUT - T)R(TUT - T) = TUT - T.$$

Then, rearranging this equality, we obtain

$$\begin{aligned} T &= TUT - T(UT - I)R(TU - I)T \\ &= T(U - UTRTU + RTU + UTR - R)T, \end{aligned}$$

so T is relatively regular. ∎

Theorem 7.3. *Let X, Y, and Z be Banach spaces and $T \in L(X,Y)$. Then the following assertions hold:*

(i) *$T \in \Phi_{\mathrm{l}}(X,Y)$ if and only if $T \in \Phi_+(X,Y)$ and* $\ker T$ *is complemented in X;*

(ii) *$T \in \Phi_{\mathrm{r}}(X,Y)$ if and only if $T \in \Phi_-(X,Y)$ and $T(X)$ is complemented in Y;*

(iii) *If $T \in \Phi_{\mathrm{l}}(X,Y)$ and $S \in \Phi_{\mathrm{l}}(Y,Z)$ then $ST \in \Phi_{\mathrm{l}}(X,Z)$. Analogously, if $T \in \Phi_{\mathrm{r}}(X,Y)$ and $S \in \Phi_{\mathrm{r}}(Y,Z)$ then $ST \in \Phi_{\mathrm{r}}(X,Z)$. The sets $\Phi_{\mathrm{l}}(X)$ and $\Phi_{\mathrm{r}}(X)$ are semi-groups in $L(X)$;*

(iv) *Suppose that $T \in L(X,Y)$, $S \in L(Y,Z)$ and $ST \in \Phi_{\mathrm{l}}(X,Z)$. Then $S \in \Phi_{\mathrm{l}}(Y,Z)$. Analogously, suppose that $T \in L(X,Y)$, $S \in L(Y,Z)$ and $ST \in \Phi_{\mathrm{r}}(X,Z)$. Then $T \in \Phi_{\mathrm{r}}(Y,Z)$;*

(v) *$\Phi(X,Y) = \Phi_{\mathrm{l}}(X,Y) \cap \Phi_{\mathrm{r}}(X,Y)$.*

Proof The proof of the assertions (i), (ii), (iii), (iv) is an useful exercise, see Problems IV.13 of Lay and Taylor [**217**], or Caradus, Pfaffenberger, and Yood [**76**, Theorem 4.3.2 and Theorem 4.3.3]. The equality (v) is clear, since all finite-dimensional subspaces and all closed finite-codimensional subspaces are complemented. ∎

The characterization (173) of inessential operators acting on a single Banach space X suggests how to extend the concept of inessential operators to operators acting between different spaces.

Definition 7.4. *A bounded operator $T \in L(X,Y)$, where X and Y are Banach spaces, is said to be an* inessential operator *if $I_X - ST \in \Phi(X)$ for all $S \in L(Y,X)$, where I_X is the identity operator on X. The class of all inessential operators is denoted by $I(X,Y)$.*

Theorem 7.5. *$I(X,Y)$ is a closed subspace of $L(X,Y)$ which contains $K(X,Y)$. Moreover, if $T \in I(X,Y)$, $R_1 \in L(Y,Z)$, and $R_2 \in L(W,X)$, where X, Y and W are Banach spaces, then $R_1TR_2 \in I(W,Z)$.*

Proof To show that $I(X,Y)$ is a subspace of $L(X,Y)$ let $T_1, T_2 \in I(X,Y)$. Then, given $S \in L(Y,X)$, we have $I_X - ST_1 \in \Phi(X)$ and hence by the Atkinson characterization of Fredholm operators $I_X - ST_1$ is invertible in $L(X)$ modulo $K(X)$. Therefore there exist operators $U_1 \in L(X)$ and $K_1 \in K(X)$ such that $U_1(I_X - ST_1) = I_X - K_1$. From the definition of inessential operators we also have $I_X - U_1ST_2 \in \Phi(X)$, so there are $U_2 \in L(X)$ and $K_2 \in K(X)$ such that $U_2(I_X - U_1ST_2) = I_X - K_2$. Then

$$U_2U_1[I_X - S(T_1 + T_2)] = U_2(I_X - K_1 - U_1ST_2) = I_X - K_2 - U_2K_1.$$

Since $K_2 + U_2K_1 \in K(X)$ this shows that $I_X - S(T_1 + T_2) \in \Phi(X)$ for all $S \in L(Y,X)$. Thus $T_1 + T_2 \in I(X,Y)$ and hence $I(X,Y)$ is a linear subspace of $L(X,Y)$.

The property of $I(X,Y)$ being closed is a simple consequence of $\Phi(X,Y)$ being an open subset of $L(X,Y)$. Moreover, $K(X,Y) \subseteq I(X,Y)$, since if $T \in K(X,Y)$ then $ST \in K(X)$ for every $S \in L(Y,X)$, and hence $I_X - ST \in \Phi(X)$.

To show the last assertion suppose that $T \in L(X,Y)$, $R_1 \in L(Y,Z)$ and $R_2 \in L(W,X)$. We need to prove that $I_W - SR_1TR_2 \in \Phi(W)$ for all $S \in L(Z,W)$. Given $S \in L(Z,W)$, as above, there exist $U \in L(X)$ and $K \in K(X)$ with $U(I_X - R_2SR_1T) = I_X - K$. Define

$$U_0 := I_W + SR_1TUR_2 \quad \text{and} \quad K_0 := SR_1TKR_2.$$

Then $K_0 \in K(W)$ and

$$\begin{aligned} U_0(I_W - SR_1TR_2) &= I_W - SR_1TR_2 + SR_1TU(I_X - R_2SR_1T)R_2 \\ &= I_W - SR_1TR_2 + SR_1T(I_X - K)R_2 \\ &= I_W - SR_1TKR_2 = I_W - K_0. \end{aligned}$$

Therefore $I_W - SR_1TR_2 \in \Phi(W)$ for all $S \in L(Z,W)$. This shows that $R_1TR_2 \in I(W,Z)$. ∎

For every operator $T \in L(X,Y)$ define by $\overline{\beta}(T)$ the codimension of the closure of $T(X)$. Clearly $\overline{\beta}(T) \le \beta(T)$, and if $\beta(T)$ is finite then $\overline{\beta}(T) = \beta(T)$ since $T(X)$ is closed by Corollary 1.15.

Lemma 7.6. *If $T \in L(X,Y)$ and $S \in L(Y,X)$ then the following equalities hold:*

(i) $\alpha(I_X - ST) = \alpha(I_Y - TS)$;

(ii) $\beta(I_X - ST) = \beta(I_Y - TS)$;

(iii) $\overline{\beta}(I_X - ST) = \overline{\beta}(I_Y - TS)$.

Proof Obviously $T(\ker(I_X - ST)) \subseteq \ker(I_Y - TS)$ and the induced operator $\widetilde{T} : \ker(I_X - ST) \to \ker(I_Y - TS)$ is invertible, with inverse induced by S. This proves (i). There is also the inclusion

$$T(I_X - ST)(X) \subseteq (I_Y - TS)(Y)$$

and the induced operator

$$\check{T} : X/(I_X - ST)(X) \to Y/(I_Y - TS)(Y)$$

is invertible with inverse induced by S. This proves (ii), and the same argument, replacing ranges by their closure, proves (iii). The last assertion is obvious. ∎

Corollary 7.7. *$T \in I(X,Y)$ if and only if $I_Y - TS \in \Phi(Y)$ for all $S \in L(Y,X)$.* ∎

Corollary 7.8. *If $T^* \in I(Y^*, X^*)$ then $T \in I(X,Y)$.*

Proof Suppose that $T \notin I(X,Y)$. Then by definition there exists an operator $S \in L(Y,X)$ such that $I_X - ST \notin \Phi(X,Y)$. By duality this implies that ${I_X}^* - T^*S^* \notin \Phi(X^*)$ and hence $T^* \notin I(Y^*, X^*)$, by Corollary 7.7. ∎

The next result will be needed later.

Corollary 7.9. *$L(X,Y) = I(X,Y)$ if and only if $L(Y,X) = I(Y,X)$.* ∎

In the sequel we describe the inessential operators in some concrete case. The description of inessential operators in these cases requires quite a bit of knowledge of the structure of some concrete Banach spaces. However, we shall appropriately refer these results.

Example 7.10. If X is reflexive and Y has the Dunford–Pettis property then $I(X,Y) = L(X,Y)$. Recall that an operator $T \in L(X,Y)$ is said to be *completely continuous* if T transforms relatively weakly compact sets into relatively compact sets, whilst T is said *weakly compact* if transforms bounded sets into relatively weakly compact sets. Note that if X or Y is reflexive then every $T \in L(X,Y)$ is weakly compact, see Goldberg [**129**, III.3.3].

A Banach space X has the *Dunford–Pettis property* if any weakly compact operator T from X into another Banach spaces Y is completely continuous. Examples of Banach space having the Dunford–Pettis property are the $C(K)$ spaces, L^1 spaces, the space of all bounded analytic functions on the unit disc $H^\infty(\mathbb{D})$, and some Sobolev spaces, see for instance Diestel [**88**] and Bourgain [**74**].
To see that $I(X,Y) = L(X,Y)$ observe first that every $T \in L(X,Y)$ is weakly compact since X is reflexive, and every operator in $L(Y,X$ is completely continuous. Given $T \in L(X,Y)$ then ST is compact for every $S \in L(Y,X)$, hence $I - ST \in \Phi(X)$.

Example 7.11. $I(X, Y) = L(X, Y)$ if X has the reciprocal Dunford–Pettis property and Y has the Schur property. Recall that X is said to have the *reciprocal Dunford–Pettis property* if every completely continuous operator from X into any Banach spaces is weakly compact, whilst Y has the *Schur property* if the identity I_Y is completely continuous. Examples of Banach spaces with this property are any $C(K)$ space, see Grothendieck [**142**], as well as Banach spaces containing copies of ℓ^1, see Emmanuele [**107**]. The proof of the equality $I(X, Y) = L(X, Y)$ is analogous to that of the previous example.

Example 7.12. $I(X, Y) = L(X, Y)$ if X contains no copies of ℓ^∞ and $Y = \ell^\infty$, $H^\infty(\mathbb{D})$, or a $C(K)$, with K σ-Stonian. In fact, given a non-weakly compact operator $U \in L(C(K), X)$, K a σ-Stonian set (note that ℓ^∞ is a $C(K)$ of this class), there exists a subspace M of $C(K)$ isomorphic to ℓ^∞, so the restriction $U|M$ is an isomorphism, see Rosenthal [**281**], and the same happens for non-compact operators defined in H^∞, see Bourgain [**73**]. In our case each $T \in L(Y, X)$ is then weakly compact. Moreover, since $C(K)$ spaces and $H^\infty(\mathbb{D})$ have the Dunford–Pettis property T is completely continuous. Thus for any $S \in L(X, Y)$ the product ST is completely continuous and weakly compact. From this we obtain that $(ST)^2$ is compact and hence a Riesz operator. From part (ii) of Theorem 3.113 we then infer that ST is a Riesz operator so that $I_Y - ST \in \Phi(Y)$. Hence T is inessential.

Example 7.13. $I(X, Y) = L(X, Y)$ if X contains no copies of c_0 and $Y = C(K)$. This may be proved as in the previous example, since any $C(K)$ space has the Dunford–Pettis property, and every non-compact operator defined in Y is an isomorphism in some subspace isomorphic to c_0, see Diestel and Uhl [**89**].

Example 7.14. $I(X, Y) = L(X, Y)$ if X contains no complemented copies of c_0 and $Y = C(K)$, or X contains no complemented copies of ℓ^1 and $Y = L^1(\mu)$. Note first that any copy of c_0 in Y is complemented.

To show the first case, take $S \in L(X, Y)$ and $T \in L(Y, X)$. If ST is not weakly compact, then it is an isomorphism in a subspace M isomorphic to c_0 and $T(M)$ is a complemented subspace isomorphic to c_0 in X.

Assume now that X contains no complemented copies of ℓ^1. Then X^* contains no copies of ℓ^∞, see Lindenstrauss and Tzafriri [**221**, 2.e.8], so for every $T \in L(X, Y)$ we have $T^* : L^\infty(\mu) \to X^*$ is weakly compact, see Rosenthal [**281**]. Then for every $S \in L(Y, X)$ it follows, arguing as in the Example 7.12, that $(S^*T^*)^2$ is compact, from which we conclude that $I_{Y^*} - S^*T^*$ is a Fredholm operator. Therefore also $I_Y - TS \in \Phi(Y)$, and consequently T is inessential.

Example 7.15. $I(X, Y) = L(X, Y)$ whenever X or Y are ℓ^p, with $1 \le p \le \infty$, or c_0, and X, Y are different. In fact, for $1 \le p < q < \infty$ every operator from ℓ^q, or c_0, into ℓ^p is compact, see Lindenstrauss and Tzafriri [**221**, 2.c.3]. The case $p = \infty$ is covered by Example 7.12.

In the sequel we shall need the following important perturbative characterization of semi-Fredholm operators.

Theorem 7.16. *Let $T \in L(X,Y)$ be a bounded operator on a Banach space X, Y. Then the following assertions hold:*

(i) *$T \in \Phi_+(X,Y)$ if and only if $\alpha(T-K) < \infty$ for all compact operators $K \in K(X,Y)$;*

(ii) *$T \in \Phi_-(X,Y)$ if and only if $\overline{\beta}(T-K) < \infty$ for all compact operators $K \in K(X,Y)$.*

Proof (i) Since $\Phi_+(X,Y)$ is stable under compact perturbations, we have only to show that if $\alpha(T-K) < \infty$ for all compact operators $K \in K(X,Y)$ then $T \in \Phi_+(X,Y)$.

Suppose that $T \notin \Phi_+(X,Y)$. Then T does not have a bounded inverse, so there exists $x_1 \in X$ with $\|x_1\| = 1$, such that $\|Tx_1\| \leq \frac{1}{2}$. By the Hahn Banach theorem we can find an element $f_1 \in X^*$ such that $\|f_1\| = 1$ and $f_1(x_1) = 1$. Let us consider a bi-orthogonal system $\{x_k\}$ and $\{f_k\}$ such that

$$\|x_k\| = 1, \quad \|Tx_k\| \leq 2^{1-2k}, \quad \text{and} \quad \|f_k\| \leq 2^{k-1} \quad \text{for all } k = 1, 2, \dots n-1.$$

Since the restriction of T to the closed subspace $N := \bigcap_{k=1}^{n-1} \ker f_k$ does not admit a bounded inverse, there is an element $x_n \in N$ such that $\|x_n\| = 1$ and $\|Tx_k\| \leq 2^{1-2n}$. Let $g \in X^*$ be such that $g(x_n) = 1$ and $\|g\| = 1$. Define $f_n \in X^*$ by the assignment:

$$f_n := g - \sum_{k=1}^{n-1} g(x_k) f_k.$$

Then $f_n(x_k) = \delta_{nk}$ for $k = 1, 2, \dots, n$ and $\|f_n\| \leq 2^{n-1}$. By means of an inductive argument we can construct two sequences $(x_k) \subset X$ and $(f_k) \subset X^*$ such that

$$\|x_k\| = 1, \quad \|f_k\| \leq 2^{k-1}, \quad f_k(x_j) = \delta_{kj}, \quad \text{and} \quad \|Tx_k\| \leq 2^{1-2k}.$$

Define $K_n \in L(X,Y)$ by

$$K_n(x) := \sum_{k=1}^{n} f_k(x) Tx_k \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly K_n is a finite-dimensional operator for every n, and for $n > m$ the following estimate holds:

$$\|(K_n - K_m)x\| \leq \sum_{k=m+1}^{n} 2^{k-1} 2^{1-2k} \|x\|,$$

so $\|K_n - K_m\| \to 0$ as $m, n \to \infty$. If we define

$$Kx := \sum_{k=1}^{\infty} f_k(x) Tx_k$$

then $K_n \to K$ in the operator norm, so K is compact. Moreover, $Kx = Tx$ for any $x = x_k$ and this is also true for any linear combination of the elements x_k. Since these elements are linearly independent we then conclude that $\alpha(T - K) = \infty$. Hence the equivalence (i) is proved.

(ii) Also here one direction of the equivalence is immediate: suppose that $T \in \Phi_-(X, Y)$ and $K \in K(X, Y)$. By duality we then have $T \in \Phi_+(Y^*, X^*)$ and $K^* \in K(Y^*, X^*)$. From this we obtain that $T^* - K^* \in \Phi_+(Y^*, X^*)$, and therefore, again by duality, $T - K \in \Phi_-(X, Y)$.

Suppose that $T \notin \Phi_-(X, Y)$. Then either $T(X)$ is closed and $\overline{\beta}(T) = \infty$ or $T(X)$ is not closed. In the first case taking $K = 0$ we obtain $\overline{\beta}(T-K) = \infty$ and we are finished. So assume that $T(X)$ is not closed.

Let (a_n) be a sequence of integers defined inductively by

$$a_1 := 2, \quad a_n := 2\left(1 + \sum_{k=1}^{n-1} a_k\right), \quad \text{for } n = 2, 3, \dots$$

We prove now that there exists a sequence (y_k) in Y and a sequence (f_k) in $Y^\star$ such that, for all $k \in \mathbb{N}$, we have

$$\|y_k\| \le a_k, \quad \|f_k\| = 1, \quad \|T^*(f_k)\| < \frac{1}{2^k a_k}, \quad \text{and} \quad f_j(y_k) = \delta_{jk}. \tag{174}$$

We proceed by induction. Since $T(X)$ is not closed, also $T^*(X^*)$ is not closed, by Theorem 1.15. Hence there exists $f_1 \in Y^*$ such that $\|f_1\| = 1$ and $\|T^* f_1\| < \dfrac{1}{4}$, and there exists $y_1 \in Y$ such that $\|y_1\| < 2$ and $f_1(y_1) = 1$.

Suppose that there exist $y_1, y_2, \dots, y_{n-1}$ in Y and $f_1, f_2, \dots f_{n-1}$ in Y^* such that (174) hold for every $j = 2, \dots, n-1$. Let $f_n \in Y^*$ be such

$$f_n(y_k) = 0 \quad \text{for } k = 1, \dots, n-1, \quad \|f_n\| = 1, \quad \|T^* f_n\| < \frac{1}{2^n a_n}.$$

There exists also $y \in Y$ such that $f_n(y) = 1$ and $\|y\| < 2$. Define

$$y_n := y - \sum_{k=1}^{n-1} f_k(y) y_k.$$

Then

$$\|y_n\| \le \|y\| \left(1 + \sum_{k=1}^{n-1} \|y_k\|\right) \le 2\left(1 + \sum_{k=1}^{n-1} a_k\right) = a_n,$$

by the induction hypothesis. Furthermore, from the choice of the elements y_n and f_n we see that $f_n(y_n) = 1$ while $f_n(y_k) = 0$ for all $k = 1, 2, \dots, n$. Finally, $f_k(y_n) = f_k(y) - f_k(y) = 0$ for all $k = 1, 2, \dots, n$, so y_n has the desired properties.

By induction then there exist the two sequences sequence (y_k) in Y and a sequence (f_k) in $Y^\star$ which satisfy (174). We now define

$$K_n x := \sum_{k=1}^{n} (T^* f_k)(x) y_k \quad \text{for } n \in \mathbb{N}.$$

For $n > m$ we obtain

$$\|K_n x - K_m x\| \leq \sum_{m+1}^{n} \|T^* f_k\| \|x\| \|y_k\| \leq \left(\sum_{m+1}^{n} \frac{1}{2^k} \right) \|x\| \leq \frac{\|x\|}{2^m},$$

and this implies that the finite-dimensional operator K_n converges to the compact operator $K \in K(X,Y)$ defined by

$$Kx := \sum_{k=1}^{\infty} (T^* f_k)(x) y_k.$$

For each $x \in X$ and each k we have $f_k(Kx) = T^* f_k(x) = f_k(Tx)$, so each of the f_k annihilates $\overline{(T-K)(X)}$. Since the f_k are linearly independent we then conclude that $\overline{\beta}(T-K) = \infty$. ■

Next we show that the class of inessential operators $I(X,Y)$ presents a perfect symmetry with respect to the defects α and β. We prove first that the inessentiall operators may be characterized only by the nullity $\alpha(T)$.

Theorem 7.17. *For a bounded operator $T \in L(X,Y$, where X and Y are Banach spaces, the following assertions are equivalent:*

(i) *T is inessential;*

(iii) *$\alpha(I_X - ST) < \infty$ for all $S \in L(Y,X)$;*

(iii) *$\alpha(I_Y - TS) < \infty$ for all $S \in L(Y,X)$.*

Proof The implication (i) $\Rightarrow$ (ii) is obvious by Corollary 7.7, whilst the equivalence of (ii) and (iii) follows from Lemma 7.6. We prove the implication (ii) $\Rightarrow$ (i).

Assume that the assertion (ii) holds. We claim that $I_X - ST \in \Phi_+(X)$ for all $S \in L(Y,X)$. Suppose that this is not true. Then there is an operator $S_1 \in L(Y,X)$ such that $I_X - S_1 T \notin \Phi_+(X)$. By part (i) of Theorem 7.16 there is then a compact operator $K \in L(X)$ such that $\alpha(I_X - S_1 T - K) = \infty$. Let $M :=$ ker $(I_X - S_1 T - K)$ and denote by J_M the embedding map from M into X. From $(I_X - S_1 T)|M = K|M$ it follows that

$$KJ_M = (I_X - S_1 T) J_M. \tag{175}$$

But $I_X - K \in \Phi(X)$, so by the Atkinson characterization of Fredholm operators the exists $U \in L(X)$ and a finite-dimensional operator $P \in L(X)$ such that

$$U(I_X - K) = I_X - P. \tag{176}$$

Multiplying the left side of (175) by U we obtain

$$UKJ_M = U(I_X - S_1 T) J_M = UJ_M - US_1 T J_M,$$

which yields

$$US_1 T J_M = UJ_M - UKJ_M = U(I_X - K) J_M.$$

From the equality (176) we obtain $US_1TJ_M = (I_X - P)J_M$ and hence

$$(I_X - US_1T)J_M = PJ_M.$$

Note that PJ_M is a finite-dimensional operator, so $\alpha(PJ_M) = \infty$. Clearly,

$$\alpha(I_X - US_1T) \leq \alpha(I_X - US_1T)J_M = \alpha(PJ_M) = \infty,$$

and this contradicts our hypothesis. Therefore our claim is proved.

In particular, $\lambda I - ST \in \Phi_+(X)$ for all non-zero $\lambda \in \mathbb{C}$, so ST is a Riesz operator by Theorem 3.111, and this implies that $I_X - ST \in \Phi(X)$. Therefore T is inessential. ∎

We show now that dually $I(X,Y)$ may be characterized only by means of the deficiencies $\beta(T)$ and $\overline{\beta}(T)$.

Theorem 7.18. *For a bounded operator $T \in L(X,Y$, where X and Y are Banach spaces, the following assertions are equivalent:*

(i) *T is inessential;*

(ii) *$\overline{\beta}(I_X - ST) < \infty$ for all $S \in L(Y,X)$;*

(iii) *$\overline{\beta}(I_Y - TS) < \infty$ for all $S \in L(Y,X)$.*

Proof (i) $\Rightarrow$ (ii) Since $\overline{\beta}(I_X - ST) \leq \beta(I_X - ST)$, if $T \in I(X,Y)$ then $\overline{\beta}(I_X - ST) < \infty$. The implication (ii) $\Rightarrow$ (iii) is clear from Lemma 7.6.

To prove the implication (iii) $\Rightarrow$ (i) suppose that $\overline{\beta}(I_Y - TS) < \infty$ for all $S \in L(Y,X)$. We show that $I_Y - TS \in \Phi_-(Y)$ for all $S \in L(Y,X)$. Suppose that this is not true. Then there exists an operator $S_1 \in L(Y,X)$ such that $(I_Y - TS_1) \notin \Phi_-(Y)$. By part (ii) of Theorem 7.16 there is then a compact operator $K \in L(Y)$ such that $\overline{\beta}(I_Y - TS_1 - K) = \infty$. Let $N := \overline{(I_Y - TS_1 - K)(Y)}$ and denote by Q_N the canonical quotient map from Y onto Y/N. Then

$$Q_N K = Q_N(I_Y - TS_1), \tag{177}$$

and since $I_Y - K \in \Phi(Y)$ by the Atkinson characterization of Fredholm operators there exists some $U \in L(Y)$ and a finite-dimensional operator $P \in L(Y)$ such that

$$(I_Y - K)U = I_Y - P. \tag{178}$$

Multiplying the right hand side of (177) by U we obtain

$$Q_N KU = Q_N(I_Y - TS_1)U = Q_N U - Q_N TS_1 U,$$

which implies

$$Q_N TS_1 U = Q_N U - Q_N KU = Q_N(I_Y - K)U.$$

From the equality (178) we have $Q_N TS_1 U = Q_N(I_Y - P)$, and consequently

$$Q_N(I_Y - TS_1U) = Q_N P.$$

The operator $Q_N P$ is finite-dimensional operator, so its range is closed, and therefore

$$Q_N(I_Y - TS_1U)(Y) = \overline{Q_N(I_Y - TS_1U)(Y)}.$$

Now,

$$\begin{aligned}\text{codim}\ \overline{(I_Y - TS_1U)(Y)} &\geq \text{codim}\ \overline{Q_N(I_Y - TS_1U)(Y)} \\ &= \text{codim}\ Q_N(I_Y - TS_1U)(Y) \\ &= \text{codim}\ (Q_NP)(Y) = \infty,\end{aligned}$$

and hence $\overline{\beta}(I_Y - TS_1U) = \infty$, contradicting our hyphotesis. Therefore $I_Y - TS \in \Phi_-(Y)$ for all $S \in L(Y,X)$. In particular, $\lambda I - TS \in \Phi_-(Y)$ for all non-zero $\lambda \in \mathbb{C}$, so TS is a Riesz operator by Theorem 3.111, and this implies that $I_Y - ST \in \Phi(Y)$. Therefore T is inessential by Corollary 7.7. ∎

Corollary 7.19. *For $T \in L(X,Y)$, where X and Y are Banach spaces, the following statements are equivalent:*

(i) *T is inessential;*

(ii) *$\beta(I_X - ST) < \infty$ for all $S \in L(Y,X)$;*

(iii) *$\beta(I_Y - TS) < \infty$ for all $S \in L(Y,X)$.*

Proof (i) ⇔ (ii) If $T \in I(X,Y)$ then $\beta(I_X - ST) < \infty$ for all $S \in L(Y,X)$. Conversely, if $\beta(I_X - ST) < \infty$ for all $S \in L(Y,X)$, from $\overline{\beta}(I_X - ST) \leq \beta(I_X - ST)$ and Theorem 7.18 we deduce that $T \in I(X,Y)$.

The equivalence (i) ⇔ (iii) is obvious by Lemma 7.7. ∎

We now see that the inessential operators may be characterized as perturbation classes. In the sequel, given two Banach spaces X and Y, by $\Sigma(X,Y)$ we shall denote any of the semi-groups $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_{\mathrm{l}}(X,Y)$ or $\Phi_{\mathrm{r}}(X,Y)$.

Definition 7.20. *If $\Sigma(X,Y) \neq \varnothing$ the* perturbation class $P\Sigma(X,Y)$ *is the set defined by*

$$P\Sigma(X,Y) := \{T \in L(X,Y) : T + \Sigma(X,Y) \subseteq \Sigma(X,Y)\}.$$

Note that $P\Sigma(X) := P\Sigma(X,X)$ is always defined since $I \in \Sigma(X) := \Sigma(X,X)$.

Theorem 7.21. *Given two Banach spaces X, Y for which $\Sigma(X,Y) \neq \varnothing$, the following assertions hold:*

(i) *$P\Sigma(X,Y)$ is a closed linear subspace of $L(X,Y)$;*

(ii) *If $T \in P\Sigma(X,Y)$, then both TS and UT belong to $P\Sigma(X,Y)$ for every $S \in L(X)$ and $U \in L(Y)$;*

(iii) *$P\Sigma(X)$ is a closed two-sided ideal of $L(X)$.*

Proof We show the assertions (i), (ii) and (iii) in the case $\Sigma(X,Y) = \Phi(X,Y)$. The proof in the other cases is similar.

(i) Clearly $P\Phi(X,Y)$ is a linear subspace of $L(X,Y)$. To show that $P\Phi(X,Y)$ is closed assume that $T_n \in P\Phi(X,Y)$ converges to $T \in L(X,Y)$. Given $S \in \Phi(X,Y)$ there exists $\varepsilon > 0$ so that $S + K \in \Phi(X,Y)$ for all

$\|K\| < \varepsilon$, see part (d) of Remark 1.54. Now, writing $T+S = T_n+(T-T_n)+S$ we see that $T + S \in \Phi(X, Y)$. Therefore $T \in P\Phi(X, Y)$.

(ii) Let $T \in P\Phi(X, Y)$ and $U \in L(Y)$. Assume first that U is invertible. For every $S \in \Phi(X, Y)$ it then follows that

$$S + UT = U(U^{-1}S + T) \in P\Phi(X, Y).$$

Consequently $UT \in P\Phi(X, Y)$. The general case that U is not invertible may be reduced to the previous case, since every operator U is the sum of two invertible operators $U = \lambda I_X + (U - \lambda I_X)$, with $0 \neq \lambda \in \rho(U)$. A similar argument shows that $TS \in P\Phi(X, Y)$ for every $S \in L(X)$.

(iii) Clear. ∎

We now characterize the inessential operators as the perturbation class of Fredholm operators in the case of operators acting on a single Banach space.

Theorem 7.22. *For every Banach space X we have*

$$I(X) = P\Phi(X) = P\Phi_{\mathrm{l}}(X) = P\Phi_{\mathrm{r}}(X).$$

Proof We only prove that $I(X) \subseteq P\Phi(X)$. Let $T \in I(X)$ and S be any operator in $\Phi(X)$. We show that $T + S \in \Phi(X)$. Since $S \in \Phi(X)$ there are by Theorem 1.53 some operators $U \in L(X)$ and $K \in K(X)$ such that $SU = I + K$. Since $T \in I(X)$ it then follows that $I + UT \in \Phi(X)$ and hence also $S(I + UT) \in \Phi(X)$, so the residual class $S(\widehat{I + UT}) := S(I+UT)+K(X)$ is invertible in the quotient algebra $L(X)/K(X)$, by the Atkinson characterization of Fredholm operators.

On the other hand, since $KT \in K(X)$ then

$$S(\widehat{I + UT}) = \widehat{S} + \widehat{SUT} = \widehat{S} + (\widehat{I + K})T = \widehat{S} + \widehat{T}\widehat{KT} = \widehat{S + T},$$

and hence $\widehat{S + T}$ is invertible. Therefore $S + T \in \Phi(X)$, and hence since S is arbitrary, $T \in \Phi(X)$.

To show the opposite implication suppose that $T \in P\Phi(X)$, namely $T + \Phi(X) \subseteq \Phi(X)$. We show first that $I - ST \in \Phi(X)$ for all $S \in \Phi(X)$.

If $S \in \Phi(X)$ there exists $U \in L(X)$ and $K \in K(X)$ such that $SU = I - K$. Since $U \in \Phi(X)$, see part (c) of Remark 1.54, from the assumption we infer that $U - T \in \Phi(X)$, and hence also

$$S(U - T) = SU - ST = I - K - ST \in \Phi(X).$$

Now, adding K to $I - K - ST$ we conclude that $I - ST \in \Phi(X)$, as claimed. To conclude the proof let $W \in L(X)$ be arbitrary. Write $W = \lambda I + (W - \lambda I)$, with $0 \neq \lambda \in \rho(T)$ then

$$I - WT = I - (\lambda I + (W - \lambda I))T = -\lambda T + (I - (\lambda I - W)T)$$

and, since $\lambda I - W \in \Phi(X)$, by the first part of the proof we obtain that $I - (\lambda I - W)T \in \Phi(X)$, so $I - WT \in -\lambda T + \Phi(X) \subseteq \Phi(X)$, which completes the proof. ∎

We now extend the result of Theorem 7.22 to inessential operators acting between different Banach spaces.

Theorem 7.23. *Let $\Sigma(X,Y)$ denote one of the semi-groups $\Phi(X,Y)$, $\Phi_{\rm l}(X,Y)$, and $\Phi_{\rm r}(X,Y)$. If $\Sigma(X,Y) \neq \varnothing$ then $P\Sigma(X,Y) = I(X,Y)$.*

Proof Also here we prove only the equality $P\Phi(X,Y) = I(X,Y)$. The other cases are analogous.

Let $S \in \Phi(X,Y)$ and $T \in I(X,Y)$. Take $U \in L(Y,X)$ and $K(X,Y)$ such that $US = I - K$. Note that $U \in \Phi(Y,X)$ by part (c) of Remark 1.54. Then $US \in \Phi(X)$, and since $UT \in I(X)$ from Theorem 7.22 we obtain $U(T+S) = UT + US \in \Phi(X)$. From this it follows, again by part (c) of Remark 1.54, that $T + S \in \Phi(X,Y)$.

For the inverse inclusion assume that $T \in L(X,Y)$ and $T \notin I(X,Y)$. Then there exists $S \in L(Y,X)$ such that $I_X - ST \notin \Phi(X)$. Taking any operator $V \in \Phi(X,Y)$ we have $V(I_X - ST) = V - VST \notin \Phi(X,Y)$, see part (c) of Remark 1.54, so $VST \notin P\Sigma(X,Y)$. This implies by part (ii) of Theorem 7.21 that $T \notin P\Sigma(X,Y)$, which concludes the proof. ■

Also in the characterization of $I(X,Y)$ established in Theorem 7.22 we may only consider one of the two defects α or β.

Theorem 7.24. *If $\Phi(X,Y) \neq \varnothing$, for every $T \in L(X,Y)$ the following assertions are equivalent:*

(i) *T is inessential;*

(ii) *$\alpha(T+S) < \infty$ for all $S \in \Phi(X,Y)$;*

(iii) *$\overline{\beta}(T+S) < \infty$ for all $S \in \Phi(X,Y)$;*

(iv) *$\beta(T+S) < \infty$ for all $S \in \Phi(X,Y)$.*

Proof (i) $\Leftrightarrow$ (ii) Let $T \in I(X,Y)$ and suppose that $S \in L(X,Y)$ is any Fredholm operator. By Theorem 7.22 it follows that $T + S \in \Phi(X,Y)$ and therefore $\alpha(T+S) < \infty$. Conversely, suppose that $\alpha(T+S) < \infty$ for each $S \in \Phi(X,Y)$. Since $K(X,Y) \subseteq I(X,Y)$ then, always by Theorem 7.22,

$$\frac{1}{\lambda}(S-K) \in \Phi(X,Y) \quad \text{for all } K \in K(X,Y) \quad \text{and } \lambda \neq 0.$$

Therefore

$$\alpha\left(T + \frac{S-K}{\lambda}\right) = \alpha(\lambda T + S - K) < \infty \quad \text{for all } K \in K(X,Y).$$

By Theorem 7.16 it follows that $\lambda T - S \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C}$. In particular, this is true for any $|\lambda| \leq 1$. Now, if $\beta(T+S)$ were infinite we would have ind $(T+S) = -\infty$, and using the stability of the index, see part (d) of Remark 1.54, we would have ind $(\lambda T + S) = -\infty$ for each $|\lambda| \leq 1$ and hence $\beta(S) = \infty$, contradicting the assumption that $S \in \Phi(X,Y)$.

Therefore $\beta(T+S) < \infty$, and consequently $T + S \in \Phi(X,Y)$. By Theorem 7.22 we then conclude that $T \in I(X,Y)$.

(i) $\Leftrightarrow$ (iii) We use an argument dual to that used in the proof of the equivalence (i) $\Leftrightarrow$ (ii). Let $T \in I(X,Y)$ and $S \in \Phi(X,Y)$. By Theorem 7.22 it follows that $\beta(T+S) = \overline{\beta}(T+S) < \infty$.

Conversely, suppose that $\overline{\beta}(T+S) < \infty$ for each $S \in \Phi(X,Y)$. As above, from the inclusion $K(X,Y) \subseteq I(X,Y)$ we obtain for all $\lambda \neq 0$ that

$$\overline{\beta}\left(T + \frac{S-K}{\lambda}\right) = \overline{\beta}(\lambda T + S - K) < \infty \quad \text{for all } K \in K(X,Y).$$

From Theorem 7.16 it follows that $\lambda T - S \in \Phi_-(X,Y)$ for all $\lambda \in \mathbb{C}$. In particular, this is true for any $|\lambda| \leq 1$. Now, if $\alpha(T+S)$ were infinite, we would have ind $(T+S) = \infty$, and from the stability of the index we would have ind $(\lambda T + S) = \infty$ for each $|\lambda| \leq 1$, and hence $\alpha(S) = \infty$, contradicting the assumption that $S \in \Phi(X,Y)$. Therefore $\alpha(T+S) < \infty$, and consequently $T + S \in \Phi(X,Y)$. By Theorem 7.22 we then deduce that $T \in I(X,Y)$.

Clearly (i) $\Rightarrow$ (iv) follows from Theorem 7.23, whilst the implication (iv) $\Rightarrow$ (iii) is obvious; so the proof is complete. ∎

We now establish a result which will be needed later.

Theorem 7.25. *Given two Banach spaces X and Y, the following assertions are equivalent:*

(i) *$L(X,Y) = I(X,Y)$;*

(ii) *Given an operator $T \in L(X \times Y)$, defined by*

$$T := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in L(X)$, $D \in L(Y)$, $B \in L(Y,X)$, and $C \in L(X,Y)$, then $T \in \Phi(X \times Y)$ if and only if A, D are Fredholm.

Proof (i) $\Rightarrow$ (ii) Suppose that $L(X,Y) = I(X,Y)$. Then by Corollary 7.9, $L(X,Y) = I(X,Y)$, so both C and B are inessential. Obviously this implies that

$$S := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in I(X \times Y).$$

Let A and D be Fredholm. If

$$K := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

then obviously $K \in \Phi(X \times Y)$, and hence by Theorem 7.23 $T = S + K \in \Phi(X \times Y)$.

Conversely, if $T \in \Phi(X \times Y)$ is Fredholm and $S \in I(X \times Y)$ is defined as above, then $K := T - S \in \Phi(X \times Y)$, from which we obtain that A and D are Fredholm.

(ii) $\Rightarrow$ (i) Suppose that $K \in L(X,Y)$ is not inessential. Then also $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \in L(X,Y)$ is not inessential, so from Theorem

7.23 there exists an operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi(X \times Y)$ such that

$$\begin{pmatrix} A & B \\ C+K & D \end{pmatrix} \notin \Phi(X \times Y),$$

so the assertion (i) fails. ■

Remark 7.26. It is clear, from the proof of Theorem 7.25, that if $L(X,Y) = I(X,Y)$ and $T := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi(X \times Y)$ then $\operatorname{ind} T = \operatorname{ind} A + \operatorname{ind} D$.

2. Ω_+ and Ω_- operators

Let us consider the following two classes of operators on a Banach space X introduced by Aiena in [**4**]:

$$\Omega_+(X) := \left\{ T \in L(X) : \begin{array}{l} T|M \text{ is an (into) isomorphism for no} \\ \text{infinite-dimensional, invariant subspace } M \text{ of } T \end{array} \right\}$$

and

$$\Omega_-(X) := \left\{ T \in L(X) : \begin{array}{l} Q_M T \text{ is surjective for no} \\ \text{infinite-codimensional, invariant subspace } M \text{ of } T \end{array} \right\}$$

Observe that if M is a closed T-invariant subspace of a Banach space X and $\widetilde{x} := x + M$ is any element in the quotient space X/M, then the induced canonical map $T^M : X/M \to X/M$, defined by

$$T^M \widetilde{x} := \widetilde{Tx} \quad \text{for every } x \in X,$$

has the same range of the composition map $Q_M T$.

Theorem 7.27. *If T is a bounded operator on a Banach space X then the following implications hold:*

(i) *If $T^* \in \Omega_-(X^*)$ then $T \in \Omega_+(X)$;*

(ii) *If $T^* \in \Omega_+(X^*)$ then $T \in \Omega_-(X)$.*

Proof (i) Suppose that $T \notin \Omega_+(X)$. We show that $T^* \notin \Omega_-(X^*)$. By hypothesis there exists an infinite-dimensional closed T-invariant subspace M of X such that $T|M$ is injective and$(T|M)^{-1}$ is continuous.

Let $M^\perp$ be the annihilator of M. Clearly, $M^\perp$ is closed and from the inclusion $T(M) \subseteq M$ it follows that $T^*(M^\perp) \subseteq M^\perp$. Moreover, the continuity of $(T|M)^{-1}$ implies that the operator $(T|M)^* : X^* \to M^*$ is surjective, see Lemma 1.30. Now, if $f \in X^*$ then the restriction $f|M \in M^*$ and there exists by surjectivity an element $g \in X^*$ such that $(T|M)^* g = f|M$, or, equivalently, $g(T|M) = f|M$. Hence, $f - gT \in M^*$, so

$$f - gT + T^* g \in M^* + T^*(X^*),$$

from which we conclude that $X^* = M^\perp + T^*(X^*)$. This last equality obvioulsy implies that $Q_{M^*} T^*$ is surjective.

To show that $T^* \notin \Omega_-(X^*)$ it suffices to prove that codim $M^* = \infty$.

Let (x_n) be a sequence linearly independent elements of M. Denote by $J : X \to X^{**}$ the canonical isomorphim. It is easily seen that

$$M^* \subseteq \bigcap_{n=1}^{\infty} \ker(Jx_n),$$

so the elements $Jx_1, Jx_2, \dots$ being linearly independent, $M^\perp$ is infinite-codimensional.

(ii) Assume that $T \notin \Omega_-(X)$. We show that $T^* \notin \Omega_+(X^*)$. By assumption there exists an infinite-codimensional closed T-invariant subspace $M \subseteq X$ such that $Q_M T$ is surjective. Therefore $X = M + T(X)$ and $T^*(M^\perp) \subseteq M^\perp$. We show that $T^*|M^\perp : M^\perp \to X^*$ is injective and has a bounded inverse.

Let us consider the canonical quotient map $T^M : X/M \to X/M$. From assumption T^M is surjective and hence its dual $(T^M)^* : (X/M)^* \to (X/M)^*$ is injective and has continuous inverse. Define $J : (X/M)^* \to M^\perp$ by

$$J(f)(x) := f(\widetilde{x}) \quad \text{for all } f \in (X/M)^*, \quad x \in X.$$

Clearly J is an isomorphism of $(X/M)^*$ onto $M^\perp$. If $x \in X$ and $\phi \in M^\perp$ then

$$\begin{aligned} (T^*\phi)(x) &= \phi(Tx) = J^{-1}(\phi)(\widetilde{Tx}) = J^{-1}(\phi)(T^M\widetilde{x}) \\ &= (T^M)^*(J^{-1}(\phi))\widetilde{x} = ((J(T^M)^*(J^{-1})(\phi))(x), \end{aligned}$$

from which we obtain $T^*|M^\perp = J(T^M)^*J^{-1}$. From this it follows that $T^*|M^\perp$ is injective and has bounded inverse. To show that $T^* \notin \Omega_+(X^*)$ we need to prove that $M^\perp$ has infinite dimension. From codim $M = \dim X/M$ we obtain that

$$\dim M^\perp = \dim J((X/M)^*) = \dim(X/M)^* = \infty,$$

so the proof is complete. ∎

Theorem 7.28. *The class of all Riesz operator $R(X)$ is contained in $\Omega_+(X) \cap \Omega_-(X)$.*

Proof We show first that $R(X) \subseteq \Omega_+(X)$. Suppose that $T \notin \Omega_+(X)$. Then there exists a closed infinite-dimensional T-invariant subspace M of X such that the $T|M$ admits a bounded inverse. Trivially, $\alpha(T|M) = 0$ and $T(M)$ is closed, so $T|M \in \Phi_+(M)$.

Let us suppose that $T \in R(X)$. By Theorem 3.113, part (iii), the restriction $T|M$ is still a Riesz operator so, by part (g) of Remark 1.54 0 cannot belong to the Fredholm resolvent $\rho_{\rm f}(T|M)$. Hence $\beta(T|M) = \infty$. From this it follows that ind $T|M = -\infty$ and the stability of the index implies that $\lambda I_M - T|M$ has index $-\infty$ in some annulus $0 < |\lambda| < \varepsilon$. This is impossible since $T|M$ is a Riesz operator. Therefore $R(X) \subseteq \Omega_+(X)$.

We show now that $R(X) \subseteq \Omega_-(X)$. We use an argument dual to that given in the first part of the proof.

Suppose that there is a Riesz operator T which does not belong to $\Omega_-(X)$. Then there exists a closed infinite-codimensional T-invariant subspace M of X such that the composition map $Q_M T : X \to X/M$ is surjective. Therefore the induced map T^M on X/M is onto and hence $T^M \in \Phi_(X/M)$.

Now, by Theorem 3.115, T^M is a Riesz operator, and since X/M is infinite-dimensional the Fredholm resolvent $\rho_{\mathrm{f}}(\widetilde{T}_M)$ cannot coincide with the whole complex field. Consequently $\alpha(T^M) = \infty$, so $\operatorname{ind} T^M = \infty$ and the stability of the index yields that $\lambda \widetilde{I}^M - T^M$ has index ∞ in some annulus $0 < |\lambda| < \varepsilon$. This contradicts the fact that T^M is a Riesz operator. Therefore $R(X) \subseteq \Omega_-(X)$. ∎

Example 7.29. The inclusion $R(X) \subseteq \Omega_+(X)$ and $R(X) \subseteq \Omega_-(X)$ are generally proper. In fact, a well known result of Read [**277**] establishes that there exists a bounded operator T on ℓ^1 which does not admit a non-trivial closed T-invariant subspace. The spectrum $\sigma(T)$ of this operator is the whole closed unit disc and hence $T \notin R(\ell^1)$, whereas, obviously, $T \in \Omega_+(\ell^1)$. Moreover, since this operator T is not surjective we also have $T \in \Omega_-(\ell^1)$.

Another interesting example which shows that the inclusion $R(X) \subseteq \Omega_+(X)$ is strict, is provided by a convolution operator on $L^1(G)$, where $G := \mathbb{T}$ is the circle group. To see this assume that $\mu \in \mathcal{M}(G)$ satisfies the following two properties:

(a) μ^n is singular for each $n \in \mathbb{N}$.

(b) $\lim_{n\to\infty} \widehat{\mu}(n) = 0$, where $\widehat{\mu}$ is the Fourier–Stieltjes transform of μ.

Note that a such measure does exist. In fact, there exists a closed independent set $E \subseteq \mathbb{T}$ which is the support of a positive measure μ such that $\lim_{n\to\infty} \widehat{\mu}(n) = 0$, see Kahane and Salem [**179**, p. 106] and the n-fold sum $E + E + \cdots + E$ has measure 0, [**179**, p. 103]. Therefore μ^n is singular for each $n \in \mathbb{N}$.

If we suppose that μ is a probability measure we have for each $f \in L^1(G)$

$$\|\mu^n + f\| = \|\mu^n\| + \|f\| \geq \|\mu^n\| = 1,$$

so the class $\widetilde{\mu} := \mu + L^1(G)$ is not quasi-nilpotent in the quotient algebra $\mathcal{M}(G)/L^1(G)$. Now, from Theorem 5.98 we know that for any measure $\nu \in \mathcal{M}(G)$ the operator $\lambda I - T_\nu$ is a Fredholm operator on $L^1(G)$ if and only if $\lambda \in \rho(\widetilde{\nu})$. Therefore T_ν is a Riesz operator precisely when $\widetilde{\nu}$ is quasi-nilpotent in $\mathcal{M}(G)/L^1(G)$. Hence our operator T_μ is not a Riesz operator.

We show now that $T_\mu \in \Omega_+(L^1(G))$. Suppose that there is a closed invariant infinite-dimensional subspace M of $L^1(G)$ such that the restriction $T_\mu|M$ is invertible. Then there exists a sequence $(n_k) \subseteq \widehat{G} := \mathbb{Z}$ and a sequence (f_k) in M such that $n_k \to 0$, as $k \to \infty$, and $\widehat{f_k}(n_k) \neq 0$.

Let U be the inverse of $T_\mu|M$. For each $p \in \mathbb{N}$ we have

$$\widehat{(U^p f_k)}(n_k) = \widehat{\mu}(n_k)^{-p} \widehat{f_k}(n_k),$$

hence

$$|\widehat{\mu}(n_k)|^{-p}|\widehat{f_k}(n_k)| \leq \|U\|^p\|f_k\|.$$

It follows that for fixed k and for each p

$$[\ |\widehat{\mu}(n_k)| \cdot \|U\|\ ]^{-p} \leq \|f_k\| \cdot |\widehat{f_k}(n_k)|^{-1},$$

from which we obtain $|\widehat{\mu}(n_k)| \cdot \|U\| \geq 1$, and hence $\|U\| \geq (|\widehat{\mu}(n_k)|)^{-1}$. Since this is true for each k and by assumption $\widehat{\mu}(n_k) =$ converges to 0, we obtain a contradiction. Thus dim $M < \infty$.

We now characterize the Riesz operators among the classes $\Omega_+(X)$ and $\Omega_-(X)$.

Theorem 7.30. *For every operator $T \in L(X)$ on a Banach space X, the following statements are equivalent:*

(i) *$T \in R(X)$;*

(ii) *$T \in \Omega_+(X)$ and $\sigma(T)$ is a finite set or a sequence which converges to* 0*;*

(iii) *$T \in \Omega_-(X)$ and $\sigma(T)$ is a finite set or a sequence which converges to* 0.

Proof (i) $\Leftrightarrow$ (ii) The spectrum of a Riesz operator is finite or a sequence which clusters at 0 so the implication (i) $\Rightarrow$ (ii) is clear from Theorem 7.28.

Conversely, suppose that there exists an operator $T \in L(X)$ such that the condition (ii) is satisfied. Let λ be any spectral point different from 0 and denote by P_λ the spectral projection associated with the spectral set $\{\lambda\}$. By Theorem 3.111 to prove that $T \in R(X)$ it suffices to show that P_λ is a finite-dimensional operator. If we let $M_\lambda := P_\lambda(X)$, from the functional calculus we know that M_λ is a closed T-invariant subspace. Furthermore, we have $\sigma(T|M_\lambda) = \{\lambda\}$, so that $0 \notin \sigma(T|M_\lambda)$ and hence $T|M_\lambda$ is invertible. Since $T \in \Omega_+(X)$ this implies that $M_\lambda = P_\lambda(X)$ is finite-dimensional, as desired.

(i) $\Leftrightarrow$ (iii) The implication (i) $\Rightarrow$ (iii) is clear, again by Theorem 7.28. Conversely, suppose that there exists an operator $T \in L(X)$ such that the condition (iii) is satisfied. Let λ be any spectral point different from 0 and, as above, let P_λ be the spectral projection associated with the spectral set $\{\lambda\}$. Again, by Theorem 3.111, to prove that $T \in R(X)$ it suffices to show that P_λ is a finite-dimensional operator. If we let $M_\lambda := P_\lambda(X)$ and $N_\lambda := (I - P_\lambda)(X)$, then M_λ and N_λ are both T-invariant closed subspaces. Furthermore, from the functional calculus we have $X = M_\lambda \oplus N_\lambda$, $\sigma(T|M_\lambda) = \{\lambda\}$ and $\sigma(T|N_\lambda) = \sigma(T) \setminus \{\lambda\}$. We claim that $Q_{N_\lambda}T : X \to X/N_\lambda$ is surjective.

To see this observe first that for every $\widehat{x} \in X/N_\lambda$ there is $z \in X$ such that $Q_{N_\lambda}z = \widehat{x}$. From the decomposition $X = M_\lambda \oplus N_\lambda$ we know that there exist $u \in M_\lambda$ and $v \in N_\lambda$ such that $z = u + v$, and consequently,

$$\widehat{x} = Q_{N_\lambda}z = Q_{N_\lambda}u.$$

Since $0 \notin \{\lambda\} = \sigma(T|M_\lambda)$, the operator $T|M_\lambda : M_\lambda \to M_\lambda$ is bijective, so there exists $w \in M_\lambda$ such that $u = (T|M_\lambda)w = Tw$. From this it follows that

$$\widehat{x} = Q_{N_\lambda} u = (Q_{N_\lambda} T)w,$$

i.e, $Q_{N_\lambda} T$ is surjective. Since, by assumption, $T \in \Omega_-(X)$ it follows that codim $N_\lambda < \infty$, and hence dim $M_\lambda < \infty$. This show that P_λ is a finite-dimensional operator, so the proof is complete. ∎

The class of Riesz operators may be also characterized as the set of operators for which the sum with a compact operator is an $\Omega_+(X)$ operator, as well as an $\Omega_-(X)$ operator.

Theorem 7.31. *If $T \in L(X)$ is an operator on a Banach space X, the following statements are equivalent:*

(i) *T is Riesz;*

(ii) *For every compact operator $K \in K(X)$ we have $T + K \in \Omega_+(X)$;*

(iii) *For every compact operator $K \in K(X)$ we have $T + K \in \Omega_-(X)$.*

Proof (i) $\Leftrightarrow$ (ii) If $T \in R(X)$ then $T + K \in R(X)$ by (iv) of Theorem 3.112, and hence $T + K \in \Omega_+(X)$ by Theorem 7.28.

Conversely, if T is not a Riesz operator there exists by Theorem 3.111 some $\lambda \neq 0$ such that $\lambda I - T \notin \Phi_+(X)$. Thus from part (i) of Theorem 7.16 we may find a compact operator $K \in L(X)$ such that $\alpha(\lambda I - T + K) = \infty$. If $M := \ker(\lambda I - T + K)$ then M is infinite-dimensional and $(T + K)x = -\lambda x$ for each $x \in M$. Therefore $T + K \notin \Omega_+(X)$.

(i) $\Leftrightarrow$ (iii) If $T \in R(X)$ then $T + K \in R(X)$ by (iv) of Theorem 3.112, and hence $T + K \in \Omega_-(X)$ by Theorem 7.28.

Conversely, if T is not a Riesz operator there exists by Theorem 3.111 some $\lambda \neq 0$ such that $\lambda I - T \notin \Phi_-(X)$. From part (ii) of Theorem 7.16 there exists a compact operator $K \in L(X)$ such that $\overline{\beta}(\lambda I - T - K) = \infty$. Let $M := \overline{(\lambda I - T - K)(X)}$. Then M is infinite-codimensional. Denote by $\widetilde{x}$ any residual class of X/M and define $(T + K)^M : X/M \to X/M$ as follows:

$$(T + K)^M \widetilde{x} := (\widetilde{T + K})x \quad \text{where } xI\widetilde{x}.$$

We show that $(T + K)^M$ is surjective. If $x \in X$ we have $(\lambda I - T - K)x \in M$, and consequently $(\lambda I \widetilde{- T} - K)x = \widetilde{0}$. Therefore $\widetilde{x} = (T + K)^M(\lambda^{-1}\widetilde{x})$ for all $x \in X$. Since the operator $(T + K)^M$ has the same range as the composition operator $Q_M(T + K)$, so, M being infinite-codimensional we conclude that $T + K \notin \Omega_-(X)$. ∎

The class of all inessential operators $I(X)$ is contained in each one of the two classes $\Omega_+(X)$ and $\Omega_-(X)$, since $I(X) \subseteq R(X)$. We see now that the structure of ideal of $I(X)$ characterizes this class in $\Omega_+(X)$ and $\Omega_-(X)$.

Theorem 7.32. *For every Banach space X $I(X)$ is the uniquely determined maximal ideal of $\Omega_+(X)$ operators. Each ideal of $\Omega_+(X)$ operators is* *contained in $I(X)$. Analogously, $I(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators. Each ideal of $\Omega_-(X)$ operators is contained in $I(X)$.*

Proof Let G be any ideal of $\Omega_+(X)$ operators. Furthermore, let T be a fixed element of G and S any bounded operator on X. Then $ST \in G$, $\ker(I - ST)$ is a closed subspace invariant under ST, and the restriction of ST to $\ker(I - ST)$ coincides with the restriction of I to $\ker(I - ST)$. Therefore $ST|\ker(I - ST)$ has a bounded inverse. From the definition of $\Omega_+(X)$ operators $\ker(I - ST)$ is finite-dimensional, and hence by Theorem 7.17 $T \in I(X)$. Hence any ideal of $\Omega_+(X)$ operators is contained in $I(X)$. On the other hand, $I(X)$ is itself an ideal of $\Omega_+(X)$ operators, so $I(X)$ is the uniquely determined maximal ideal of $\Omega_+(X)$ operators.

To show that $I(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators, let us consider any ideal G of $\Omega_-(X)$ operators. If $T \in G$ and S is any bounded operator on X then $TS \in G$. Let $M := \overline{(I - TS)(X)}$. Then

$$[(TS(I - TS)](X) = [(I - TS)TS](X) \subseteq (I - TS)(X).$$

From this it follows that M is invariant under TS. Let us consider the induced quotient map $(TS)^M : X/M \to X/M$. The map $(TS)^M$ is surjective; in fact, if $x \in X$ then from $(I - TS)x \in M$ we obtain that $\widetilde{x} = \widetilde{(TS)}x$. From the definition of $\Omega_-(X)$ operators it follows that

$$\overline{\beta}(I - TS) = \text{codim}\ \overline{I - TS)(X)} < \infty,$$

and thus by Theorem 7.18 $T \in I(X)$. Since $I(X)$ is itself an ideal of $\Omega_-(X)$ operators we conclude that $I(X)$ is the uniquely determined maximal ideal of $\Omega_-(X)$ operators. ∎

The following useful result characterizes the inessential operators acting between different Banach space by means of the two classes $\Omega_+(X)$ and $\Omega_-(X)$.

Theorem 7.33. *If $T \in L(X, Y)$, where X and Y are Banach spaces, then the following assertions are equivalent:*

(i) *T is inessential;*

(ii) *$ST \in \Omega_+(X)$ for all $S \in L(Y, X)$;*

(i) *$TS \in \Omega_-(Y)$ for all $S \in L(Y, X)$.*

Proof (i) ⇔ (ii) Let $T \in I(X, Y)$ and $S \in L(Y, X)$. Then for each $\lambda \neq 0$ $\lambda I_X - ST \in \Phi(X)$, and hence $ST \in R(X) \subseteq \Omega_+(X)$.

Conversely, let us suppose that $T \notin I(X, Y)$. Then by Theorem 7.17 there exists an operator $S \in L(Y, X)$ such that $\alpha(I_X - ST) = \infty$. Let $M := \ker(I_X - ST)$. Clearly $ST|M = I_X|M$, and since M is infinite-dimensional we conclude that $ST \notin \Omega_+(X)$, and the proof of the equivalence (i) ⇔ (ii) is complete.

(i) ⇔ (iii) Let $T \in I(X, Y)$ and $S \in L(Y, X)$. Then by Corollary 7.7 for each $\lambda \neq 0$ we have $\lambda I_Y - TS \in \Phi(Y)$, and hence $TS \in R(X) \subseteq \Omega_-(Y)$.

Conversely, let us suppose that $T \notin I(X,Y)$. Then by Theorem 7.18 there exists an operator $S \in L(Y,X)$ such that $\overline{\beta}(I_Y - TS) = \infty$. Let $M := \overline{(I-TS)(Y)}$. The induced quotient operator $(TS)^M : Y/M \to Y/M$ is surjective. In fact, for each $y \in Y$ we have $(I_Y - TS)y \in M$ and hence $(TS)^M \widetilde{y} = \widetilde{(TS)y}$. Since M is infinite-codimensional and $(TS)^M$ has the same range of $Q_M TS$ it then follows that $TS \notin \Omega_-(Y)$. ∎

3. Strictly singular and strictly cosingular operators

In this section we shall introduce two other important classes of operators in perturbation theory.

Definition 7.34. *Given two Banach spaces X and Y, an operator $T \in L(X,Y)$ is said to be* strictly singular *if no restriction TJ_M of T to an infinite-dimensional closed subspace M of X is an isomorphism.*

The operator $T \in L(X,Y)$ is said to be strictly cosingular *if there is no infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective.*

We denote by $SS(X,Y)$ and $SC(X,Y)$ the classes of all strictly singular operators and strictly cosingular operators, respectively.

For the detailed study of the basic properties of the two classes $SS(X,Y)$ and $SC(X,Y)$ we refer the reader to Pietsch [**262**], Section 1.9 and Section 1.10 (these operators are also called *Kato operators* and *Pełczyński operators*, respectively). We are interested in studyng the relationships between strictly singular and strictly cosingular operators and semi-Fredholm operators.

We remark, however, that $SS(X,Y)$ and $SC(X,Y)$ are closed linear subspaces of $L(X,Y)$. Furthermore, if $T \in L(X,Y)$, $S \in SS(X,Y)$ (respectively, $S \in SC(X,Y)$ and $U \in L(Y,Z)$) then $UST \in SS(X,Y)$ (respectively, $UST \in SC(X,Y)$). Therefore $SS(X) := SS(X,X)$ and $SC(X) := SC(X,X)$ are closed ideals of $L(X)$. That $SS(X)$ and $SC(X)$ are closed ideals in $L(X)$ may be shown by means of the *measure of non-strict singularity* defined by

$$\mu_S(T) := \sup_M \inf_{N \subseteq M} \|Q_N T\|,$$

where the sup is taken over all infinite-dimensional closed subspaces N, M of X, see for instance Schechter [**288**], and, analogously, by means of the *measure of non-strict cosingularity*, defined by

$$\mu_C(T) := \sup_M \inf_{N \supseteq M} \|T|N\|,$$

see González and Martinón [**136**]. Obviously

$$SS(X) \subseteq \Omega_+(X) \quad \text{and} \quad SC(X) \subseteq \Omega_-(X),$$

and hence by Theorem 7.32 $SS(X) \subseteq I(X)$ and $SC(X) \subseteq I(X)$.

In the next theorems we shall give a more precise localization of the two classes $SS(X,Y)$ and $SC(X,Y)$ with respect to the other subspaces. We first need a preliminary result on compact operators.

Theorem 7.35. *If X and Y are Banach spaces and $K \in L(X,Y)$ is a compact operator having closed range then K is finite-dimensional.*

Proof Suppose first that K is compact and bounded below, namely K has closed range and is injective. Let (y_n) be a bounded sequence of $T(X)$ and let (x_n) be a sequence of X for which $Kx_n = y_n$ for all $n \in \mathbb{N}$. Since K is bounded below there exists a $\delta > 0$ such that $\|y_n\| = \|Kx_n\| \geq \delta\|x_n\|$ for all $n \in \mathbb{N}$, so (x_n) is bounded. The compactness of K then implies that there exists a subsequence (x_{n_k}) of (x_n) such that $Kx_{n_k} = y_{n_k}$ converges as $k \to \infty$. Hence every bounded sequence of $T(X)$ contains a convergent subsequence and by a basic result of functional analysis this implies that $T(X)$ is finite-dimensional.

Assume now the more general case that $T \in K(X,Y)$ has closed range. If $T_0 : X \to T(X)$ is defined by $T_0 x := Tx$ for all $x \in X$, then T_0 is a compact operator from X onto $T(X)$. Therefore the dual ${T_0}^* : T(X)^* \to X^*$ is a compact operator by the classical Schauder theorem, see [**159**, Proposition 42.2]. Moreover, T_0 is onto, so ${T_0}^*$ is bounded below by Lemma 1.30. The first part of the proof then gives that ${T_0}^*$ is finite-dimensional, and hence $T(X)^*$ is finite-dimensional because it is isomorphic to the range of ${T_0}^*$. From this it follows that also $T(X)$ is finite-dimensional. ■

Theorem 7.36. *If X and Y are Banach spaces then*

$$K(X,Y) \subseteq SS(X,Y) \cap SC(X,Y).$$

Proof Suppose that $T \notin SS(X,Y)$ and $T \in K(X,Y)$. Then there exists a closed infinite-dimensional subspace M of X such that TJ_M admits a bounded inverse. Since TJ_M is compact and has closed range $T(M)$, it follows that $T(M)$ is finite-dimensional by Theorem 7.35, and hence also M is finite-dimensional, a contradiction. This shows the inclusion $K(X,Y) \subseteq SS(X,Y)$.

Analogously, suppose that $T \notin$ and $T \in K(X,Y)$. Then there exists a closed infinite-codimensional subspace N of Y such that $Q_N T$ is onto Y/N. But $T \in K(X,Y)$ so $Q_N T$ is compact, and since its range is closed then, again by Theorem 7.35, Y/N is finite-dimensional. Thus N is finite-codimensional, a contradiction. Hence $K(X,Y) \subseteq SS(X,Y)$. ■

We give in the sequel some examples of strictly singular and strictly cosingular operators. For some other example we refer to Goldberg [**129**].

Example 7.37. Let $p, q \in \mathbb{N}$ be such that $1 < p, q < \infty$ and $p \neq q$. Then $SS(\ell^p, \ell^q) = SC(\ell^p, \ell^q) = L(\ell^p, \ell^q)$. In fact, let $T \in L(\ell^p, \ell^q)$ be such that $T \notin SS(\ell^p, \ell^q)$. Then there exists a closed infinite-dimensional subspace M of ℓ^p such that $T|M$ has a bounded inverse. Therefore M is isomorphic to a subspace of ℓ^q, hence by a theorem owed to Banach [**55**, Theorem 1, p.194] ℓ^q is isomorphic to a closed subspace of M and hence to a subspace of ℓ^p, contradicting [**55**, Theorem 7, p.205]. This shows the equality $SS(\ell^p,\ell^q) = L(\ell^p,\ell^q)$. The equality $SC(\ell^p,\ell^q) = L(\ell^p,\ell^q)$ will follow from Theorem 7.51.

Example 7.38. If X is a Banach space which does not contain an infinite-dimensional reflexive subspace and Y is a reflexive Banach space, then $SS(X,Y) = L(X,Y)$ and $SS(Y,X) = L(Y,X)$. In fact, if $T \in L(X,Y)\setminus SS(X,Y)$ then X contains a closed infinite-dimensional subspace M which is isomorphic to a closed subspace of Y. Thus M is reflexive, and this is impossible. A similar reasoning shows that $SS(Y,X) = L(Y,X)$. Note that this result proves that

$$SS(\ell^1,\ell^2) = L(\ell^1,\ell^2) \quad \text{and} \quad SS(\ell^2,\ell^1) = L(\ell^2,\ell^1),$$

and these sets coincide with $SC(\ell^1,\ell^2)$ and $SC(\ell^2,\ell^1)$, respectively, by Theorem 7.49.

Example 7.39. A strictly singular operator can have a non-separable range, see Goldberg and Thorp [**128**, p. 335], in contrast to the well known result that every compact operator has separable range. Hence the inclusion $K(X,Y) \subseteq SS(X,Y)$ in general is proper. Another example of a non-compact strictly singular operator is provided by identity map J injecting ℓ^1 into ℓ^2. In fact, J is strictly singular, as noted in Example 7.38. Also the embedding from ℓ^1 into c_0 is an example of a non-compact strictly singular operator, see Lacey Withley [**194**]. An example of non-compact strictly singular integral operators from $L^1[0,1]$ into $L^p[0,1]$, $1 \le p < \infty$, are given in Goldberg [**129**, Example III.3.10 and Example III.3.12].

Example 7.40. The canonical embedding $i : c_0 \to \ell^\infty$ is strictly cosingular and non-compact. This easily follows once it is observed that every separable quotient of ℓ^∞ is reflexive and any reflexive quotient is finite-dimensional, see Lindenstrauss and Tzafriri [**221**]. Note that since $c_0{}^*$ is complemented in $\ell^{\infty *}$ that i^* is not strictly singular.

Example 7.41. If X and Y are Hilbert spaces then

$$K(X,Y) = SS(X,Y) = SC(X,Y) = I(X,Y).$$

The equality $SS(X,Y) = SC(X,Y) = I(X,Y)$ is a consequence of the next Theorem 7.51, whilst for the equality $K(X,Y) = S(X,Y)$, see Kato [**182**]. A classical result of Calkin [**75**] demonstrates that for $X := \ell^2$ the set $K(X)$ is the unique closed ideal of $L(X)$. An extension of this result was given by Gohberg, Markus and Fel'dman [**127**], who obtained the same conclusion for $X = \ell^p$, with $1 \le p < \infty$ and $X := c_0$, see also Herman [**158**] or the monograph [**76**, Section 5.4].

Example 7.42. Let $T : L^2[0,1] \to L^1[0,1]$ denote the natural inclusion. We show that T is strictly cosingular and but not strictly singular. Suppose that N is a closed subspace of $L^1[0,1]$ such that $Q_N T$ is surjective. Then

Q_N is a surjection onto a reflexive space and hence is weakly compact.

Now, since $L^1[0,1]$ has the Dunford–Pettis property it follows that Q_N is completely continuous and hence $Q_N T$ is compact. From this it follows that the quotient $L^1[0,1]/N$ is finite-dimensional, from which we conclude that T is strictly cosingular. To see that T is not strictly singular it is enough to observe that the restriction of T to the subspace generated by the *Rademacher functions* is an isomorphism, see Beauzamy [**64**, Proposition VI.1.1].

Example 7.43. Given two Banach spaces X and Y let $Z := X \oplus Y$ be canonically normed by $\|(x,y)\| := \max\{\|x\|, \|y\|\}$. If $T \in L(X,Y)$ let $L : Z \to Z$ be defined by $L(x,y) := (0, Tx)$. Then L is the composition mapping UTW, where W is the natural map from Z to X and U is the natural map from Y to Z. Therefore L is strictly singular (respectively, strictly cosingular, compact) if and only if T is strictly singular (respectively, strictly cosingular, compact). Note that this result can be used to obtain from the previous examples endomorphisms which show that the inclusion $K(Z) \subseteq SS(Z))$ and $K(Z) \subseteq SC(Z))$ are, in general, proper.

Theorem 7.44. *If X and Y are Banach spaces then*

$$SS(X,Y) \cup SC(X,Y) \subseteq I(X,Y).$$

Proof If $T \in SS(X,Y)$, for any $S \in L(Y,X)$ we have $ST \in SS(X,Y) \subseteq \Omega_+(X)$, so by Theorem 7.33 $T \in L(X,Y)$. The inclusion $SC(X,Y) \subseteq I(X,Y)$ follows in a similar way from Theorem 7.33. ∎

Suppose now that $\Phi_+(X,Y) \neq \varnothing$ and let us consider the perturbation class :

$$P\Phi_+(X,Y) := \{T \in L(X,Y) : T + \Phi_+(X,Y) \subseteq \Phi_+(X,Y)\},$$

and, analogously, if $\Phi_-(X,Y) \neq \varnothing$ let

$$P\Phi_-(X,Y) := \{T \in L(X,Y) : T + \Phi_-(X,Y) \subseteq \Phi_-(X,Y)\}.$$

Note that $K(X,Y) \subseteq P\Phi_+(X,Y)$ and $K(X,Y) \subseteq P\Phi_-(X,Y)$, see Remark 1.54, part (f).

Lemma 7.45. *If X and Y are infinite-dimensional Banach spaces and $T \in L(X,Y)$ then the following implications hold:*

$$T \in \Phi_+(X,Y) \Rightarrow T \notin SS(X,Y), \quad T \in \Phi_-(X,Y) \Rightarrow T \notin SC(X,Y).$$

Proof If $T \in \Phi_+(X,Y)$ then $X = \ker T \oplus M$ for some closed subspace M of X, and $T_0 : M \to T(X)$, defined by $T_0 x := Tx$ for all $x \in M$, is an isomorphism between infinite-dimensional closed subspaces. Hence $T \notin SS(X,Y)$.

Analogously, if $T \in \Phi_-(X,Y)$ then $Y = T(X) \oplus N$ for some infinite-codimensional closed subspace N. Evidently $Y/N = Q_N(Y) = Q_N T(X)$, so $T \notin SC(X,Y)$. ∎

Theorem 7.46. *If X and Y are Banach spaces and $\Phi_+(X,Y) \neq \varnothing$ then*

$$SS(X,Y) \subseteq P\Phi_+(X,Y) \subseteq I(X,Y). \tag{179}$$

Analogously, if $\Phi_-(X,Y) \neq \varnothing$ then

$$SC(X,Y) \subseteq P\Phi_-(X,Y) \subseteq I(X,Y). \tag{180}$$

Proof To show the first inclusion of (179) assume that $\Phi_+(X,Y) \neq \varnothing$ and suppose that $T \in SS(X,Y)$ and $S \in \Phi_+(X,Y)$. We need to show that $T+S \in \Phi_+(X,Y)$. Suppose that $T+S \notin \Phi_+(X,Y)$. By part (i) of Theorem 7.16 there exists a compact operator $K \in K(X,Y)$ such that $\alpha(T+S-K) = \infty$. Set $M := \ker(T+S-K)$. Then M is a closed infinite-dimensional subspace and $TJ_M = (K-S)J_M$. If $U := K-S$ then $U \in \Phi_+(X,Y)$ because $S \in \Phi_+(X,Y)$ and K is compact. Hence $TJ_M = UJ_M \in \Phi_+(M,Y)$, and from Lemma 7.45 we obtain that $TS \notin SS(M,Y)$. From this it follows that $T \notin SS(X,Y)$, a contradiction. Therefore the inclusion $\mathcal{SS}(X,Y) \subseteq P\Phi_+(X,Y)$ is proved.

To show the second inclusion of (179) assume that $T \in L(X,Y)$ and $T \notin I(X,Y)$. By Theorem 7.17 there exists an operator $S \in L(Y,X)$ such that $I_X - ST \notin \Phi_+(X)$. On the other hand, for every operator $U \in \Phi_+(X,Y)$ we have $U(I_X - ST) = U - UST \notin \Phi_+(X,Y)$, otherwise we would have $I_X - ST \in \Phi_+(X,Y)$, see part (c) of Remark 1.54. But $U \in \Phi_+(X,Y)$ so $UST \notin P\Phi_+(X,Y)$, and hence by part (ii) of Theorem 7.21 we conclude that $T \notin P\Phi_+(X,Y)$, and this shows the second inclusion of (179).

To show the first inclusion of (180) assume that $\Phi_-(X,Y) \neq \varnothing$ and suppose that $T \in SC(X,Y)$ and $S \in \Phi_-(X,Y)$. We need to show that $T+S \in \Phi_-(X,Y)$. Suppose that $T+S \notin \Phi_-(X,Y)$. By Theorem 7.16, part (ii), there exists a compact operator $K \in K(X,Y)$ such that

$$M := \overline{(T+S-K)(X)}$$

is a closed infinite-codimensional subspace of Y. Denote by $\widetilde{x}$ the residual class $x+M$ in X/M. Since $(T+S-K)x \in M$ it follows that $Q_M(T+S-K)x = \widetilde{0}$ for all $x \in X$. Therefore $Q_M T = Q_M(K-S)$. Clearly, if $U := K-S$ then $U \in \Phi_-(X,Y)$, and therefore $Q_M U \in \Phi_-(X,Y/M)$. By Lemma 7.45 we infer that $Q_M T = Q_M U \notin SC(X,Y/M)$ and this implies that $T \notin SC(X,Y)$.

To show the second inclusion of (180) assume that $T \in L(X,Y)$ and $T \notin I(X,Y)$. By Theorem 7.18 there exists $S \in L(Y,X)$ such that $I_Y - TS \notin \Phi_-(Y)$. For every operator $U \in \Phi_-(Y,X)$ it follows that

$$U(I_Y - TS) = U - UTS \notin \Phi_-(Y,X).$$

But $U \in \Phi_-(Y,X)$, so $UTS \notin P\Phi_-(Y,X)$, and hence $T \notin \oplus_-(X,Y)$, so also the second inclusion in (180) is proved. ∎

We show next that if someone of the two Banach spaces X or Y possesses many complemented subspaces, in the sense of the following definitions, the inclusions of Theorem 7.46 actually are equalities.

Definition 7.47. *A Banach space X is said to be* subprojective *if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace which is complemented in X.*

The space X is said to be superprojective *if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace which is complemented in X.*

Theorem 7.48. *Let X be a Banach space. Then the following assertions hold:*

(i) *If X is subprojective and N is an infinite-codimensional reflexive subspace of X^* then there exists an infinite-codimensional complemented subspace of X^* which contains N;*

(ii) *If X is superprojective and N is an infinite-dimensional reflexive subspace of X^* then there exists an infinite-dimensional complemented subspace of X^* contained in N.*

Proof (i) Note that $\dim {}^\perp N = \text{cod } ({}^\perp N)^\perp = \infty$. By the subprojectivity of X there exists an infinite-dimensional subspace M which is complemented in X and is contained in ${}^\perp N$. Then $X^* = M^\perp \oplus W$ and $M^\perp \supseteq ({}^\perp N)^\perp \supset N$, and $\text{cod } M^\perp = \dim (X^*/M^\perp) = \dim M^* = \infty$.

(ii) Note that $\text{cod } {}^\perp N = \dim (X/{}^\perp N)^* = \dim ({}^\perp N)^\perp = \infty$. By the superprojectivity of X there exists an infinite-codimensional complemented subspace M which contains in ${}^\perp N$. Then $M^\perp$ is complemented in X^*, $M^\perp \subseteq ({}^\perp N)^\perp = N$, and $\dim M^\perp = \dim (X/M)^* = \infty$. ∎

Corollary 7.49. *Let X be a reflexive Banach space. Then X is subprojective (respectively, superprojective) if and only if X^* is superprojective (respectively, subprojective).*

Example 7.50. We list some examples of subprojective and superprojective Banach spaces. Further examples may be found in Whitley [**328**], Pełczyński [**259**], Aiena and González [**18**].

(a) Evidently every Hilbert space is both subprojective and superprojective. The spaces ℓ^p with $1 < p < \infty$, and c_0 are subprojective, see Whitley [**328**, Theorem 3.2]. Corollary 7.49 shows that all spaces ℓ^p with $1 < p < \infty$ are also superprojective.

(b) The spaces $L^p[0,1]$, with $2 \le p < \infty$, are subprojective and $L^p[0,1]$, with $1 < p < 2$ are not subprojective, see Whitley [**328**, Theorem 3.4]. Corollary 7.49 shows that all spaces $L^p[0,1]$ with $1 < p \le 2$ are subprojective and are not superprojective for $2 < p < \infty$.

(c) The spaces $L^1[0,1]$ and $C[0,1]$ are neither subprojective nor superprojective, see Whitley [**328**, Corollary 3.6].

Theorem 7.51. *Let X, Y be Banach spaces. The following statements hold:*

(i)*) If Y is subprojective then $SS(X,Y) = P\Phi_+(X,Y) = I(X,Y)$;*

(ii) *If X is superprojective then $SC(X,Y) = P\Phi_-(X,Y) = I(X,Y)$.*

Proof (i) Let $T \in L(X,Y)$, $T \notin SS(X,Y)$. Then there exists an infinite-dimensional closed subspace M of X such that the restriction TJ_M is bounded below. Since Y is subprojective we can assume that the subspace $T(M)$ is complemented in Y, namely $Y = T(M) \oplus N$. Then $X = M \oplus T^{-1}(N)$ and defining $A \in L(Y,X)$ equal to T^{-1} in $T(M)$ and equal to 0 in N it easily follows that AT is a projection of X onto M. Therefore $\ker(I_X - AT) = (AT)(X) = M$ is infinite-dimensional, so we have $I_X - AT \notin \Phi(X)$ and consequently $T \notin I(X,Y)$.

(ii) Let $T \in L(X,Y)$, $T \notin SC(X,Y)$. Then there exists a closed infinite-codimensional subspace N of Y such that $Q_N T$ is surjective. Note that $T^{-1}(N)$ is infinite-codimensional because Y/N is isomorphic to $X/T^{-1}(N)$. So since X is subprojective we can assume that $T^{-1}(N)$ is complemented in X. Clearly $\ker T \subset T^{-1}(N)$, and hence if $X = M \oplus T^{-1}(N)$ then $T(M) \cap N = \{0\}$. Moreover, $Y = T(X) + N$ implies $Y = T(M) + N$, from which it follows that $T(M)$ is closed, see Theorem 1.14. Therefore the topological direct sum $Y = T(M) \oplus N$ is satisfied. Now, if we define $A \in L(Y,X)$ as in part (i) we then easily obtain that $T \notin I(X,Y)$. ■

The following examples show that the property for an operator of being strictly singular or strictly cosingular is not preserved under conjugation.

Example 7.52. Let $T \in L(\ell^1, \ell^2)$ be onto. Note that such a map does exist. In fact, by a well known result of Banach and Mazur [**56**, p. 2], given any separable Banach space X there exists a continuous linear operator which maps ℓ^1 onto X. From Example 7.38 we know that $T \in SS(\ell^1, \ell^2)$ whilst its dual T^* has a bounded inverse and hence is not strictly singular.

To see that the dual of a strictly cosingular operator may be not strictly cosingular, let us consider the natural inclusion T of $L^2[0,1]$ into $L^\infty[0,1]$. We have already proved in Example 7.42 that T is strictly cosingular and not strictly singular. On the other hand, T^* cannot be strictly cosingular because this would imply, as we show now in the next Theorem 7.53, that T is strictly singular.

Theorem 7.53. *If X and Y are Banach spaces and $T^* \in SS(Y^*, X^*)$, then $T \in SC(X,Y)$. Analogously, if $T^* \in SC(Y^*, X^*)$ then $T \in SS(X,Y)$.*

Proof The proof is analogous to that of Theorem 7.27. ■

We complement the result of Theorem 7.53 by mentioning a result established by Pełczyński [**260**], which shows that T^* is strictly singular whenever $T \in L(X,Y)$ is strictly cosingular and weakly compact. It is an open problem if T^* is strictly cosingular whenever $T \in L(X,Y)$ is strictly singular and weakly compact, or, equivalently, if T is strictly singular and Y is reflexive.

Theorem 7.54. *Let X and Y be Banach spaces and $T \in L(X,Y)$. Then the following statements hold:*

(i) *If* $T^* \in SS(Y^*, X^*)$ *and* Y *is subprojective then* $T \in SS(X, Y)$*;*

(ii) *If* $T \in SS(Y, X)$*, where* X *is reflexive and* X^* *is subprojective, then* $T^* \in SS(Y^*, X^*)$*.*

Proof (i) Suppose that $T \notin SS(X, Y)$. Then there is an infinite-dimensional closed subspace M of X such that $T|M$ has a bounded inverse. Since Y is subprojective we can find a closed infinite-dimensional subspace N of $T(M)$ and a bounded projection P of Y onto N. Note that $N = T(W)$, where W is a closed infinite-dimensional subspace of M. Since P maps Y onto N its dual P^* is an injective continuous map of N^* onto an infinite-dimensional subspace Z of Y^*. We prove that $T^*|Z$ has a bounded inverse and is therefore not strictly singular. For any $f \in Z$ we have

$$\|T^* f\| = \sup_{0 \neq x \in X} \frac{\|(T^* f)(x)\|}{\|x\|} \geq \sup_{0 \neq x \in W} \frac{\|f(Tx)\|}{\|x\|}.$$

Since $T|M$ has a bounded inverse there is $K > 0$ such that $\|Tx\| \geq K\|x\|$ for all $x \in W \subseteq M$. Moreover, since $f \in P^*(N^*)$ there is $g \in N^*$ such that $P^* g = f$. Combining this information, we deduce that

$$\|T^* f\| \geq \sup_{0 \neq y \in N} \frac{K\|(P^* g)y\|}{\|y\|} = \sup_{0 \neq y \in N} \frac{K\|g(Py)\|}{\|y\|} = K\|g\|.$$

Finally, from $\|P^*\|\|g\| \geq \|P^* g\| = \|f\|$ we obtain $\|T^* f\| \geq (K/\|P^*\|)\|f\|$. This shows that T^* has a bounded inverse on the subspace Z of Y^*, so that $T^* \notin SS(Y^*, X^*)$.

(ii) Note that $T^{**} = J_Y T {J_X}^{-1}$, where J_X and J_Y are the canonical embeddings of, respectively, X onto X^{**} and Y into Y^{**}. Then if T is strictly singular so is T^{**}, and by part (i) also T^* is strictly singular. ∎

Corollary 7.55. *If* X *is a Hilbert space and if* $T \in SS(X, Y)$ *then also* T^* *is strictly singular. If* Y *is a Hilbert space and if* $T^* \in SS(Y^*, X^*)$*, then* T *is strictly singular.* ∎

Observe that every separable Banach space is isomorphic to a subspace of $C[0, 1]$. So the hypothesis of the following result implies that $\Phi_+(X, Y) \neq \varnothing$.

Theorem 7.56. *Suppose that* X *is separable and* Y *contains a complemented subspace isomorphic to* $C[0, 1]$*. Then* $P\Phi_+(X, Y) = SS(X, Y)$*.*

Proof Since $C[0, 1]$ is isomorphic to $C[0, 1] \times C[0, 1]$ there are closed subspaces W and Z of Y such that W is isomorphic to Y, Z is isomorphic to $C[0, 1]$ and $Y = W \oplus Z$. Let $r > 0$ be such that $\|a + b\| \geq r \max\{\|a\|, \|b\|\}$ for every $a \in W$ and $b \in Z$, and let $U \in L(Y)$ be an isomorphism with range equal to W.

Suppose that $K \in L(X, Y)$ is not strictly singular. Let $K_1 := UK \in L(X, Y)$. Without loss of generality we assume that $\|K_1\| = 1$. Then there exist an infinite-dimensional subspace M of X and $c > 0$ such that

$\|K_1 m\| \geq c\|m\|$ for every $m \in M$. We denote $d := \min\{\frac{c}{3}, \frac{1}{3}\}$. Since X is separable there exists an isomorphism V from X/M into Y with range contained in Z. We define $S \in L(X,Y)$ by $S := VQ_M$. Without loss of generality we assume that $\|Sx\| \geq \|Q_M x\|$ for every $x \in X$.

Let us see that the operator $S + K_1$ is an isomorphism into. Indeed, let $x \in X$ be a vector of norm-one. If $\|Q_M x\| \geq d$ then

$$\|(S + K_1)x\| \geq r\|Sx\| \geq rd.$$

Otherwise $\|Q_M x\| < d$, and we can choose $m \in M$ such that $\|x - m\| < d$, hence $\|m\| > 2/3$. Thus

$$\|(S + K_1)x\| \geq r\|K_1 x\| \geq r(\|K_1 m\| - \|x - m\|) \geq r(3d(2/3) - d) = rd.$$

Thus $K_1 \notin P\Phi_+(X,Y)$ because $S + K_1$ is upper semi-Fredholm, but S is not. Hence $K \notin P\Phi_+(X,Y)$ by part (ii) of Theorem 7.21. ∎

It is an interesting question to see if it is possible in Theorem 7.56 to remove the requirement for $C(0,1)$ to be complemented.

Observe that every separable Banach space is isomorphic to a quotient of ℓ^1. So the hypothesis of the following result implies that $\Phi_-(X,Y) \neq \varnothing$.

Theorem 7.57. *Suppose that X contains a complemented subspace isomorphic to ℓ^1 and Y is separable. Then*

$$P\Phi_-(X;Y) = SC(X,Y).$$

Proof Since ℓ^l is isomorphic to $\ell^1 \times \ell^1$ there are closed subspaces W and Z of X such that W is isomorphic to X, Z is isomorphic to ℓ^1, and $X = W \oplus Z$. Let $U \in L(X)$ be an operator which is an isomorphism from W onto X, with kernel equal to Z. Suppose that $K \in L(X,Y)$ is not strictly cosingular. Then there exists a closed infinite-codimensional subspace M of Y such that the operator $Q_M K$ is surjective; that is, $M + R(K) = Y$. We consider the operator $K_1 := KU \in L(X,Y)$.

Since Y is separable there exists an operator $S \in L(X,Y)$ with kernel equal to W and range equal to M. Clearly $S + K_1$ is surjective, but S is not a lower semi-Fredholm operator. Thus $K_1 \notin P\Phi_-(X,Y)$. Hence $K \notin P\Phi_-(X,Y)$, by part (ii) of Theorem 7.21. ∎

Also here an interesting question is that if it is possible in Theorem 7.57 to remove the requirement for ℓ^1 to be complemented.

It has been for a long time an open problem whether or not $\mathcal{SS}(X,Y) = P\Phi_+(X,Y)$ and $\mathcal{SC}(X,Y) = P\Phi_-(X,Y)$ for all Banach spaces X, Y. Next we show a very recent result of González [**131**] which gives an example of a Banach space for which these equalities do not hold. First we need a preliminary work.

Lemma 7.58. *Suppose that $T \in L(X)$ has range $T(X)$ not closed. Then for each $\varepsilon > 0$ there exists an infinite-dimensional closed subspace M of X such that $\|T|M\| < \varepsilon$.*

Proof Suppose that $T(X)$ is not closed. We now observe that if M is a closed finite-codimensional subspace of X then $T|M$ does not have a bounded inverse. In fact, if we assume that $T|M$ has a bounded inverse, then $T(M)$ is certainly closed. Since $X = M \oplus N$, with N finite-dimensional, then $T(X) = T(M) + T(N)$ with $T(M)$ closed and $T(N)$ finite-dimensional. Hence $T(X)$ is closed and this is a contradiction.

Now, since T does not have a bounded inverse there exists an $x_1 \in X$ such that $\|x_1\| = 1$ and $\|Tx_1\| < \varepsilon/3$. By the Hahn–Banach theorem there exists some $f_1 \in X^*$ such that $\|f_1\| = 1$ and $f_1(x_1) = \|x_1\| = 1$. Since $\ker f_1$ has codimension 1 in X we can find an $x_2 \in \ker f_1$ such that $\|x_2\| = 1$ and $\|Tx_2\| < \varepsilon/3^2$. Again, there exists $f_2 \in X^*$ such that $\|f_2\| = 1$ and $f_2(x_2) = \|x_2\| = 1$. Since $\ker f_1 \cap \ker f_2$ has finite codimension in X there exists an $x_3 \in X$ such that $\|x_3\| = 1$ and $\|Tx_3\| < \varepsilon/3^3$. Repeating this process we can construct a two sequences $(x_k) \subseteq X$ and $(f_k) \subseteq X^*$ such that

$$\|x_k\| = \|f_k\| = f_k(x_k) = 1, \quad \|Tx_k\| < \varepsilon/3^k \quad \text{for all } k \in \mathbb{N}, \tag{181}$$

and

$$f_i(x_k) = 0 \quad \text{for } i < k. \tag{182}$$

The elements x_k are linearly independent, hence if M denotes the subspace spanned by these elements M is infinite-dimensional. We show now that $T|M$ and hence also $T|\overline{M}$ has norm less than ε. Let us consider $x := \sum_{i=1}^n \lambda_i x_i \in M$. From (181) and (182) we have

$$|f_i(x)| = \left| f_i(\sum_{i=1}^n \lambda_i x_i) \right| = |\lambda_1| \leq \|f_i\| \|x\| = \|x\|.$$

We show by induction that

$$|\lambda_k| \leq 2^{k-1}\|x\|, \quad k = 1, \dots, n. \tag{183}$$

For $k = 1$ this inequality has been observed above. Suppose that (183) is true for $k \leq j < n$. Then from (181) and (182) we obtain

$$f_{j+1}(x) = \sum_{i=1}^j \lambda_i f_{j+1}(x_i) + \lambda_{j+1}.$$

Using the induction hypothesis we the obtain

$$\begin{aligned} |\lambda_{j+1}| &\leq |f_{j+1}(x)| + \sum_{i=1}^j |\lambda_i||f_{j+1}(x_i)| \\ &\leq \|x\| + \sum_{j=1}^j 2^{i-1}\|x\| \leq 2^j \|x\|. \end{aligned}$$

Thus (183) follows by induction. Moreover, for every $x \in M$ we have

$$\|Tx\| \leq \sum_{i=1}^{n} |\lambda_i| \|Tx_i\| \leq \sum_{i=1}^{n} 2^{i-1}\varepsilon 3^i \|x\| = \frac{\varepsilon}{2}\|x\|.$$

Since this is true for every $x \in M$ we then conclude that $\|T|M\| \leq \varepsilon/2 < \varepsilon$ and hence $\|T|\overline{M}\| < \varepsilon$, as desired. ∎

Definition 7.59. *A Banach space X is said to be* decomposable *if it contains a pair of infinite-dimensional closed subspaces M and N, such that $X = M \oplus N$. Otherwise X is said to be* indecomposable. *A Banach space X is said to be* hereditarily indecomposable *if every closed subspace of X is indecomposable, i.e., there no exist infinite-dimensional closed subspaces M and N for which $M \cap N = \{0\}$ and $M + N$ is closed. Dually, a Banach space X is said to be* quotient hereditarily indecomposable *if every quotient of X is indecomposable.*

Finite-dimensional Banach spaces are trivial examples of indecomposable spaces. Note that the existence of infinite-dimensional indecomposable Banach spaces has been a long standing open problem and has been positively solved by Gowers and Maurey [**137**] and [**138**], who construct an example of a reflexive hereditarily indecomposable Banach space X_{GM}. Thus the dual space X_{GM}^* is quotient hereditarily indecomposable. Moreover, Gowers and Maurey constructed in [**138**] a whole family of other indecomposable Banach spaces, amongst them we shall find some Banach spaces useful in order to construct our counterexamples.

Roughly speaking one can say that indecomposable Banach spaces are Banach spaces with small spaces of operators. In fact, it is shown in [**137**] that if X is hereditarily indecomposable then $L(X) = \{\mathbb{C}I_X\} \oplus SS(X)$, whilst if X is quotient hereditarily indecomposable, then $L(X) = \{\mathbb{C}I_X\} \oplus SC(X)$. The Gowers Maurey construction is rather technical and requires some techniques of analysis combined with involved combinatorial arguments. We do not describe this space (a beautiful discussion of it may be found in Bollobás [**71**]), but in the sequel we shall point out some of the properties of these spaces, from the point of view of Fredholm theory, needed for the construction of our counterexamples.

Theorem 7.60. *Let X be a Banach space. Then we have:*

i $L(X, Y) = \Phi_+(X, Y) \cup SS(X, Y)$ *for all Banach spaces Y if and only if X is hereditarily indecomposable;*

(ii) $L(Y, X) = \Phi_-(Y, X) \cup SC(Y, X)$ *for all Banach spaces Y if and only if Y is quotient hereditarily indecomposable.*

Proof (i) Suppose that there exists an operator $T \in L(X, Y)$ for which $T \notin \Phi_+(X, Y) \cup SS(X, Y)$. We may assume $\|T\| = 1$. Since $T \notin SS(X, Y)$ T is not invertible on infinite-dimensional closed subspaces of X, so there exist $\varepsilon > 0$ and a closed infinite-dimensional subspace M such that $\|Tu\| \geq \varepsilon\|x\|$

for all $u \in M$. On the other hand, since $T \notin \Phi_+(X,Y)$ there exists by Lemma 7.58 a closed infinite-dimensional subspace N of X such that $\|Tv\| < \varepsilon/2\|v\|$ for all $v \in N$. Let $u \in M$ and $v \in N$ such that $\|u\| = \|v\| = 1$. Then

$$\|u+v\| \geq \|Tu+Tv\| \geq \|Tu\| - \|Tv\| > \varepsilon/2.$$

Therefore the operator $S : M \times N \to X$, defined by $S(u,v) := u+v$ for all $u \in M$ and $v \in N$, has a continuous inverse and consequently $M+N$ is a direct sum. Hence the closed subspace $M \oplus N$ is not decomposable, so X is not hereditarily indecomposable.

Conversely, assume that not all the closed subspaces of X are indecomposable. Then we can find two closed infinite-dimensional subspaces M and N of X such that $M \cup N = \{0\}$ and $M+N$ is closed. From this it follows that the quotient map $Q_N \in L(X, X/N)$ does not belong to $SS(X,Y) \cup \Phi_+(X,Y)$, since Q_N is an isomorphism on M and N is infinite-dimensional.

(ii) Assume that $T \in L(Y,X)$ but $T \notin \Phi_-(Y,X) \cup SC(Y,X)$. Since T is not strictly cosingular there exists an infinite-codimensional closed subspace M of X such that $Q_M T$ is onto. By the open mapping theorem there exists a $\delta > 0$ such that $\delta \mathbf{D}_{X/M} \subseteq Q_M T(\mathbf{D}_Y)$, where $\mathbf{D}_{X/M}$ and $\mathbf{D}_Y$ denote the closed unit ball in X/M and Y, respectively. Therefore $\|T^* f\| \geq \delta \|f\|$ for every $f \in M^\perp$. On the other hand, since $T \notin \Phi_-(Y,X)$ there exists an infinite codimensional closed subspace N of X such that $\|Q_N T x\| < \delta/2\|x\|$ for every $x \in Y$. Therefore $\|T^*\| < \delta/2\|f\|$ for every $f \in N^\perp$. We have $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed. Then $X = M+N$ and $M \cap N$ is infinite-codimensional in M and N. This shows that the quotient

$$X/(M \cap N) = M/(M \cap N) \oplus N/(M \cap N)$$

is decomposable, and hence X is not quotient hereditarily indecomposable.

Conversely, if not all the quotient subspaces of X are indecomposable, then we can find two closed infinite-codimensional subspaces M and N of X such that $M \cap N = U$, $M+N = X$ and both the quotients M/U and N/U are infinite-dimensional, hence $X/U = M/U \oplus N/U$ is not indecomposable. Then the injection $J_M \in L(M,X)$ does not belong to $SC(M,X) \cup \Phi_-(M,X)$, because $M+N = X$ and both M and N are infinite-codimensional. ∎

Corollary 7.61. *Suppose that X is hereditarily indecomposable, Y is any Banach space and $\Phi_+(X,Y) \neq \varnothing$. Then $P\Phi_+(X,Y) = SS(X,Y)$. Analogously, if $\Phi_-(X,Y) \neq \varnothing$ then $P\Phi_-(X,Y) = SC(X,Y)$.*

Proof For the first equality, by Theorem 7.46 we need only to show that $P\Phi_+(X,Y) \subseteq SS(X,Y)$. Suppose that there is $T \in P\Phi_+(X,Y) \setminus SS(X,Y)$. By Theorem 7.60 then $T \in \Phi_+(X,Y)$ and hence $0 = T - T \in \Phi_+(X)$, which is impossible. Therefore $P\Phi_+(X,Y) = SS(X,Y)$.

The equality $P\Phi_-(X,Y) = SC(X,Y)$ by using a similar argument. ∎

Theorem 7.62. *If X is a hereditarily indecomposable space then for every Banach space Y $\Phi_+(X,Y)$ is a connected subset of $L(X,Y)$. In particular, the index is constant on $\Phi_+(X,Y)$.*

Proof Let T_0 and T_1 be in $\Phi_+(X,Y)$. For every $\lambda \in [0,1]$, consider the operators $T_\lambda := \lambda T_1 + (1-\lambda)T_0$. We consider two cases.

First case: Suppose that $T_\lambda \in \Phi_+(X,Y)$ for all $\lambda \in [0,1]$. Then clearly T_0 and T_1 belong to the same connected component of $\Phi_+(X,Y)$.

Second case: Suppose that $T_\lambda \notin \Phi_+(X,Y)$ for some $\mu \in (0,1)$. By part (i) then $T \in SS(X,Y)$ and we can write $\mu T_1 = (\mu-1)T_0 + T_\mu$. Since $\Phi_+(X,Y)$ is stable under strictly singular perturbations, by Theorem 7.46 it follows that $(\mu-1)T_0+\lambda T_\mu \in \Phi_+(X,Y)$ for all $\lambda \in (0,1)$. Hence $(\mu-1)T_0$ and μT_1 belong to the same connected component of $\Phi_+(X,Y)$. But $L(X,Y)$ is a complex Banach space, so for any operator $S \in \Phi_+(X,Y)$ and $\lambda \neq 0$, both S and λS belong to the same connected component of $\Phi_+(X,Y)$. Therefore again T_0 and T_1 belong to the same connected component of $\Phi_+(X,Y)$.

The last assertion is clear, the continuity of the index implies that the index is constant on connected components, see Goldberg [**129**, 7. V.1.6 Theorem]. ∎

Corollary 7.63. *If X is hereditarily indecomposable and $T \in \Phi_\pm(X)$ then* $\operatorname{ind} T = 0$.

Proof The index of the identity I_X is 0. ∎

Corollary 7.64. *If X is hereditarily indecomposable and $T \in L(X)$ then $\sigma_{\mathrm{f}}(T)$ is a singleton.*

Proof Since $F(X) \subseteq SS(X) \subseteq I(X)$, as it was observed in the beginning of this chapter $T \in \Phi(X)$ precisely when T is invertible in $L(X)$ modulo $SS(X)$. We know also that $\sigma_{\mathrm{f}}(T)$ is non-empty, so there exists some $\lambda \in \mathbb{C}$ such that $\lambda I_X - T$ is not Fredholm. By Theorem 7.60 and by the continuity of the index we obtain that $\lambda I_X - T$ is not semi-Fredholm. Hence $\lambda I_X - T \in SS(X)$ and $\sigma_{\mathrm{f}}(T) = \{\lambda\}$. ∎

We can now establish González's result.

Theorem 7.65. *Let X be a reflexive hereditarily indecomposable Banach space and Y be a closed subspace of X such that* $\dim Y = \operatorname{codim} Y = \infty$. *If $Z := X \times Y$ then $P\Phi_+(Z) \neq SS(Z)$ and $P\Phi_-(Z) \neq SC(Z)$.*

Proof We show first that $L(X,Y) = SS(X,Y)$. By Theorem 7.60 it suffices to prove that $\Phi_+(X,Y) = \varnothing$.

Suppose that there exists $T \in \Phi_+(X,Y)$. If J_Y denotes the canonical embedding of Y then $J_Y T \in \Phi_+(X)$ and ind $T = -\infty$, which is impossible by Corollary 7.63. Therefore $L(X,Y) = SS(X,Y)$, and from the inclusion $SS(X,Y) \subseteq I(X,Y) \subseteq L(X,Y)$ we then conclude that $L(X,Y) = I(X,Y)$. Note that by Corollary 7.9 we also have $L(Y,X) = I(Y,X)$.

We now prove that $\Phi_+(Z) = \Phi(Z)$. Clearly we have only to show that $\Phi_+(Z) \subseteq \Phi(Z)$. To see this inclusion set $T \in L(Z)$ as

$$T := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in L(X)$, $D \in L(Y)$, $B \in L(Y, X)$ and $C \in L(X, Y)$. Since $L(X, Y) = I(X, Y)$, by Theorem 7.25 we have that

$$\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{f}}(A) \cup \sigma_{\mathrm{f}}(D).$$

Thus $\sigma_{\mathrm{f}}(T)$ contains at most two points by Corollary 7.64. This also shows that if T is not Fredholm then there exists $\delta > 0$ such that $\lambda I_Z - T \in \Phi(Z)$ for all $0 < |\lambda| < \delta$. By the continuity of the index then $T \notin \Phi_+(Z)$. Hence $\Phi_+(Z) = \Phi(Z)$.

From the last equality and Theorem 7.22 we obtain that

$$P\Phi_+(Z) = P\Phi(Z) = I(Z),$$

so the operator

$$S = \begin{pmatrix} 0 & J_Y \\ 0 & 0 \end{pmatrix},$$

provides an example of an operator $S \in P\Phi_+(Z)$ which does not belong to $SS(Z)$. This shows that $P\Phi_+(Z) \neq SS(Z)$.

To see that $P\Phi_-(Z) \neq SC(Z)$ observe that $S \in P\Phi_-(Z)$ because Z is reflexive, and an operator $T \in \Phi_+(Z)$ if and only if $T \in \Phi_-(Z)$. However, S is not strictly singular since J_Y is surjective. ■

4. Improjective operators

The characterizations of inessential operators T given in the previous sections are given in terms of the properties of the product of T by a large class of operators, or also as the perturbation class of Fredholm operators. It is a problem of a certain interest to find an *intrinsic* characterization of inessential operators, for instance in terms of their action on certain subspaces. A characterization of this type is, for instance, obtained in the cases $I(X, Y) = SS(X, Y)$ or $I(X, Y) = SC(X, Y)$.

In this section we shall see that for many classic Banach spaces the class of the inessential operators coincide with a class of operators, the improjective operators, which are defined by means of their action on certain complemented subspaces. In order to see that, we first define an interesting class of operators introduced by Tarafdar [**305**] and [**306**].

Definition 7.66. *An operator $T \in L(X, Y)$ is said to be* improjective *if there exists no infinite-dimensional closed subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is a complemented subspace of Y. The set of all improjective operators from X into Y will be denoted by* $\mathrm{Imp}(X, Y)$ *and we set* $\mathrm{Imp}(X) := \mathrm{Imp}(X, X)$.

Trivially, the identity on an infinite-dimensional Banach space X and any isomorphism between infinite-dimensional Banach spaces are examples of operators which are not improjective. Moreover, the restriction $T|M$ of an improjective operator $T \in \mathrm{Imp}(X, Y)$ to a closed infinite-dimensional subspace M of X is also improjective.

The next result will be useful for studying the improjective operators and their relationships with other classes of operators.

Lemma 7.67. *Let X and Y be Banach spaces and $T \in L(X, Y)$. Then the following assertions hold:*

(i) *If M is a closed subspace of X such that TJ_M is an isomorphism, $T(M)$ is complemented in Y, and N is a closed complement of $T(M)$, then M is complemented in X and $T^{-1}(N)$ is a closed complement of M;*

(ii) *If N is a closed subspace of Y such that $Q_N T$ is surjective, $T^{-1}(N)$ is complemented in X, and M is a closed complement of $T^{-1}(N)$, then N is complemented in Y and $T(M)$ is a closed complement of N.*

Proof (i) If N is a closed complement of $T(M)$ in Y and TJ_M is an isomorphism then $T^{-1}(N) \cap M = \{0\}$, both $T^{-1}(N)$ and M are closed subspaces and $X = T^{-1}(N) \oplus M$. Hence the result is a direct consequence of the closed graph theorem.

If M is a closed complement of $T^{-1}(N)$ in X, since $\ker T$ is contained in $T^{-1}(N)$, we have $T(M) \cap N = \{0\}$. Moreover, since $Q_N T$ is surjective we obtain that $T(M) \oplus N = T(X) + N = Y$, and it follows from Theorem 1.14 that $T(M)$ is closed; hence N is complemented in Y. ∎

The class $\mathrm{Imp}(X, Y)$ admits the following dual characterization in terms of quotient maps.

Theorem 7.68. *An operator $T \in L(X, Y)$ is improjective if and only if there is no infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of X.*

Proof Assume that $T \in L(X, Y)$ is improjective and let N be a closed subspace of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of X. By Lemma 7.67 if M is a closed complement of $T^{-1}(N)$ then $T(M)$ is a closed complement of N. Since the restriction of T to M is an isomorphism and T is improjective it then follows that $T(M)$ is finite-dimensional; hence N is finite-codimensional.

Conversely, assume that T is not improjective and take an infinite-dimensional closed subspace M of X such that TJ_M is an isomorphism and $T(M)$ is complemented in Y. Given a closed complement N of $T(M)$ we have that N is infinite-codimensional and $Q_N T$ is surjective. Hence by Lemma 7.67 we conclude that M is a closed complement of $T^{-1}(N)$. ∎

Theorem 7.69. *For every pair of Banach spaces X and Y we have $I(X, Y) \subseteq \mathrm{Imp}(X, Y)$.*

Proof If $T \in L(X,Y)$ is not improjective then there exists an infinite-dimensional closed subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is a complemented subspace of Y. By Lemma 7.67 we have that M is also complemented in X and

$$X = M \oplus T^{-1}(N) \text{ and } Y = T(M) \oplus N,$$

where N and $T^{-1}(N)$ are closed subspaces of Y and X, respectively. So we can define an operator $S \in L(Y,X)$ by

$$Sy := \begin{cases} (T|_M)^{-1}y & \text{if } y \in T(M), \\ 0 & \text{if } y \in N. \end{cases}$$

We have that $\ker(I_X - ST) = M$; hence by Theorem 7.17 the operator T is not inessential. ■

Theorem 7.70. *Let $A \in L(Y,Z)$, $T \in \mathrm{Imp}(X,Y)$ and $B \in L(W,X)$. Then $TB \in \mathrm{Imp}(W,Y)$ and $AT \in \mathrm{Imp}(X,Z)$.*

Proof Assume first that AT is not improjective. Then we can find an infinite-dimensional closed subspace M of X such that ATJ_M is an isomorphism and $AT(M)$ is complemented in Z. Note that TJ_M is also an isomorphism; hence $T(M)$ is closed. Since $AJ_{T(M)}$ is an isomorphism it follows from Lemma 7.67 that $T(M)$ is complemented in Y. Hence T is not improjective.

In the case in which TB is not improjective Theorem 7.68 allows us to select an infinite-codimensional closed subspace N of Y such that $Q_N TB$ is surjective and $(TB)^{-1}(N)$ is a complemented subspace of W. Denoting $M := T^{-1}(N)$ we have that M is an infinite- codimensional closed subspace of X such that $Q_M B$ is surjective and $B^{-1}(M)$ is complemented. It follows from Lemma 7.67 that M is complemented in X. Moreover $Q_N T$ is surjective; hence it follows from Theorem 7.68 that T is not improjective. ■

To determine the structure of the set $\mathrm{Imp}(X,Y)$ we now introduce the following concept, that will be useful in our discussions.

Let $\mathcal{L}$ denotes the class of all bounded operator acting between Banach spaces, and let $\mathcal{F}$ denote the class of all operators with finite-dimensional range acting between Banach spaces. Given a subclass $\mathcal{A}$ of $\mathcal{L}$, the subsets

$$\mathcal{A}(X,Y) := \mathcal{A} \cap L(X,Y),$$

are called the *components* of $\mathcal{A}$. Moreover, we set $\mathcal{A}(X) := \mathcal{A}(X,X)$.

Definition 7.71. *A subclass $\mathcal{A}$ of L is said to be a* quasi-operator ideal *if it satisfies the following two conditions:*

(a) $\mathcal{F} \subseteq \mathcal{A}$;

(b) $A \in L(Y,Z)$, $K \in \mathcal{A}(X,Y)$, $B \in L(W,X) \Rightarrow AKB \in \mathcal{A}(W,Z)$.

A quasi-operator ideal is an *operator ideal* (in the sense of Pietsch [**263**]) if and only if $\mathcal{A}(X,Y)$ is a subspace of $L(X,Y)$ for every pair X, Y of Banach spaces. It is easily seen the the closure of a quasi-operator ideal, defined as the union of all closures of components, is a quasi-operator ideal. Evidently, I, SS, SC, K are operator ideals, while Imp is a quasi-operator ideal by Theorem 7.70.

Theorem 7.72. *The class of all improjective operators* Imp *is the largest quasi-operator ideal $\mathcal{A}$ for which $I_X \notin \mathcal{A}(X)$ for every infinite-dimensional Banach space X.*

Proof As observed before, Imp is a quasi-operator ideal and $I_x \notin \mathrm{Imp}(X)$.

To complete the proof, let $\mathcal{A}_\alpha$ denote the family of all quasi-operator ideals for which $I_X \notin \mathcal{A}(X)$. Note that this family is non-empty since Imp belongs to it. Let $\mathcal{A}_0 := \bigcup A_\alpha$. Clearly $\mathcal{A}_0$ is a quasi-operator ideal which contains every quasi-operator ideal for which $I_X \notin \mathcal{A}(X)$ for every infinite-dimensional Banach space X.

We show that $\mathcal{A}_0 = \mathrm{Imp}$. To see this it suffices to prove that $\mathcal{A}_0 \subseteq \mathrm{Imp}$. Suppose that there exist two Banach spaces X, Y and an operator $T \in \mathcal{A}_0(X, Y)$ such that $T \notin \mathrm{Imp}(X, Y)$. Then there exists a closed infinite-dimensional subspaces M of X such that $T|M$ has a bounded inverse and $T(M)$ is complemented in Y. The equality $T|M = TJ_M$ yields that $T|M \in \mathcal{A}_0(M, Y)$, since $\mathcal{A}_0$ is a quasi-operator ideal.

Now, let Q denote the projection of Y onto $T(M)$. Clearly, $S := QTJ_M \in \mathcal{A}_0(M, T(M)$ is an isomorphism. From this it follows that $I_M = S^{-1}S \in \mathcal{A}_0(M)$, and since M is infinite-dimensional we obtain a contradiction. Therefore $T \in \mathrm{Imp}(X, Y)$. ∎

Corollary 7.73. $\mathrm{Imp}(X, Y)$ *is a closed subset of* $L(X, Y)$.

Proof Imp is a closed quasi-operator ideal, by maximality. ∎

We now investigate the cases where $\mathrm{Imp}(X, Y) = I(X, Y)$. The next result shows a symmetry of this equality.

Theorem 7.74. *Let X, Y be Banach spaces. Then $I(X, Y) = \mathrm{Imp}(X, Y)$ if and only if $I(Y, X) = \mathrm{Imp}(Y, X)$.*

Proof Suppose that $I(X, Y) = \mathrm{Imp}(X, Y)$ and let $T \in \mathrm{Imp}(Y, X)$. We have only to show that $T \in I(Y, X)$.

Given $S \in L(X, Y)$, by Theorem 7.70 we have that $STS \in \mathrm{Imp}(X, Y) = I(X, Y)$; hence $(TS)^2 \in I(Y, X)$. Therefore

$$I_X - (TS)^2 = (I_X + TS)(I_X - TS) \in \Phi(X),$$

and hence $\ker(I_X - TS) \subseteq \ker(I_X - (TS)^2)$ is finite dimensional. By Theorem 7.17 we conclude that T is inessential. ∎

The following Lemma will be the key to characterize the inessential operators amongst the improjective operators.

Lemma 7.75. *For every $T \in L(X, Y)$ and $S \in L(Y, X)$ the following statements hold:*

(i) *If the subspace* $M := \ker(I_X - ST)$ *is complemented in* X *with closed complement* U *then both subspaces* $T(M)$ *and* $S^{-1}(U)$ *are complemented in* Y. *In fact, we have* $Y = T(M) \oplus S^{-1}(U)$;

(ii) *If the subspace* $N := \overline{(I_Y - TS)(Y)}$ *is complemented in* Y *with closed complement* V *then both subspaces* $T^{-1}(N)$ *and* $S(V)$ *are complemented in* X. *In fact, we have* $X = S(V) \oplus T^{-1}(N)$.

Proof (i) Let P denote the projection from X onto M along U; thus $P(X) = M$ and $\ker(P) = U$. Since $(I_X - ST)P = 0$ we have that $P = STP$. Therefore on defining $Q := TPS$ we have

$$Q^2 = TP(STP)S = TP^2S = Q;$$

i.e., Q is a projection in Y. From the equality $P = STP$ we easily obtain that $\ker(T) \cap P(X) = \{0\}$; thus

$$\ker(Q) = \ker(PS) = S^{-1}(U).$$

Moreover, $P = STP$ implies that $ST(M) = M$; hence $S^{-1}(U) \cap T(M) = \{0\}$. On the other hand, from $Q = TPS$ it follows that $Q(Y) \subset T(M)$; thus we conclude that $Q(Y) = T(M)$.

(ii) Let us denote by Q the projection from Y onto V along N. Then we have $Q(I_Y - TS) = 0$; hence $Q = QTS$. Therefore, as in the previous part, $P := SQT$ defines a projection in X.

From $Q = QTS$ we obtain that $\ker(S) \cap Q(Y) = \{0\}$; thus

$$\ker(P) = \ker(QT) = T^{-1}(N).$$

Moreover, $Q = QTS$ implies that $(TS)^{-1}(N) = N$; hence $S(V) \cap T^{-1}(N) = \{0\}$. On the other hand, from $P = SQT$, it follows that $P(X) \subset S(V)$; thus we conclude that $P(X) = S(V)$. ■

We now establish several characterizations of the inessential operators among the improjective operators in terms of the complementability of some subspaces.

Theorem 7.76. *For an operator* $T \in L(X, Y)$ *the following assertions are equivalent:*

(i) T *is inessential;*

(ii) $T \in \mathrm{Imp}(X, Y)$ *and* $\ker(I_X - ST)$ *is complemented for every* $S \in L(Y, X)$;

(iii) $T \in \mathrm{Imp}(X, Y)$ *and* $\ker(I_Y - TS)$ *is complemented for every* $S \in L(Y, X)$;

(iv) $T \in \mathrm{Imp}(X, Y)$ *and* $\overline{(I_X - ST)(X)}$ *is complemented for every* $S \in L(Y, X)$;

(v) $T \in \mathrm{Imp}(X, Y)$ *and* $\overline{(I_Y - TS)(Y)}$ *is complemented for every* $S \in L(Y, X)$.

Proof First we show that (i) implies the other assertions.

Assume that T is inessential. By Theorem 7.69 T is improjective. Moreover, by Theorem 7.17, for every $S \in L(Y,X)$ $\ker(I_X - ST)$ and $\ker(I_Y - TS)$ are finite-dimensional, and $\overline{(I_X - ST)(X)}$ and $\overline{(I_Y - TS)(Y)}$ are finite-codimensional; hence all of them are complemented.

(ii) $\Rightarrow$ (i) Assume that $T \in \mathrm{Imp}(X,Y)$ and $M := \ker(I_X - ST)$ is complemented. We have that T is an isomorphism on M, and by Lemma 7.75 $T(M)$ is complemented. Hence M is finite-dimensional. We have seen that (ii) implies that $\ker(I_X - ST)$ is finite- dimensional for every $S \in L(Y,X)$. By Theorem 7.17 we conclude that T is inessential.

(iii) $\Rightarrow$ (i) Assume that $T \in \mathrm{Imp}(X,Y)$ and $N := \ker(I_Y - TS)$ is complemented. We have that S is an isomorphism on N (so that $S(N)$ is closed), T is an isomorphism on $S(N)$ and $T(S(N)) = N$ is complemented. Hence N is finite-dimensional. We have seen that the condition (iii) implies that $\ker(I_Y - TS)$ is finite-dimensional for every $S \in L(Y,X)$. By Theorem 7.17, we may conclude that $T \in I(X,Y)$.

(iv) $\Rightarrow$ (i) Assume that $T \in \mathrm{Imp}(X,Y)$ and $M := \overline{(I_X - ST)(X)}$ is complemented. Since $Q_M(I_X - ST) = 0$ we have that $Q_M ST = Q_M$; in particular $(ST)(X) + M = X$. Then $T(X) + S^{-1}(M) = Y$; i.e., $Q_{S^{-1}(M)}T$ is surjective. Moreover, we have that

$$\begin{aligned} T^{-1}S^{-1}(M) &= (ST)^{-1}(M) =^{\perp} \Big((ST)^*(M^{\perp})\Big) \\ &= \Big(T^*S^*(\ker^{\perp}(I_{X^*} - T^*S^*)\Big) \\ &= {}^{\perp}(\ker(I_{X^*} - T^*S^*)) = M \end{aligned}$$

is complemented. From Theorem 7.68 it then follows that $S^{-1}(M)$ is finite-codimensional; hence $M = T^{-1}S^{-1}(M)$ is also finite-codimensional. Therefore (iv) implies that $\overline{(I_X - ST)(X)}$ is finite-codimensional for every $S \in L(Y,X)$. Again, by Theorem 7.18 we conclude that $T \in I(X,Y)$.

(v) $\Rightarrow$ (i) Assume that $T \in \mathrm{Imp}(X,Y)$ and $N := \overline{(I_Y - TS)(Y)}$ is complemented. Since $Q_N(I_Y - TS) = 0$ we have that $Q_N TS = Q_N$. In particular $Q_N T$ is surjective, and by Lemma 7.75 $T^{-1}(N)$ is complemented. Applying Theorem 7.68 we conclude that N is finite-codimensional. Therefore (v) implies that $\overline{(I_Y - TS)(Y)}$ is finite-codimensional for every $S \in L(Y,X)$, and hence by Theorem 7.18 we conclude that $T \in I(X,Y)$. ∎

To give some sufficient conditions for the equality $I(X,Y) = \mathrm{Imp}(X,Y)$, we consider the two classes of operators Ω_+ and Ω_- introduced in Section 2 of this chapter.

Theorem 7.77. *Let $\mathcal{A}$ be a quasi-operator ideal and let Y be a Banach space.*

(i) *If $\mathcal{A}(Y) \subseteq \Omega_+(Y)$, then $\mathcal{A}(Y,Z) \subseteq I(Y,Z)$ holds for every Banach space Z;*

(ii) *If* $\mathcal{A}(Y) \subseteq \Omega_-(Y)$*, then* $\mathcal{A}(X,Y) \subseteq I(X,Y)$ *holds for every Banach space* X*.*

Proof (i) Suppose that there exists $T \in \mathcal{A}(Y,Z)\setminus I(Y,Z)$. By Theorem 7.17 we can find $S \in L(Z,Y)$ so that $M := \ker\,(I_Y - ST)$ is infinite-dimensional. However, since $ST \in \mathcal{A}(Y) \subseteq \Omega_+(Y)$ and ST coincides with the identity on M it follows that M is finite-dimensional; a contradiction.

(ii) Suppose that there exists $T \in \mathcal{A}(X,Y) \setminus I(X,Y)$. By Theorem 7.18 we can find $S \in L(Y,X)$ such that $N := \overline{(I_Y - TS)(Y)}$ is infinite-codimensional. However, since $TS \in \mathcal{A}(Y) \subseteq \Omega_-(Y)$, N is an invariant subspace of TS and $Q_N TS$ is surjective, we also obtain that N is finite-codimensional; a contradiction. ∎

In the case $\mathcal{A} = \mathrm{Imp}$, we obtain further characterizations of pairs X, Y satisfying $I(X,Y) = \mathrm{Imp}(X,Y)$.

Corollary 7.78. *For a Banach space* X *the following statements are equivalent:*

(i) $I(X) = \mathrm{Imp}(X)$;

(ii) $\mathrm{Imp}(X) \subseteq \Omega_+(X)$;

(iii) $\mathrm{Imp}(X) \subseteq \Omega_-(X)$;

(iv) $\mathrm{Imp}(X) \subseteq R(X)$;

(v) $I(X,Y) = \mathrm{Imp}(X,Y)$ *for every Banach space* Y*.*

Proof The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious since by Theorem 7.32 we have $I(X) \subseteq \Omega_+(X) \cap \Omega_-(X)$.

The implication (ii) $\Rightarrow$ (v) follows from Theorem 7.77, once it is observed that $I(X,Y) \subseteq \mathrm{Imp}(X,Y)$, whilst the implication (iii) $\Rightarrow$ (v) follows from Theorem 7.77, the inclusion $I(Y,X) \subseteq \mathrm{Imp}(Y,X)$ and the equivalence $I(Y,X) = \mathrm{Imp}(Y,X)$ if and only if $I(X,Y) = \mathrm{Imp}(X,Y)$ are proved in Theorem 7.74.

The implication (v) $\Rightarrow$ (i) is trivial. Therefore (i), (ii), (iii), (v) are equivalent. Finally, (iv) $\Rightarrow$ (ii) by Theorem 7.28, whilst from (i) we obtain that $\mathrm{Imp}(X) = I(X) \subseteq R(X)$, so (i) $\Rightarrow$ (iv).

∎

Theorem 7.79. *Let* X *be an infinite-dimensional complex Banach space and let* $T \in \mathrm{Imp}(X)$*. Then we have:*

(i) $0 \in \sigma(T)$;

(ii) *If* $\sigma \subseteq \sigma(T)$ *is a spectral set and* $0 \notin \sigma$ *then the spectral projection associated to* σ *has finite-dimensional range.*

Proof (i) The assertion is evident, since an invertible operator in an infinite-dimensional space cannot be improjective.

(ii) Let P denote the spectral projection associated with σ. Then taking $M := P(X)$ we have that $T(M) = M$, and the restriction TJ_M is an isomorphism. Since $T \in \mathrm{Imp}(X)$ we then conclude that M is finite-dimensional subspace of X. ∎

Next we show tha, for complex Banach spaces the equality $I(X) = \mathrm{Imp}(X)$ can be formulated in terms of the spectral properties of improjective operators.

Theorem 7.80. *Let X be a complex Banach space. Then* $\mathrm{Imp}(X) = I(X)$ *if and only if for every $T \in \mathrm{Imp}(X)$, the spectrum $\sigma(T)$ is either a finite set or a sequence which clusters at* 0.

Proof The direct implication follows from the fact that each inessential operator $T \in I(X))$ is a Riesz operator, so by Theorem 3.111 the spectrum $\sigma(T)$ is either a finite set or a sequence which clusters at 0.

For the converse, let us fix an operator $T \in \mathrm{Imp}(X)$. For every $S \in L(X)$ we have that $ST \in I(X)$. Now, by the hypothesis and Theorem 7.79 we obtain that either $I_X - ST$ is bijective or 1 is an isolated point in $\sigma(ST)$ and the spectral projection associated to the spectral set $\{1\}$ has finite-dimensional range. In any case $\ker(I_X - ST)$ is finite-dimensional, and applying Theorem 7.17 we conclude that T is inessential. ∎

The next result shows that if one of the spaces is subprojective or superprojective, all the improjective operators are inessential.

Theorem 7.81. *Assume that one of the spaces X, Y is subprojective or superprojective . Then we have* $\mathrm{Imp}(X, Y) = I(X, Y)$.

Proof Assume first that Y is superprojective. If $T \in L(X, Y)$ is not inessential then we can find an operator $S \in L(Y, X)$ such that $M := \overline{(I_Y - TS)(Y)}$ is infinite-codimensional in Y. We take an infinite-codimensional complemented subspace N of Y containing M, and we select a projection P with kernel $\ker P = N$. We have that $P(Y)$ is infinite-dimensional. Moreover, since $P(I_Y - TS) = 0$ we obtain that PTS restricted to $P(Y)$ coincides with the identity operator. Then PTS is not improjective, and by Theorem 7.70 T is not improjective.

Now we consider the case in which X is subprojective. If $T \in L(X, Y)$ is not inessential then we can find an operator $S \in L(Y, X)$ such that $M := \ker(I_X - ST)$ is infinite-dimensional in X. We take an infinite-dimensional complemented subspace N of X contained in M. Since ST restricted to the subspace N coincides with the identity we have that $T(N)$ is closed, the restriction $SJ_{T(N)}$ is an isomorphism, and $S(T(N))$ is complemented. Hence by Lemma 7.67 the subspace $T(N)$ is complemented. Since TJ_N is an isomorphism we conclude that T is not improjective.

For the remaining cases it is enough to apply Theorem 7.74 and the previously proved cases. ∎

Corollary 7.82. *Suppose that X and Y are Banach spaces. Then the following assertions hold:*

(i) *If Y is subprojective then $I(X,Y) = SS(X,Y)$;*

(ii) *If X is superprojective then* $\mathrm{Imp}(X,Y) = SC(X,Y)$.

Proof Combine Theorem 7.81 with Theorem 7.51. ∎

The following examples show that we cannot change the order of the spaces X, Y in Theorem 7.82. To be precise, in part (i) if X is subprojective then $\mathrm{Imp}(X,Y) = SS(X,Y)$ is not true in general, and analogously in part (ii) if Y is superprojective, then $\mathrm{Imp}(X,Y) = SC(X,Y)$ is in general not true.

Example 7.83. (a) The space ℓ^2 is subprojective, so by Theorem 7.81 and taking into account what was established in Example 7.12, we obtain

$$L(\ell^2, \ell^\infty) = \mathrm{Imp}(\ell^2, \ell^\infty) = I(\ell^2, \ell^\infty);$$

On the other hand, we also have $L(\ell^2, \ell^\infty) \neq SS(\ell^2, \ell^\infty)$ because ℓ^∞ contains a closed subspace isomorphic to ℓ^2, see Beauzamy [**64**, Theorem IV.II.2] .

Another example may be derived from the natural inclusion of $L^2[0,1]$ in $L^1[0,1]$ being not strictly singular, as observed in Example 7.42. However, see Example 7.14, we have

$$L(L^2[0,1], L^1[0,1]) = \mathrm{Imp}(L^2[0,1], L^1[0,1]) = I(L^2[0,1], L^1[0,1]).$$

(b) The space ℓ^2 is superprojective and, as already seen, every operator $T \in L(\ell^1, \ell^2)$ is strictly singular. Therefore we have

$$L(\ell^1, \ell^2) = \mathrm{Imp}(\ell^1, \ell^2) = I(\ell^1, \ell^2).$$

However, $L(\ell^1, \ell^2) \neq SC(\ell^1, \ell^2)$, because ℓ^1 has a quotient isomorphic to ℓ^2, see Beauzamy [**64**, Theorem IV.II.1].

In order to see that the inessential operators and the improjective operators coincide for many other classical Banach spaces, we introduce the following two concepts of S-system and C-system.

Definition 7.84. *A class $\mathcal{N}$ of closed infinite-dimensional subspaces of Y is said to be a S-*system *of Y with respect to a Banach space X if $T \in L(X,Y)$ is strictly singular if and only if for every subspace M of X with $T(M) \in \mathcal{N}$ we have that TJ_M is not an isomorphism.*

A S-system $\mathcal{N}$ is said to be complemented *if every element of $\mathcal{N}$ is complemented in Y. Note that by Lemma 7.67 if $T(M)$ is complemented in Y then M is complemented in X .*

Example 7.85. In the sequel we shall give several examples of S-systems between concrete examples of Banach spaces.

(a) If Y is a subprojective Banach space and X an arbitrary Banach space then the class

$$\mathcal{N} := \{N : N \text{ is an infinite-dimensional complemented subspace of } Y\}$$

is a S-system of Y with respect to X, by Theorem 7.81.

(b) Let Y be an arbitrary separable Banach space and $X := C(K)$, K compact. As has been shown by Pełczyński [**260**] $T \in L(X,Y)$ is strictly singular if and only if there are no embeddings $J_1 : c_o \to X$ and $J_2 : c_o \to Y$ such that $TJ_1 = J_2$. From Theorem 8.3 of Przeworska and Rolewicz [**266**] we then conclude that

$$\mathcal{N} := \{N : N \text{ a subspace of } Y \text{ isomorphic to } c_o\}$$

is a complemented S-system of Y with respect to X.

In the case that K is σ-Stonian a similar characterization of strictly singular operators shows that the subspaces of Y isomorphic to ℓ^∞ form a complemented S-system of Y with respect to X.

(c) If Y is a $\mathcal{L}^p$-space, see Lindenstrauss and Tzafriri [**220**] for definitions, with $2 \le p < \infty$, then the class

$$\mathcal{N} := \{N : N \text{ a subspace of } Y \text{ isomorphic to } \ell^2 \text{ or to } \ell^p\}$$

forms a complemented S-system of Y with respect to every Banach space X. This is a consequence of the property that for $2 \le p < \infty$, $L^p[0,1]$ is subprojective and every $\mathcal{L}^p$-space is complemented in some $L^p(\mu)$.

(d) If $Y := L^1(\mu)$ and X has the Dunford–Pettis property then

$$\mathcal{N} := \{N : N \text{ a complemented subspace of } Y \text{ isomorphic to } \ell_1\}$$

is a complemented S-system of Y with respect to X. Indeed, see Pełczyński [**260**], $T \in L(X,Y)$ is strictly singular if and only if there are no embeddings $J_1 : \ell_1 \to X$ and $J_2 : \ell_1 \to Y$ such that $TJ_1 = J_2$.

Theorem 7.86. *Let X and Y be two Banach spaces. Then Y has a complemented S-system with respect to X if and only if* $\mathrm{Imp}(X,Y) = SS(X,Y)$.

Proof Suppose $T \notin SS(X,Y)$ and Y admits a complemented S-system with respect to X. Then there exists a subspace M for which $T(M)$ is complemented in Y and TJ_M is not an isomorphism. Hence $T \notin \mathrm{Imp}(X,Y)$.

Conversely, if $\mathrm{Imp}(X,Y) = SS(X,Y)$ then the class $\mathcal{M}$ of all infinite-dimensional complemented subspaces of Y is a complemented S-system of Y with respect to X. ■

Corollary 7.87. *Suppose that one of the two Banach spaces X and Y admits a complemented S-system with respect to the other. Then*

$$\mathrm{Imp}(X,Y) = I(X,Y).$$

In particular, this equality holds whenever X and Y are the Banach spaces in parts (a), (b), (c), *and* (d) *of Example 7.85.*

Proof If Y has a complemented S-system with respect to X then $\mathrm{Imp}(X,Y) = I(X,Y) = SS(X,Y)$. If X has a complemented S-system with respect to Y then $\mathrm{Imp}(Y,X) = I(Y,X) = SS(X,Y)$ and therefore, from Proposition 7.74, $\mathrm{Imp}(X,Y) = I(X,Y)$ ∎

Definition 7.88. *A class $\mathcal{N}$ of closed infinite-codimensional subspaces of X is said to be a C-*system *of X with respect to Y if $T \in L(X,Y)$ is strictly cosingular if and only if for every subspace M of Y with $T^{-1}(M) \in \mathcal{N}$ we have that $Q_M T$ is not surjective.*

A C-system $\mathcal{N}$ is said to be complemented *if every element of $\mathcal{N}$ is complemented in X. Note that if $T^{-1}(M)$ is complemented in X then M is complemented in Y by Lemma 7.67.*

Example 7.89. In the sequel we shall give several examples of C-systems between some Banach spaces.

(e) If X is a superprojective Banach space and Y an arbitrary Banach space then the class

$$\mathcal{N} := \{N : N \text{ is a complemented subspace of } X\}$$

is a C-system of X with respect to Y by Theorem 7.81.

(f) Let Y be an arbitrary separable Banach space which has the Dunford–Pettis property and $X := C(K)$, K compact. Then the class

$$\mathcal{N} := \{N : N \text{ a subspace of } C(K) \text{ such that } C(K)/N \text{ is isomorphic to } c_0\}$$

is a complemented C-system of X with respect to Y.

In the case in which K is σ-Stonian and Y is a (not necessarily separable) Banach space which has the Dunford–Pettis property the subspaces N of X such that $C(K)/N$ is isomorphic to ℓ^∞ form a complemented C-system of X with respect to Y.

Indeed, every separable non-reflexive quotient of $C(K)$ has a quotient isomorphic to c_0. The case K σ-Stonian a similar.

(g) If X is a $\mathcal{L}_p$-space, $1 < p < 2$, the class

$$\mathcal{N} := \{N : N \text{ a subspace of } \mathcal{L}_p \text{ such that } X/N \text{ is isomorphic to } \ell^2 \text{ or to } \ell^p\}$$

forms a complemented C-system of X with respect to every Banach space Y. This can be obtained by duality from (c) of Example 7.85.

(h) If $Y := L^1(\mu)$ and X is arbitrary then

$$\mathcal{N} := \{N : N \text{ a subspace of } X \text{ such that } X/N \text{ is isomorphic to } \ell_1\}$$

is a complemented C-system of X with respect to Y.

This is a consequence of every non-reflexive subspace of $L^1(\mu)$ containing a complemented copy of ℓ^1, see Wojtaszczyk [**329**, p.144].

Theorem 7.90. *Let X and Y be two Banach spaces. Then X has a complemented C-system with respect to Y if and only if* $\mathrm{Imp}(X,Y) = SC(X,Y)$.

Proof If $T \notin SC(X, Y)$ and X admits a complemented C-system with respect to Y, there exists a subspace M of Y for which $T^{-1}(M)$ is complemented in X and $Q_M T$ is not surjective. Hence by Theorem 7.68 $T \notin \mathrm{Imp}(X, Y)$.

Conversely, if $\mathrm{Imp}(p(X, Y) = SC(X, Y)$ then the class $\mathcal{M}$ of all subspaces M of Y for which $T^{-1}(M)$ is complemented in X is a complemented C-system of X with respect to Y. ∎

Corollary 7.91. *Suppose that one of the spaces X and Y admits a complemented C-system with respect to the other. Then*

$$\mathrm{Imp}(X, Y) = I(X, Y).$$

In particular, this equality holds whenever X and Y are the Banach spaces in parts (e), (f), (g, and (h) of Example 7.89

Proof If X has a complemented C-system with respect to Y then $\mathrm{Imp}(X, Y) = I(X, Y) = SC(X, Y)$. If Y has a complemented C-system with respect to X then $\mathrm{Imp}(Y, X) = I(Y, X) = SC(X, Y)$, and therefore from Proposition 7.74 $\mathrm{Imp}(X, Y) = I(X, Y)$. ∎

Let us see some cases in which the components of Imp coincide with those of the biggest proper operator ideal.

Theorem 7.92. *Let $\mathcal{A}$ be a proper operator ideal. Suppose that one of the spaces X and Y admits a S-system or a C-system with respect to the other. Then*

$$\mathcal{A}(X, Y) \subseteq \mathrm{Imp}(X, Y) = I(X, Y).$$

Proof Clearly $\mathcal{A} \subseteq \mathrm{Imp}$ since Imp is the largest proper quasi-ideal. Therefore from Corollary 7.87 or Corollary 7.91 we have $\mathcal{A}(X, Y) \subseteq \mathrm{Imp}(X, Y) = I(X, Y)$. ∎

Corollary 7.93. *If X and Y are two Banach spaces and $T \in L(X, Y)$, then the following statements hold:*

(i) *If Y admits a S-system with respect to X then $T^* \in SS(Y^*, X^*) \Rightarrow T \in SS(X, Y)$;*

(ii) *If X admits a C-system with respect to Y then $T^* \in SC(Y^*, X^*) \Rightarrow T \in SC(X, Y)$.* ∎

Remark 7.94. Note that the property of having a certain Banach space a S-system, as well as a C-system with respect to another Banach space is not symmetric. For instance, since ℓ^2 is subprojective ℓ^∞ has an S-system with respect to ℓ^2, whilst ℓ^2 does not admit any S-system with respect to ℓ^∞. Indeed,

$$\mathcal{L}(\ell^2, \ell^\infty) = \mathrm{Imp}(\ell^2, \ell^\infty) = I(\ell^2, \ell^\infty),$$

see Example 7.15, whereas $\mathcal{L}(\ell^2, \ell^\infty) \neq SS(\ell^2, \ell^\infty)$ since ℓ^∞ contains a copy of ℓ^2. Analogously the superprojective space ℓ^2 admits a C-system with respect to ℓ^∞, whilst ℓ^∞ does not admit a C-system with respect to ℓ^2. Indeed, $\mathcal{L}(\ell^\infty, \ell^2) = \mathrm{Imp}(\ell^\infty, \ell^2)$ whilst $\mathcal{L}(\ell^\infty, \ell^2) \neq SC(\ell^\infty, \ell^2)$, since ℓ^∞ has a quotient isomorphic to ℓ^2.

We wish to show now that the class of inessential operators is in general strictly smaller than the class of improjective operators. As in the perturbation theory of Section 3, we solve this question in the framework of indecomposable Banach spaces. Note that by Corollary 7.78 for a Banach space X we have $I(X) \neq \mathrm{Imp}(X)$ if and only if $I(X,Y) \neq \mathrm{Imp}(X,Y)$ for all Banach spaces Y. We have the following characterization of indecomposable spaces in terms of improjective operators.

Theorem 7.95. *For a Banach space Y, the following statements are equivalent:*
(i) *Y is indecomposable;*
(ii) *$L(Y,Z) = \Phi_{\mathrm{l}}(Y,Z) \cup \mathrm{Imp}(Y,Z)$ for every Banach space Z;*
(iii) *$L(X,Y) = \Phi_{\mathrm{r}}(X,Y) \cup \mathrm{Imp}(X,Y)$ for every Banach space X;*
(iv) *$L(Y) = \Phi(Y) \cup \mathrm{Imp}(Y)$.*

Proof (i) $\Rightarrow$ (ii) Assume that Y is indecomposable. If $T \in L(Y,Z)$ is not improjective then there exists an infinite-dimensional subspace M of Y such that TJ_M is an isomorphism and $T(M)$ is complemented. By Lemma 7.67 M is also complemented; hence M is finite-codimensional since Y is indecomposable.

Now, it is clear that ker T is finite-dimensional and $T(X)$ is the direct sum of $T(M)$ and a finite-dimensional subspace, hence $T(X)$ is complemented. Thus $T \in \Phi_{\mathrm{l}}(Y,Z)$ by Theorem 7.3, part (ii).

(i) $\Rightarrow$ (iii) Assume that Y is indecomposable. If $T \in L(X,Y)$ is not improjective then by Theorem 7.68 and Lemma 7.67 there exists an infinite-codimensional complemented subspace N of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is also complemented in X. Since Y is indecomposable N is finite-dimensional. Therefore $T(X)$ is finite-codimensional, hence closed, and $T^{-1}(N)$ is the direct sum of $\ker T$ and a finite-dimensional subspace. This argument shows that ker T is complemented. Thus $T \in \Phi_{\mathrm{r}}(X,Y)$ by part (i) of Theorem 7.3.

(ii) $\Rightarrow$ (iv) Assume that $T \in L(Y) \setminus \mathrm{Imp}(Y)$. Then $T \in \Phi_{\mathrm{l}}(Y)$ and $T(X)$ is infinite-dimensional. Now we can select an operator $S \in L(Y)$ so that ST is a projection with $\ker ST = \ker T$ and $Y = \ker S \oplus T(X)$. It is easy to see that the restriction $SJ_{T(X)}$ is an isomorphism and $S(T(X)) = (ST)(X)$ is complemented. Then $S \notin \mathrm{Imp}(Y)$; hence $S \in \Phi_{\mathrm{l}}(Y)$. In particular, ker S is finite-dimensional. Therefore $T(X)$ is finite-codimensional; thus $T \in \Phi(Y)$.

(iii) $\Rightarrow$ (iv) Assume that $T \in L(Y) \setminus \mathrm{Imp}(Y)$. Then $T \in \Phi_{\mathrm{r}}(Y)$ and ker T is infinite-codimensional. Now we can select an operator $S \in L(Y)$ so that TS is a projection with $(TS)(Y) = T(X)$ and $Y = \ker T \oplus S(Y))$. Also here we have that $Q_{\ker T} S$ is surjective and $S^{-1}(\ker T) = \ker TS$ is complemented. Then $S \notin \mathrm{Imp}(Y)$; hence $S \in \Phi_{\mathrm{r}}(Y)$. Now, since we have $\ker T \cap S(Y) = \{0\}$ it follows that $\ker T$ is finite-dimensional; thus $T \in \Phi(Y)$.

(iv) $\Rightarrow$ (i) Suppose that Y is decomposable. Then we can find a projection $P \in L(Y)$ such that both the subspaces $\ker P$ and $P(Y)$ are infinite-dimensional; therefore $P \notin \Phi(Y) \cup \mathrm{Imp}(Y)$. ∎

Theorem 7.96. *Let X be a Banach space. Assume that either X is hereditarily indecomposable or quotient hereditarily indecomposable. Then $L(X) = \Phi(X) \cup I(X)$.*

Proof Since strictly singular operators and strictly cosingular operators are inessential, by Theorem 7.60 we have

$$L(X) = \Phi_{+}(X) \cup \Phi_{-}(X) \cup I(X).$$

Assume that there exists $T \in L(X) \setminus I(X)$. Then X is infinite-dimensional, $T \in \Phi_{+}(X) \cup \Phi_{-}(X)$ and T has index not finite. It easily follows from Remark 1.54, part (d), that the set of all semi-Fredholm operators $\Phi_{+}(X) \cup \Phi_{-}(X)$ with index equal to a fixed value is open. Therefore since $\lambda I_X - T$ is invertible for $|\lambda| > \|T\|$ we can find real numbers μ, ν such that $\mu < 0 < \nu$ and the operators $\mu I_X - T$ and $\nu I_X - T$ are not in $\Phi_{+}(X) \cup \Phi_{-}(X)$, and consequently they belong to $I(X)$. Since $I(X)$ is a subspace of $L(X)$ we conclude that $(\mu - \nu) I_X \in I(X)$; hence X is finite-dimensional, and so we obtain a contradiction. ∎

Theorem 7.97. *Let X be Banach space. Assume that $L(X) = \Phi(X) \cup I(X)$. Then* $\mathrm{Imp}(X, Y) = I(X, Y)$ *for every Banach space Y.*

Proof The result is clear if $I(X, Y) = L(X, Y)$. So we assume that there exists $T \in L(X, Y) \setminus I(X, Y)$, and we show that $T \notin \mathrm{Imp}(X, Y)$.

We take $S \in L(Y, X)$ such that ker $(I_X - ST)$ is infinite dimensional. Since $I_X - ST$ is not Fredholm then by the hypothesis it is inessential; hence $ST = I_X - (I_X - ST)$ is Fredholm. In particular, $T \in \Phi_{\mathrm{l}}(X, Y)$, so by Theorem 7.3 we have $X = \ker T \oplus M$, the restriction $T J_M$ is an isomorphism, and $T(M) = T(X)$ is complemented; hence T is not improjective. ∎

Theorem 7.97, Corollary 7.96, and Proposition 7.97 suggest that if we want to find improjective operators which are not inessential, we should look for an indecomposable Banach space which is not hereditarily subspace and quotient indecomposable. Again, Gowers and Maurey ([**138**]) provide the required example.

Theorem 7.98. ([**138**]) *There exists an indecomposable Banach space Z which is neither hereditarily indecomposable nor quotient hereditarily indecomposable. This space has a Schauder basis and the associated right shift S is an isometry on Z.* ∎

We can show Aiena and González's counter example ([**21**]).

Theorem 7.99. *There exists a complex Banach space Z with the following properties:*

a) $I(Z) \neq \mathrm{Imp}(Z)$*;*

b) *There exists* $T \in \mathrm{Imp}(Z)$ *which is not Riesz;*

c) $\mathrm{Imp}(Z)$ *is not a subspace of* $L(Z)$.

Proof We consider the space Z mentioned in Theorem 7.98. This space has a Schauder basis and the associated right shift S is an isometry on Z. Moreover, Z is indecomposable, so $L(Z) = \Phi(Z) \cup \mathrm{Imp}(Z)$ by Theorem 7.95.

Let $\lambda \in \mathbb{C}$ be a complex number. Clearly, for $|\lambda| > 1$ the operator $\lambda I_Z - S$ is invertible; in particular, it is a Fredholm operator with ind $(\lambda I_Z - S) = 0$. Moreover, for $|\lambda| < 1$ the operator $\lambda I_Z - S$ is Fredholm with ind $(\lambda I_Z - S) = -1$. Therefore by the continuity of the index of Fredholm operators, see Remark 1.54, for $|\lambda| = 1$ the operator $\lambda I_Z - S$ is not Fredholm; hence for $|\lambda| = 1$ the operator $\lambda I_Z - S$ belongs to $\mathrm{Imp}(Z)$ but it is not Riesz. In particular, $I_Z - S \in \mathrm{Imp}(Z) \setminus I(Z)$.

On the other hand, we have

$$(I_Z - S) - (-I_Z - S) = 2I_Z \notin \mathrm{Imp}(Z),$$

since Z is infinite-dimensional. Hence $\mathrm{Imp}(Z)$ is not a subspace of $L(Z)$. ∎

Theorem 7.99 answers in the negative some questions of Tarafdar [**305, 306**]. In particular, $\mathrm{Imp}(X, Y)$ is not, in general, a linear subspace of $L(X, Y)$, the class Imp of all improjective operators between Banach spaces is not an operator ideal.

5. Incomparability between Banach spaces

An important field in which operator ideals find a natural application is that of the incomparability of Banach spaces. There are several notions of incomparability; for an excellent survey we refer to González and Martinón [**135**]. Roughly speaking, two Banach spaces X and Y are incomparable if there is no isomorphism between *certain* infinite-dimensional subspaces.

Definition 7.100. *Given two Banach spaces X and Y are said* projection incomparable*, or also* totally dissimilar*, if no infinite-dimensional complemented subspace of X is isomorphic to a complemented subspace of Y.*

The next result shows that the notion of incomparabilty defined above may be given in terms of improjective operators.

Theorem 7.101. *Two Banach spaces X and Y are projection incomparable precisely when* $L(X, Y) = \mathrm{Imp}(X, Y)$.

Proof Suppose first that X, Y are projection incomparable and suppose that there exists $T \in L(X, Y)$ which is not improjective. Then there exists an infinite-dimensional closed subspace M of X such that the restriction

$T|M$ is an isomorphism and $T(M)$ is complemented. From this it follows by part (i) of Lemma 7.67 that also M is complemented in X, so we have a contradiction.

Conversely, it is easy to see that if X and Y are not projection incomparable then there exists a bounded operator $T \in L(X,Y)$ which is not improjective. Indeed, assume that M is an infinite-dimensional complemented subspace of X isomorphic to a complemented subspace N of Y. If S is the isomorphism of M onto N, P a projection of X onto M, and J_M the injection of M into X, then $T := J_M SP \in L(X,Y)$ and $T \notin \mathrm{Imp}(X,Y)$, since the restriction $T|M$ is an isomorphism of M onto N, which is complemented in Y. ■

Definition 7.102. *Given two Banach spaces X and Y are said to be* essentially incomparable *if $L(X,Y) = I(X,Y)$.*

From the inclusion $I(X,Y) \subseteq \mathrm{Imp}(X,Y)$ we immediately obtain:

$$X, Y \text{essentially incomparable } \Rightarrow \ X, Y \text{ projection incomparable.}$$

Moreover, since the existence of $T \in \Phi(X,Y)$ implies that $\ker T$ has an infinite-dimensional complement M isomorphic to $T(X)$ we also have:

$$X, Y \text{projection incomparable } \Rightarrow \Phi(X,Y) = \varnothing.$$

The last implication, in general, cannot be reversed. In fact, if $X = L^p[0,1]$ and $Y = L^q[0,1]$, with $1 < p < q < \infty$, then $\Phi(X,Y) = \varnothing$, whereas $I(X,Y) \neq L(X,Y)$, since both $L^p[0,1]$ and $L^q[0,1]$ have a complemented subspace M isomorphic to ℓ^2, M the subspace spanned by the Rademacher functions, see Lindenstrauss and Tzafriri [**220**].

Of course, all the examples in which $L(X,Y) = I(X,Y)$, given in Section 2, provide pairs of Banach spaces which are essentially incomparable. Analogously, all the examples of Banach spaces for which $L(X,Y) = \mathrm{Imp}(X,Y)$ provide examples of projection incomparable Banach spaces. The next result is an obvious consequence of Theorem 7.81.

Theorem 7.103. *Suppose that X or Y is a subprojective Banach space. Then X and Y are projection incomparable precisely when X and Y are essentially incomparable. Analogously, if X or Y is superprojective then X and Y are projection incomparable if and only if X and Y are essentially incomparable.* ■

We show now that the two kind of incomparability are not the same. In fact, the example given in Theorem 7.99 also allows us to show that the inequality $I(Z,Y) \neq L(Z,Y)$ does not imply that Z has an infinite-dimensional complemented subspace isomorphic to a complemented subspace of Y. Moreover, we see also that $\mathrm{Imp}(Z,Y)$ being a subspace of $L(Z,Y)$ does not imply $I(Z,Y) = \mathrm{Imp}(Z,Y)$.

Theorem 7.104. *There exist a pair of Banach spaces Z, Y for which we have*

$$I(Z, Y) \neq \mathrm{Imp}(Z, Y) = L(Z, Y).$$

Proof Let Z be the Banach space considered in the proof of Theorem 7.99. We consider the operator $T = I_Z - S \in L(Z)$. Again from Remark 1.54 we obtain that $T \notin \Phi_-(Z)$. Therefore, by part (ii) of Theorem 7.16 we can find a compact operator $K \in L(Z)$ such that the closure of $(T-K)(Z)$ is infinite-codimensional in Z. We take $Y = \overline{(T-K)(Z)}$ and consider the operator $U \in L(Z, Y)$ defined by

$$Uz := Tz - Kz \quad \text{for all } z \in Z.$$

Clearly U is not inessential, because $T - K = J_{\overline{(T-K)(Z)}}U$ is not inessential either. Moreover, every infinite-dimensional complemented subspace of Z is isomorphic to Z, see [**138**, Theorem 13], but Z is not isomorphic to any of its infinite- codimensional closed subspaces [**138**, Theorem 16]. Hence $\mathrm{Imp}(Z, Y) = L(Z, Y)$. ∎

One may ask whether an improjective operator T on a complex Banach space X has its spectrum $\sigma(T)$ as either a finite set or a sequence which clusters at 0. Again, assuming that for $T \in \mathrm{Imp}(X)$ the spectrum is either a finite set or a sequence which clusters at 0, one may ask whether T is inessential. The following examples allow us to give a negative answer to these questions.

Example 7.105. The operator $T = I_Z - S \in L(Z)$ in the proof of Theorem 7.99 is improjective, but it is not Riesz. Moreover, taking $Y = \overline{R(T-K)}$ as in the proof of Theorem 7.104 and $A : Y \times Z \to Y \times Z$ defined by

$$A(y, z) := (Tz, 0) \quad \text{for all } y \in Y, z \in Z,$$

we have that A is improjective and its spectrum is the set $\{0\}$. Therefore A is Riesz, but clearly A is not inessential.

Observe that if all the improjective operators in a complex Banach space are Riesz, then they are also inessential.

The last result provides some infomation about the structural properties of products of essentially incomparable Banach spaces.

Theorem 7.106. *Suppose that X and Y are essentially incomparable Banach spaces. For every complemented subspace M of $X \times Y$ there exists a bijective isomorphism $U \in L(X \times Y)$ and complemented subspaces X_0 of X and Y_0 of Y such that $U(M) = X_0 \times Y_0$.*

Proof We may only consider the case of complex Banach spaces, since the real case can be reduced to the complex case by using the complexifications of the spaces involved. Also we shall assume that M is infinite-dimensional and infinite-codimensional in $X \times Y$. Otherwise the result is immediate.

Suppose that $P \in L(X \times Y)$ is a projection onto M, and write

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in L(X)$, $D \in L(Y)$, $B \in L(Y,X)$ and $C \in L(X,Y)$. The operator defined as

$$S := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

is inessential, since by assumption B and C are inessential. By Theorem 7.25, the operator

$$Q = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

has the same Fredholm spectrum , namely $\lambda I - P$ is Fredholm if and only if $\lambda I - Q$ is Fredholm. In particular, the spectrum of Q is countable, has at most 0 or 1 as cluster points, and for all $\lambda \in \sigma(Q)$ with $0 \neq \lambda \neq 1$ the corresponding spectral projection is finite-dimensional. Let us denote by Γ the boundary of a rectangle which does not meet $\sigma(Q)$, with 1 in the interior and 0 in the exterior of the rectangle. Let

$$P_1 := \frac{1}{2\pi i}\int_\Gamma (\lambda I - Q)^{-1}\, \mathrm{d}\lambda.$$

Clearly, P_1 is a projection that commutes with Q and

$$P_1 = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix},$$

where A_1 is a projection in X and D_1 is a projection in Y. Moreover, as is easy to see,

$$P := -\frac{1}{2\pi i}\int_\Gamma (\lambda I - P_1)^{-1}\, \mathrm{d}\lambda.$$

Moreover, for every $\lambda \in \Gamma$ we have $(\lambda I - P) - (\lambda I - Q) \in I(X \times Y)$, and from the equality

$$(\lambda I - P)^{-1} - (\lambda I - Q)^{-1} = (\lambda I - Q)^{-1}\left((\lambda I - Q) - (\lambda I - P)\right)(\lambda I - P)^{-1}$$

we see that $(\lambda I - P)^{-1} - (\lambda I - Q)^{-1}$ belongs to $I(X \times Y)$. From this it follows that $P_1 - P \in I(X \times Y)$, since $I(X,Y)$ is closed and is an operator ideal by Theorem 7.5.

Now define $V := I - P + P_1$. Then V is a Fredholm operator having index 0. Furthermore $V(M) = P_1(M) \subseteq P_1(X)$. Since $I - P_1 + P$ is also Fredholm and

$$(I - P_1 + P)(X \times Y) \subseteq (I - P_1)(X \times Y) + P(X \times Y) = \ker P_1 + M,$$

we then infer that the sum $\ker P_1 + M$ is a subspace of finite-codimension in $P_1(X \times Y)$. Therefore V is a Fredholm operator of index 0, which maps the subspace M into a subspace of finite codimension of $P_1(X \times Y)$. Now we can find a subspace M_1 of finite-codimension in M such that V isomorphically maps M_1 into $X_1 \times Y_1$ for some subspaces $X_1 \subseteq A_1(X)$ and $Y_1 \subseteq D_1(Y)$. Note that either X_1 has infinite codimension in X, or Y_1 has infinite-codimension in Y. We may assume the first possibility since the second case will follow in an analogous way.

Let N be a complement of M_1 in M. Let us consider a finite-dimensional operator L defined in $X \times Y$ with kernel $\ker L = M_1 \oplus \ker P$ and mapping N into a subspace of X disjoint from X_1. Clearly, $V + L$ is a Fredholm operator having index 0 and $\ker(V + L) \cap M = \{0\}$. Now we can find another finite-dimensional operator K such that M is contained in $\ker K$ and $U := V + L + K$ is bijective. Setting $X_0 := X_1 \oplus L(X \times Y)$ and $Y_0 := Y_1$, we have clearly that U maps isomorphically M onto $X_0 \times Y_0$, so the proof is complete. ■

Let now consider an integer $n \in \mathbb{N}$, and for any Banach space X set

$$\Phi_{\mathrm{n}}(X) := \{T \in \Phi(X) : \operatorname{ind} T = n\}.$$

If we consider an operator $T \in \Phi(X)$, by the Atkinson characterization of Fredholm operators there exists $A \in L(X)$, $K_1, K_2 \in K(X)$ such that

$$AT = I_X - K_1 \quad \text{and} \quad TA = I_X - K_2.$$

From the index theorem we have $\operatorname{ind} A = -\operatorname{ind} T$. Moreover, since for any $T \in \Phi(X)$ we also have $T^k \in \Phi(X)$ for all $k \in \mathbb{N}$, we obtain $\operatorname{ind} T^k = k \operatorname{ind} T$. From this it follows that if $\Phi_{\mathrm{n}}(X)$ is non-empty $\Phi_{\mathrm{nk}}(X)$ is also non-empty. However, there is a Banach space X, defined by Kalton and Peck [**180**], such that $\Phi_2(X)$ is non-empty, but it is not known whether $\Phi_1(X)$ is non-empty. In general it is not known whether, for every Banach space X and every $n > 0$, the class $\Phi_{\mathrm{n}}(X)$ is non-empty.

Theorem 7.107. *Let X, Y be essentially incomparable Banach spaces, and suppose that $\Phi_1(X)$ and $\Phi_1(Y)$ are non-empty. Then for every $n \in \mathbb{N}$, the class $\Phi_n(X \times Y)$ is not connected.*

Proof Given an integer $n \in \mathbb{N}$ there exist $A_0 \in \Phi_{n-1}(X)$, $A_1 \in \Phi_n(X)$ and $D_0 \in \Phi_1(X)$. We have that

$$\begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_1 & 0 \\ 0 & I_Y \end{pmatrix} \in \Phi(X \times Y),$$

but they cannot be continuously connected by a family of Fredholm operators $\begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$, otherwise $\operatorname{ind} A_t$ and $\operatorname{ind} D_t$ would be constant. ■

5.1. Comments. The concept of inessential operators was introduced first by Kleinecke [**188**] in order to find the largest ideal of Riesz operators. The characterization of $I(X, Y)$ as the class of all perturbations of Fredholm operators, here established in Theorem 7.22, is owed to Schechter [**286**], which also shows the characterization of $I(X, Y)$ in terms of the nullity α

given in Theorem 7.24. The class of Atkinson operators was introduced by Atkinson [**51**]. The characterization of inessential operators given in Theorem 7.17 is owed to Pietsch [**262**], whilst all the characterizations in terms of the defects β or $\overline{\beta}$ established in Theorem 7.18 and Theorem 7.22 are owed to Aiena [**1**], [**4**] in the case $X = Y$. Succesively these characterizations were extended to different Banach spaces by Aiena and González [**18**]. The examples where $L(X,Y) = I(X,Y)$ are taken from González [**130**].

The two classes of operators $\Omega_+(X)$ and $\Omega_-(X)$ were introduced by Aiena in [**2**], [**1**] and [**4**], which established all the results of the second section, except for the duality result of Theorem 7.27, which was proved by Volkmann and Wacker [**312**], whilst Theorem 7.31 and Theorem 7.33 were proved in Aiena and González [**18**]; for related results see also Harte [**150**] and [**151**].

The class of strictly singular operators was introduced by Kato [**182**] in his treatment of perturbation theory, whilst the class of strictly cosingular operators was introduced by Pełczyński [**260**]. These classes of operators and the class of inessential operators in the monograph of Pietsch [**263**] form important examples of operator ideals and have been studied by several authors, see, for instance, Schechter [**288**]. An useful class of operators associated with an operator ideal is that of operator semi-group in the sense of Aiena, González, and Martínez-Abejón [**23**]. This concept is, in a sense, opposite to that of operator ideal and includes most of the classical semigroups. To every operator ideal one may associate several semigroups, for instance Fredholm and semi-Fredholm operators are associated with the operator ideal of all compact operators, whilst the Tauberian and coTauberian operators are associated with the operator ideal of weakly compact operators, see, for instance, González and Martínez-Abejón [**132**], [**133**], [**134**], Aiena, González, and Martínez-Abejón [**25**].

The material on subprojective and superprojective Banach spaces is modeled after Whitley [**328**], Pfaffenberger [**261**], Goldberg and Thorp [**128**].

Theorem 7.56 and Theorem 7.57 are taken from Aiena, González, and Martinón [**26**]. The perturbation classes problem which consists in determining whether or not $SS(X,Y) = P\Phi_+(X,Y)$ and $SC(X,Y) = P\Phi_-(X,Y)$ can be traced back to results of Kato [**182**], Gohberg, Markus, and Fel'dman [**127**], and Vladimirskii [**311**]. These problems have been also treated in Caradus, Pfaffenberger and Yood [**76**], Pietsch [**263**] and Tylli [**307**]. The negative answers to these problems, here given in Theorem 7.65, have been given only recently by González [**131**].

Weis in [**317**] studied the perturbation classes for not necessarily continuous, closed semi-Fredholm operators, and gave some other examples of Banach spaces X for which the corresponding perturbation classes coincide with $SS(X)$ or $SC(X)$. However, these perturbation classes for the closed semi-Fredholm operators can be smaller than the classes $P\Phi_+(X)$, $P\Phi_-(X)$

in the case of bounded operators.

The existence of indecomposable Banach spaces was proved by Gowers and Maurey [**137**] and [**138**] and has solved many open questions in Banach space theory. Several of these questions, such as the *hyperplane problem* and the *unconditional basic sequence problem*, were raised by Banach in the early 1930s.

The class of improjective operators was introduced by Tarafdar, [**305**] and [**306**], which showed that all inessential operators are improjective, and in most of the classical Banach spaces the two classes of operators are the same. For this reason, for a quite long period it has been an open problem whether the two classes of operators coincide. The class $\mathrm{Imp}(X, Y)$ was studied in Aiena and González [**20**], which reformulated in several ways the original Tarafdar problem, and succesively the inequality of these classes was proved by Aiena and González in [**21**]. The crucial result of Theorem 7.60, which characterizes the Banach spaces which are hereditarily indecomposable or quotient hereditarily indecomposable in terms of semi-Fredholm operators, is essentially owed to Weis [**317**]. Note that at the time Weis proved this result the existence of hereditarily indecomposable or quotient hereditarily indecomposable Banach spaces was an open problem. To the dissertation thesis of Weis [**316**] is also owed the concept of S-system and its dual concept of C-system. The last section on incomparable Banach spaces is modeled after González [**130**], Aiena and González [**21**], and Tarafdar [**306**].

Note that the notions of incomparability may be also used to define families of semigroups, see Aiena, González, and Martínez-Abejón [**24**].

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